Rough boundaries and wall laws *

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Abstract

We consider Laplace and Stokes operators in domains with rough boundaries and search for an effective boundary condition. The method of homogenization, coupled with the boundary layers, is used to obtain it. In the case of the homogeneous Dirichlet condition at the rough boundary, the effective law is Navier's slip condition, used in the computations of viscous flows in complex geometries. The corresponding effective coefficient is determined by upscaling. It is given by solving an appropriate boundary layer problem. Finally we address application to the drag reduction. In this review article we will explain how those results are obtained, give precise references for technical details and present open problems.

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1 Introduction

Boundary value problems involving rough boundaries arise in many applications, like flows on surfaces with fine longitudinal ribs, rough periodic surface diffraction, cracks for elastic bodies in such situations etc.

An important class of problems is modeling reinforcement by thin layers with oscillating thickness (see e.g. Buttazzo and Kohn [25] and references therein). Reinforcement is described by an important contrast in the coefficients and in the Γ-limit a Robin type boundary condition, with coefficients of order 1, is obtained. Its value is calculated using finite cell auxiliary problems.

Next we can mention homogenization of elliptic problems with the Neumann boundary conditions in domain with rapidly oscillating locally periodic boundaries, depending on small parameter. For more details we refer to [27].

The main goal of this review is to discuss the effective boundary conditions for the Laplace equation and the Stokes system with homogeneous Dirichlet condition at the rough boundary.

In fluid mechanics the widely accepted boundary condition for viscous flows is the no-slip condition, expressing that fluid velocity is zero at an immobile solid boundary. It is only justified where the molecular viscosity is concerned. Since the fluid cannot penetrate the solid, its normal velocity is equal to zero. This is the condition of non-penetration. To the contrary, the absence of slip is not very intuitive. For the Newtonian fluids, it was established experimentally and contested even by Navier himself (see [44]). He claimed that the slip velocity should be proportional to the shear stress. The kinetic-theory calculations have confirmed Navier's boundary condition, but they give the slip length proportional to the mean free path divided by the continuum length (see [47]). For practical purposes it means a zero slip length, justifying the use of the no-slip condition.

In many cases of practical significance the boundary is rough. An example is complex boundaries in the geophysical fluid dynamics. Compared with the characteristic size of a computational domain, such boundaries could be considered as rough. Other examples involve sea bottoms of random roughness and artificial bodies with periodic distribution of small bumps. A numerical simulation of the flow problems in the presence of a rough boundary is very difficult since it requires many mesh nodes and handling of many data. For computational purposes, an artificial smooth boundary, close to the original one, is taken and the equations are solved in the new domain. This way the rough boundary is avoided, but the boundary conditions at the artificial boundary are not given by the physical principles. It is clear that the non-penetration condition $v \cdot n = 0$ should be kept, but there are no reasons to keep the full no-slip condition. Usually it is supposed that the shear stress is a non-linear function F of the tangential velocity. F is determined empirically and its form varies for different problems. Such relations are called the *wall laws* and classical Navier's condition is one example. Another well-known example is modeling of the turbulent boundary layer close to the rough surface by a *logarithmic velocity profile*

$$v_\tau = \sqrt{\frac{\tau_w}{\rho}} \Big(\frac{1}{\kappa} \ln \big(\frac{y}{\mu} \sqrt{\frac{\tau_w}{\rho}} \big) + C^+(k_s^+) \Big) \tag{1}$$

where v_τ is the tangential velocity, y is the vertical coordinate and τ_w the shear stress. ρ denotes the density and μ the viscosity. $\kappa \approx 0.41$ is the von Kármán's constant and C^+ is a function of the ratio k_s^+ of the roughness height k_s and the thin wall sublayer thickness $\delta_v = \frac{\mu}{v_\tau}$. For more details we refer to the book of Schlichting [49] .

Justifying the logarithmic velocity profile in the overlap layer is mathematically out of reach for the moment. Nevertheless, after recent results [35] and [37] we are able to justify the Navier's condition for the laminar incompressible viscous flows over periodic rough boundaries. In [37] the Navier law was obtained for the Couette turbulent boundary layer. We note generalization to random rough boundaries in [15].

In the text which follows, we are going to give a review of rigorous results on Navier's condition.

Somewhat related problem is the homogenization of the Poisson equation in a domain with a periodic oscillating boundary and we start by discussing that situation.

2 Wall law for Poisson's equation with the homogeneous Dirichlet condition at the rough boundary

In our knowledge, mathematically rigorous investigations of the effective wall laws started with the paper by Achdou and Pironneau [1]. They considered

Poisson's equation in a ring with many small holes close to the exterior boundary. They create an oscillating perforated annular layer close to the outer boundary. The amplitude and the period of the oscillations are of order ε and the homogeneous Dirichlet condition is imposed on the solution. In the paper by Achdou and Pironneau [1] the homogenized problem was derived. The rough boundary was replaced by a smooth artificial one and the corresponding wall law was the Robin boundary condition, saying that the effective solution u was proportional to the characteristic roughness ε times its normal derivative. The proportionality constant was calculated using an auxiliary problem for Laplace's operator in a finite cell. Nevertheless, in [1] the conductivity of the thin layer close to the boundary is not small and, contrary to [25], the homogenized boundary condition contains an ε. Consequently, it is not clear that using the finite cell for the auxiliary problem gives the the H^1-error estimate from [1]. Despite this slight criticism, the reference [1] is a pioneering work since it was first to point out that a) keeping homogeneous Dirichlet boundary condition gives an approximation; b) the wall law is a correction of the previous approximation and c) the wall laws are valid for curved rough boundaries.

The readable error estimate for the wall laws, in the case of Poisson's equation and the flat rough boundary is in the paper by Allaire and Amar [4] . They considered a rectangular domain having one face which was a periodic repetition of $\varepsilon\Gamma_g$ and the same boundary value problem as in [1] except periodic lateral conditions. Then they introduced the following auxiliary boundary layer problem in the infinite strip $\Gamma_g \times]0, +\infty[$:

Find a harmonic function ψ, $\nabla\psi \in L^2$, periodic in $y' = (y_1, \dots, y_{n-1})$ and having a value on Γ_g equal to its parametric form. The classical theory (see e.g. [46] or [39]) gives existence of a unique solution which decays exponentially to a constant d. The conclusion of [4] was that the homogenized solution $\bar{u}^\varepsilon$ obeyed the wall law $\bar{u}^\varepsilon = \varepsilon d \dfrac{\partial \bar{u}^\varepsilon}{\partial x_n}$ on the artificial boundary and gave an interior H^1-approximation of order $\varepsilon^{3/2}$. We note the difference in determination of the proportionality constant in the wall law between papers [1] and [4].

It should be pointed out that there is a similarity between the homogenization of Poisson's equation in partially perforated domain and obtaining wall laws for the same equations in presence of rough boundaries. In [30] an effective Robin condition, analogous to one from [1] and [4] was obtained for the artificial boundary in the case of partially perforated domain.

Other important work on Laplace's operator came from the team around Y. Amirat and J. Simon. They were interested in the question if presence of the roughness diminishes the hydrodynamical drag. We will be back to this question in Sec. §4. In [7] and [8] they undertook study on the Couette flow over a rough plate. For the special case of longitudinal grooves, the problem is reduced to the Laplace operator. This research for the case of Laplace operator and for complicated roughness was continued in the doctoral thesis [28] and articles [11], [12] and [21].

Even if the homogeneous Dirichlet condition at the rough boundary is meaningful mostly for flow problems, it makes sense to study the case of Poisson equation. Following Bechert and Bartenwerfer [17] we can interpret it as simplified Stokes system for longitudinal ribs at the outer boundary. Mathematically, it is much easier to treat Laplace's operator than technically complicated Stokes system. We start with a simple problem, which would serve us to present the main ideas.

2.1 The geometry and statement of the model problem

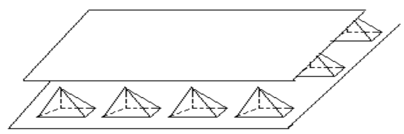

Figure 1: Domain Ω^ε with the rough boundary $\mathcal{B}^\varepsilon$

We consider the Poisson equation in a domain $\Omega^\varepsilon = P \cup \Sigma \cup R^\varepsilon$ consisting of the parallelepiped $P = (0, L_1) \times (0, L_2) \times (0, L_3)$, the interface $\Sigma = (0, L_1) \times (0, L_2) \times \{0\}$ and the layer of roughness $R^\varepsilon = \Big(\cup_{\{k\in\mathbb{Z}^2\}} \varepsilon\big(Y + (k_1 b_1, k_2 b_2, -b_3)\big)\Big) \cap \Big((0, L_1) \times (0, L_2) \times (-\varepsilon b_3, 0)\Big)$. The canonical cell of roughness $Y \subset (0,b_1)\times(0,b_2)\times(0,b_3)$ is defined in Subsection §2.2. Let $\Upsilon = \partial Y \setminus \Sigma$. For simplicity we suppose that $L_1/(\varepsilon b_1)$ and $L_2/(\varepsilon b_2)$ are integers. Let $\mathcal{I} = \{\ k \in \mathbb{Z}^2\ :\ 0 \le k_1 \le L_1/b_1;\ 0 \le k_2 \le L_2/b_2\ \}$. Then, the rough boundary is $\mathcal{B}^\varepsilon = \cup_{\{k\in\mathcal{I}\}}\varepsilon\big(\Upsilon + (k_1b_1, k_2b_2, -b_3)\big)$. It consists of a large number of periodically distributed humps of characteristic length and amplitude ε, small compared with a characteristic length of the macroscopic domain. Finally, let $\Sigma_2 = (0,L_1)\times(0,L_2)\times\{L_3\}$.

We suppose that $f \in C^\infty(\overline{\Omega^\varepsilon})$, periodic in (x_1,x_2) with period (L_1,L_2), and consider the following problem:

$$-\triangle v^\varepsilon = f \qquad \text{in } \Omega^\varepsilon, \tag{2}$$
$$v^\varepsilon = 0 \qquad \text{on } \mathcal{B}^\varepsilon \cup \Sigma_2, \tag{3}$$
$$v^\varepsilon \quad \text{is periodic in } (x_1,x_2) \text{ with period } (L_1,L_2). \tag{4}$$

Obviously problem (2)-(4) admits a unique solution in $H(\Omega^\varepsilon)$, where

$$H(\Omega^\varepsilon) = \{\varphi \in H^1(\Omega^\varepsilon)\ :\ \varphi = 0 \text{ on } \mathcal{B}^\varepsilon \cup \Sigma_2,$$
$$\varphi \text{ is periodic in } x' = (x_1,x_2) \text{ with period } (L_1,L_2)\ \}. \tag{5}$$

By elliptic regularity, $v^\varepsilon \in C^\infty(\Omega^\varepsilon)$. Every element of $H(\Omega^\varepsilon)$ is extended by zero to $(0,L_1)\times(0,L_2)\times(-b_3,0)\setminus R^\varepsilon$.

STEP1: ZERO ORDER APPROXIMATION.

We consider the problem

$$-\triangle u_0 = f \qquad \text{in } P, \tag{6}$$
$$u_0 = 0 \qquad \text{on } \Sigma \cup \Sigma_2, \tag{7}$$
$$u_0 \quad \text{is periodic in } (x_1,x_2) \text{ with period } (L_1,L_2). \tag{8}$$

Obviously problem (6)-(8) admits a unique solution in $H(P)$ and, after extension by zero to $(0,L_1)\times(0,L_2)\times(-b_3,0)$, it is also an element of $H(\Omega^\varepsilon)$. Obviously

$$v^\varepsilon \rightharpoonup u_0, \qquad \text{weakly in} \quad H(P).$$

We wish to have an error estimate.

First we need estimates of the L^2-norms of the function in a domain and at a boundary using the L^2-norm of the gradient. Here the geometrical structure is used in essential way. We have:

Proposition 1. *Let $\varphi \in H(\Omega^\varepsilon)$. Then we have*

$$\|\varphi\|_{L^2(\Sigma)} \leq C\varepsilon^{1/2} \|\nabla_x \varphi\|_{L^2(\Omega^\varepsilon \setminus P)^3}, \tag{9}$$
$$\|\varphi\|_{L^2(\Omega^\varepsilon \setminus P)} \leq C\varepsilon \|\nabla_x \varphi\|_{L^2(\Omega^\varepsilon \setminus P)^3}. \tag{10}$$

This result is well-known and we give its proof only for the comfort of the reader.

Proof. Let $\tilde{\varphi}(y) = \varphi(\varepsilon y), y \in Y+(k_1, k_2, -b_3)$. Then $\tilde{\varphi} \in H^1(Y+(k_1, k_2, -b_3)), \forall k$, and $\varphi = 0$ on $\Upsilon + (k_1, k_2, -b_3)$. Therefore by the trace theorem and the Poincaré's inequality

$$\int_{\{y_3=0\} \cap \bar{Y}+(k_1,k_2)} | \tilde{\varphi}(\tilde{y}, 0) |^2 \ d\tilde{y} \leq C \int_{Y+(k_1,k_2,-b_3)} | \nabla_y \tilde{\varphi} |^2 \ dy.$$

Change of variables and summation over k gives

$$(\int_\Sigma | \varphi(\tilde{x}, 0) |^2 \ d\tilde{x})^{1/2} \leq C\varepsilon^{1/2} (\int_{R^\varepsilon} | \nabla_x \varphi(x) |^2 \ dx)^{1/2}$$

and (9) is proved.

(10) is well-known (see e.g. Sanchez-Palencia [48]). □

Next we introduce $w = v^\varepsilon - u_0$. Then we have

$$-\Delta w = \begin{cases} 0, & \text{in } P \\ f, & \text{in } R^\varepsilon, \end{cases} \tag{11}$$

and $w \in H(\Omega^\varepsilon)$ satisfies the variational equation

$$-\int_\Sigma \frac{\partial u_0}{\partial x_3} \varphi \ dS + \int_{\Omega^\varepsilon} \nabla w \nabla \varphi \ dx = \int_{R^\varepsilon} f \varphi \ dx, \quad \forall \varphi \in H(\Omega^\varepsilon). \tag{12}$$

After testing (12) by $\varphi = w$, and using Proposition 1 we get

$$\int_{\Omega^\varepsilon} |\nabla w|^2 \ dx \leq | \int_{R^\varepsilon} f w \ dx| + | \int_\Sigma \frac{\partial u_0}{\partial x_3} w \ dS| \leq C\sqrt{\varepsilon} ||w||_{L^2(R^\varepsilon)}. \tag{13}$$

We conclude that

$$||\nabla(v^\varepsilon - u_0)||_{L^2(\Omega^\varepsilon)} \leq C\sqrt{\varepsilon}. \tag{14}$$

Could we get some more precise error estimates?

Answer is positive. First, after recalling that the total variation of ∇w is given by

$$\int_{\Omega^\varepsilon} |\nabla w| \ dx = \sup\{ \int_{\Omega^\varepsilon} w \ \text{div} \ \mathbf{s} \ dx \ : \mathbf{s} \in C_0^1(\Omega^\varepsilon; \mathbb{R}^3), \ |\mathbf{s}(x)| \leq 1, \ \forall x \in \Omega^\varepsilon \},$$

we conclude that

$$||v^\varepsilon - u_0||_{BV(\Omega^\varepsilon)} \leq C\varepsilon. \tag{15}$$

Next, we need the notion of the *very weak solution* of the Poisson equation:

Definition 2. *Function $B \in L^2(P)$ is called a very weak solution of the problem*

$$\begin{cases} -\Delta B = G \in H^{-1}(P), & in \, P \\ B = \xi \in L^2(\Sigma \cup \Sigma_2), & on \, \Sigma \cup \Sigma_2 \\ B \ \ is\ periodic\ in \ \ (x_1, x_2) & with\ period \ \ (L_1, L_2). \end{cases} \tag{16}$$

if

$$-\int_P B \Delta \varphi \, dx - \int_{\Sigma_2} \frac{\partial \varphi}{\partial x_3} \xi \, dS + \int_{\Sigma} \frac{\partial \varphi}{\partial x_3} \xi \, dS = \int_P G \varphi \, dx, \quad \forall \varphi \in H(P) \cap C^2(\bar{P}).$$

We recall the following result on very weak solutions to Poisson equation, which is easily proved using transposition:

Lemma 3. *The problem (16) has a unique very weak solution such that*

$$\begin{cases} ||B||_{L^2(\Sigma)} \leq C\{||\xi||_{L^2(\Sigma \cup \Sigma_2)} + ||G||_{H^{-1}(P)}\}; \\ ||B||_{L^2(P)} \leq C\{||\xi||_{L^2(\Sigma \cup \Sigma_2)} + ||G||_{H^{-1}(P)}\}. \end{cases} \tag{17}$$

Direct consequence of Lemma 3 is the estimate

$$\begin{cases} ||v^\varepsilon - u_0||_{L^2(\Sigma)} \leq C\varepsilon; \\ ||v^\varepsilon - u_0||_{L^2(P)} \leq C\varepsilon. \end{cases} \tag{18}$$

Now we see that if we want to have a better estimate, an additional correction is needed.

2.2 Laplace's boundary layer

The effects of roughness occur in a thin layer surrounding the rough boundary. In this subsection we construct the 3D boundary layer, which will be used in taking into account the effects of roughness.

We start by prescribing the geometry of the layer. Let b_j, $j = 1, 2, 3$ be 3 positive constants. Let $Z = (0, b_1) \times (0, b_2) \times (0, b_3)$ and let Υ be a Lipschitz surface $y_3 = \Upsilon(y_1, y_2)$, taking values between 0 and b_3. We suppose that the rough surface $\cup_{k\in\mathbb{Z}^2}\big(\Upsilon + (k_1 b_1, k_2 b_2, 0)\big)$ is also a Lipschitz surface. We introduce the canonical cell of roughness (the canonical hump) by $Y = \big\{y \in Z : \ b_3 > y_3 > \max\{0, \Upsilon(y_1, y_2)\}\big\}$.

The crucial role is played by an auxiliary problem. It reads as follows:
Find β that solves

$$-\triangle_y \beta = 0 \qquad \text{in } Z^+ \cup (Y - b_3\vec{e}_3) \tag{19}$$

$$\big[\beta\big]_S(\cdot, 0) = 0 \quad \text{and} \quad \big[\frac{\partial \beta}{\partial y_3}\big]_S(\cdot, 0) = 1 \tag{20}$$

$$\beta = 0 \quad \text{on } (\Upsilon - b_3\vec{e}_3), \qquad \beta \text{ is } y' = (y_1, y_2) - \text{periodic}, \tag{21}$$

where $S = (0, b_1) \times (0, b_2) \times \{0\}$, $Z^+ = (0, b_1) \times (0, b_2) \times (0, +\infty)$, and $Z_{bl} = Z^+ \cup S \cup (Y - b_3\vec{e}_3)$.

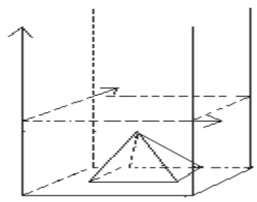

Figure 2: Boundary layer containing the canonical roughness

Let $V = \{z \in L^2_{loc}(Z_{bl}) : \nabla_y z \in L^2(Z_{bl})^3;\ z = 0 \text{ on } (\Upsilon - b_3\vec{e}_3);\ \text{and } z$ is $y' = (y_1, y_2)$-periodic $\}$. Then, by Lax-Milgram lemma, there is a unique

$\beta \in V$ satisfying

$$\int_{Z_{bl}} \nabla\beta\nabla\varphi \; dy = -\int_S \varphi \; dy_1 dy_2, \qquad \forall\varphi \in V. \tag{22}$$

By the elliptic theory, any variational solution β to (19)-(21) satisfies $\beta \in V \cap C^\infty(Z^+ \cup (Y - b_3\vec{e}_3))$.

Lemma 4. *For every $y_3 > 0$ we have*

$$\int_0^{b_1}\int_0^{b_2} \beta(y_1, y_2, y_3) \; dy_1 dy_2 = C^{bl} = \int_S \beta \; dy_1 dy_2 = -\int_{Z_{bl}} |\nabla\beta(y)|^2 \; dy < 0. \tag{23}$$

Next, let $a > 0$ and let β^a be the solution for (19)-(21) with S replaced by $S_a = (0, b_1) \times (0, b_2) \times \{a\}$ and Z^+ by $Z_a^+ = (0, b_1) \times (0, b_2) \times (a, +\infty)$. Then we have

$$C^{a,bl} = \int_0^{b_1}\int_0^{b_2} \beta^a(y_1, y_2, a) \; dy_1 = C^{bl} - ab_1b_2. \tag{24}$$

Proof. Integration of the equation (19) over the section, gives for any $y_3 > a$

$$\frac{d^2}{dy_3^2}\int_0^{b_1}\int_0^{b_2} \beta^a(y_1, y_2, y_3) \; dy_1 dy_2 = 0 \quad \text{on} \quad (a, +\infty). \tag{25}$$

Since $\beta^a \in V$, we conclude that $\int_0^{b_1}\int_0^{b_2} \beta^a(y_1, y_2, y_3) \; dy_1 dy_2$ is constant on $(a, +\infty)$. Then the variational equation (22) yields (23).

Next we have

$$C^{a,bl} = \int_0^{b_1}\int_0^{b_2} \beta^a(y_1, y_2, c) \; dy_1 dy_2, \qquad \forall c \geq a.$$

Let $0 \leq c_1 < a < c_2$. Integration of the equation (19) over (c_1, c_2) gives

$$\int_0^{b_1}\int_0^{b_2} \Big\{ \frac{\partial\beta^a}{\partial y_3}(y_1, y_2, c_2) - \frac{\partial\beta^a}{\partial y_3}(y_1, y_2, a + 0) + \frac{\partial\beta^a}{\partial y_3}(y_1, y_2, a - 0) - \frac{\partial\beta^a}{\partial y_3}(y_1, y_2, c_1) \Big\} \; dy_1 dy_2 = 0.$$

Hence from (20) and (25) we get

$$\frac{d}{dy_3}\int_0^{b_1}\int_0^{b_2} \beta^a(y_1, y_2, y_3) \; dy_1 dy_2 = -b_1b_2, \qquad \text{for} \quad c_1 < y_3 < a$$

and

$$\int_0^{b_1}\int_0^{b_2} \beta^a(y_1,y_2,y_3)\; dy_1dy_2 = (a-y_3)b_1b_2 + C^{a,bl}, \qquad \text{for} \quad 0\le y_3\le a. \tag{26}$$

The variational equation for $\beta^a - \beta$ reads

$$\int_{Z_{BL}} \nabla(\beta^a-\beta)\nabla\varphi\, dy = -\int_0^{b_1}\int_0^{b_2}(\varphi(y_1,y_2,a)-\varphi(y_1,y_2,0))\, dy_1dy_2, \qquad \forall\varphi\in V.$$

Testing with $\varphi=\beta^a-\beta$ and using (26) yields

$$\int_{Z_{BL}} |\nabla(\beta^a-\beta)|^2\, dy = -\int_0^{b_1}\int_0^{b_2}(\beta^a(y_1,y_2,a)-\beta^a(y_1,y_2,0))\, dy_1dy_2 = ab_1b_2.$$

From the other hand

$$\int_{Z_{BL}} |\nabla(\beta^a-\beta)|^2\, dy = \int_{Z_{BL}} |\nabla\beta^a|^2\, dy + \int_{Z_{BL}} |\nabla\beta|^2\, dy - 2\int_{Z_{BL}} \nabla\beta^a\nabla\beta\, dy = C^{bl} - C^{a,bl}$$

and formula (24) is proved. □

Next we search to establish the exponential decay. For the Laplace operator the result is known for long time. General reference for the decay of solutions to boundary layer problems corresponding to the operator $-\text{div}\,(A\nabla u)$, with bounded and positively definite matrix A is [46], where a Saint Venant type estimate was proved. A very readable direct proof for similar setting and covering our situation, is in [4] and in [6]. Nevertheless one of the first known proofs for the case of second order elliptic operators in divergence form is in [39]. Here we will present the main steps of that approach from late seventies.

This early result is based on the following Tartar's lemma:

Lemma 5. *(Tartar's lemma) Let V and V_0 be two real Hilbert spaces such that $V_0\subset V$ with continuous injection. Let a be a continuous bilinear form on $V\times V_0$ and M a surjective continuous linear map between V and V_0. We assume that*

$$a(u,Mu)\ge \alpha||u||_V^2, \quad \alpha>0, \quad \forall u\in V \tag{27}$$

and $f\in V_0'$. Then there exists a unique $u\in V$ such that

$$a(u,v) = < f,v >_{V_0',V_0}, \quad \forall v\in V_0. \tag{28}$$

Proof. For the proof see [39]. We note that this is a variant of Lax-Milgram lemma. $\square$

Now we suppose that

$$\begin{gathered} A = A(y) \quad \text{is a matrix such that} \quad A(y)\xi\cdot\xi \geq C_A|\xi|^2, \quad \text{a.e. and} \\ ||A_{ij}||_\infty \leq \bar{C}_A; \quad g \in H^1_{per}(S); \quad e^{\delta_0 y_3} f \in L^2(Z^+) \;\text{ for some }\; \delta_0 > 0, \end{gathered} \tag{29}$$

and consider the problem

$$-\text{div}\,_y(A(y)\nabla_y \beta) = f \qquad \text{in } Z^+ \tag{30}$$

$$\beta \quad \text{is} \quad y' = (y_1, y_2) - \;\text{periodic}\;; \qquad \beta = g \quad \text{on } S. \tag{31}$$

We have the following result

Proposition 6. *Under conditions (29) the problem (30)-(31) admits a unique solution such that for some $\delta \in (0, \delta_0)$ we have*

$$\begin{cases} \displaystyle\int_0^\infty \int_0^{b_1} \int_0^{b_2} e^{2\delta y_3} |\nabla_y \beta|^2 \; dy < +\infty; \\ \displaystyle\int_0^\infty \int_0^{b_1} \int_0^{b_2} e^{2\delta y_3} |\beta - \frac{1}{b_1 b_2} \int_0^{b_1} \int_0^{b_2} \beta(t, y_3)\; dt|^2 \; dy < +\infty. \end{cases} \tag{32}$$

Proof. We just repeat the main steps from the proof from [39]. It relies on Tartar's lemma.

We introduce the spaces V and V_0 by

$$V = \{z \in L^2_{loc}((0,+\infty); H^1_{per}((0,b_1)\times(0,b_2))) \;:\; e^{\delta y_3}\nabla z \in L^2(Z^+) \;\text{ and }\; z|_S = 0\}$$
$$V_0 = \{z \in V \;:\; e^{\delta y_3} z \in L^2(Z^+)\}.$$

the associated bilinear form is

$$a(u,v) = \int_{Z^+} A\nabla u \nabla(e^{2\delta y_3} v)\; dy, \qquad u \in V,\; v \in V_0, \tag{33}$$

and the linear form is

$$< f, v >_{V_0', V_0} = \int_{Z^+} e^{2\delta y_3} f v \; dy, \qquad v \in V_0. \tag{34}$$

Obviously, the linear form is continuous for $\delta \leq \delta_0$. Same property holds for the bilinear form a.

In the next step we introduce the operator M by setting

$$Mu(y) = u(y) - \frac{2\delta}{b_1 b_2} \int_0^{y_3} \int_0^{b_1} \int_0^{b_2} e^{-2\delta(y_3 - t)} u(y_1, y_2, t) \; dy_1 dy_2 dt. \tag{35}$$

Using Poincaré's inequality in $H^1_{per}((0, b_1) \times (0, b_2))$ we get

$$e^{\delta y_3} Mu \in L^2(Z^+) \qquad \text{and} \quad Mu \in V_0 \;\; \text{for} \;\; \delta < \delta_0. \tag{36}$$

We note that M is surjective since the equation $Mu = v$, $v \in V_0$, admits a solution $u = v + 2\delta \int_0^{y_3} < v >_{(0,b_1)\times(0,b_2)} (t) \; dt \in V$.

Concerning ellipticity, a direct calculation yields

$$a(u, Mu) \geq (\alpha - 2\delta C_P ||A||_\infty) ||e^{\delta y_3} \nabla u||_{L^2(Z^+)}, \tag{37}$$

where C_p is the constant in Poincaré's inequality in $H^1_{per}((0, b_1) \times (0, b_2))$. Therefore, for $\delta < \min\{\delta_0, \dfrac{\alpha}{2C_p} \dfrac{1}{||A||_\infty)}\}$ we have the ellipticity and the Proposition is proved. □

Next, by refining the result of Proposition 6 we get the pointwise exponential decay, as in [46].

2.3 Rigorous derivation of the wall law

After constructing the boundary layer, we are ready for passing to the next order

STEP 2: NEXT ORDER CORRECTION:

From the proof of (13) we see that the main contribution comes from the term corresponding to the artificial interface Σ. Therefore one should eliminate the term $\displaystyle\int_\Sigma \frac{\partial u_0}{\partial x_3} \varphi \; dS$. The correction is given through a new unknown $u^{BL,\varepsilon}$ and we search for $u^{BL,\varepsilon} \in H(\Omega^\varepsilon)$ such that

$$\int_\Sigma \frac{\partial u_0}{\partial x_3} \varphi \; dS + \int_{\Omega^\varepsilon} \nabla u^{BL,\varepsilon} \nabla \varphi \; dx = 0, \qquad \forall \varphi \in H(\Omega^\varepsilon). \tag{38}$$

Since the geometry is periodic this problem can be written as

$$\sum_{\{k\in\mathbb{Z}^2 \; : \; (\varepsilon k_1, \varepsilon k_2) \in (0,L_1)\times(0,L_2)\}} \left\{ \int_{\Upsilon + (\varepsilon k_1 b_1, \varepsilon k_2 b_2)} \frac{\partial u_0}{\partial x_3} \, |_{x_3=0} \varphi|_{x_3=0} \; dS + \right.$$
$$\left. \int_{\varepsilon Z_{bl} + (\varepsilon k_1 b_1, \varepsilon k_2 b_2, 0)} \nabla u^{BL,\varepsilon} \nabla \varphi \; dx \right\} = 0. \tag{39}$$

For $\frac{\partial u_0}{\partial x_3}|_\Sigma$ constant, by uniqueness, the solution to (39) would read $u^{BL,\varepsilon} = \varepsilon\beta(x/\varepsilon)\frac{\partial u_0}{\partial x_3}|_\Sigma$, where β is the solution for (22). In general this is not the case, but this is the candidate for a good approximation. also, the boundary layer function β does not satisfy the homogeneous Dirichlet boundary condition at Σ_2. In order to have correct boundary condition we introduce an auxiliary function v by

$$\begin{cases} -\Delta v = 0 \text{ in } \ P \\ v = \frac{\partial u_0}{\partial x_3}|_\Sigma \ \text{ on } \ \Sigma \ \text{ and } \ v = 0 \ \text{ on } \ \Sigma_2, \\ v \ \text{ is } \ (y_1, y_2) - \text{periodic}. \end{cases} \tag{40}$$

Therefore we search for $u^{BL,\varepsilon}$ in the form

$$u^{BL,\varepsilon} = \varepsilon\Bigg((\beta(\frac{x}{\varepsilon}) - \frac{C^{bl}}{b_1 b_2} H(x_3))\frac{\partial u_0}{\partial x_3}|_\Sigma + \frac{C^{bl}}{b_1 b_2} v(x) H(x_3) \Bigg) - w_\varepsilon, \tag{41}$$

where $C^{bl} < 0$ is a uniquely determined constant such that $e^{\delta y_3}(\beta(y) - \frac{C^{bl}}{b_1 b_2}) \in L^2(Z^+)$ (the boundary layer tail). By Proposition 6 we know that such constant exists and is uniquely determined.

Next by direct calculation, as in [35], we get

- div $(\nabla(\beta(\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_3}|_\Sigma))$ is bounded by $C\varepsilon^{3/2}$ in H^{-1}.
- Jump of the normal derivative of εv at Σ leads also to a term which is bounded by $C\varepsilon^{3/2}$ in H^{-1}.
- Corresponding terms in R^ε are even smaller.

Then after testing by $w_\varepsilon = v^\varepsilon - u_0 + \varepsilon\Bigg((\beta(\frac{x}{\varepsilon}) - \frac{C^{bl}}{b_1 b_2} H(x_3))\frac{\partial u_0}{\partial x_3}|_\Sigma + \frac{C^{bl}}{b_1 b_2} v(x) H(x_3) \Bigg)$, we get that

$$\begin{cases} ||\nabla w_\varepsilon||_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{3/2}, \\ ||w_\varepsilon||_{L^2(\Sigma)} + ||w_\varepsilon||_{L^2(\Omega)} \leq C\varepsilon^2. \end{cases} \tag{42}$$

STEP 3: Derivation of the wall law

Having obtained a good approximation for the solution of the original problem, we get the wall law. We start by a formal derivation:

At the interface Σ we have

$$\frac{\partial v^\varepsilon}{\partial x_3} = \frac{\partial u_0}{\partial x_3} - \frac{\partial \beta(\frac{x}{\varepsilon})}{\partial x_3}\frac{\partial u_0}{\partial x_3} + O(\varepsilon)$$

and

$$\frac{v^\varepsilon}{\varepsilon} = \frac{u_0}{\varepsilon} - \frac{\partial u_0}{\partial x_3}\beta(\frac{x}{\varepsilon}) + O(\varepsilon).$$

After averaging, and using that $C^{bl} = \int_0^{b_1}\int_0^{b_2} \beta(y_1, y_2, 0)\, dy_1 y_2$ and that the mean of the normal derivative is zero, we obtain the familiar form of *the wall law*

$$u^{eff} = -\varepsilon \frac{C^{bl}}{b_1 b_2}\frac{\partial u^{eff}}{\partial x_3} \quad \text{on} \quad \Sigma, \tag{43}$$

where u^{eff} is the average over the impurities and $C^{bl} < 0$ is defined by (23). The higher order terms are neglected.

Let us now give a rigorous justification of *the wall law* (43). First we introduce the effective problem:

$$\begin{cases} -\Delta u^{eff} = f \text{ in } \ P \\ u^{eff} = -\varepsilon \frac{C^{bl}}{b_1 b_2}\frac{\partial u^{eff}}{\partial x_3} = \varepsilon \frac{C^{bl}}{b_1 b_2}\frac{\partial u^{eff}}{\partial \mathbf{n}} \ \text{ on } \ \Sigma \ \text{ and } \ u^{eff} = 0 \ \text{ on } \ \Sigma_2, \\ u^{eff} \ \text{ is } \ (y_1, y_2) - \text{periodic}. \end{cases} \tag{44}$$

How close is u^{eff} to v^ε? In the difference $v^\varepsilon - u^{eff} = w_\varepsilon + u_0 - \varepsilon\Bigg((\beta(\frac{x}{\varepsilon}) - \frac{C^{bl}}{b_1 b_2}H(x_3))\frac{\partial u_0}{\partial x_3}|_\Sigma + \frac{C^{bl}}{b_1 b_2}v(x)H(x_3)\Bigg) - u^{eff}$, the error estimate (42) implies that w_ε is negligible. Next $\varepsilon\Bigg(\beta(\frac{x}{\varepsilon}) - \frac{C^{bl}}{b_1 b_2}\Bigg)\frac{\partial u_0}{\partial x_3}|_\Sigma$ is $O(\varepsilon^{3/2})$ in $L^2(P)$ and $O(\varepsilon^2)$ in $L^1(P)$. Therefore it is enough to consider the function $z_\varepsilon = u_0 - \varepsilon \frac{C^{bl}}{b_1 b_2} v(x) - u^{eff}$. What do we know about this function?

First, we have $\Delta(u_0 - \varepsilon \frac{C^{bl}}{b_1 b_2} v(x) - u^{eff}) = 0$ in P. Then on the lateral boundaries and on Σ_2 it satisfies homogeneous boundary conditions. Finally on Σ we have

$$z_\varepsilon = -\varepsilon \frac{C^{bl}}{b_1 b_2} \frac{\partial z_\varepsilon}{\partial x_3} + \varepsilon^2 (\frac{C^{bl}}{b_1 b_2})^2 \frac{\partial v}{\partial x_3}.$$

Hence z_ε solves the variational equation

$$\int_P \nabla z_\varepsilon \nabla \varphi \, dx - \frac{b_1 b_2}{\varepsilon C^{bl}} \int_\Sigma z_\varepsilon \varphi \, dS = -\frac{\varepsilon C^{bl}}{b_1 b_2} \int_\Sigma \frac{\partial v}{\partial x_3} \varphi \, dS, \quad \forall \varphi \in H(P). \quad (45)$$

Testing (45) by $\varphi = z_\varepsilon$ yields

$$||\nabla z_\varepsilon||_{L^2(P)} \le C\varepsilon^{3/2}, \quad ||z_\varepsilon||_{L^2(\Sigma)} \le C\varepsilon^2 \quad \text{and} \quad ||z_\varepsilon||_{L^2(P)} \le C\varepsilon^2. \quad (46)$$

Using (42), (46) and estimates for the boundary layer β we conclude that

$$\begin{cases} ||v^\varepsilon - u^{eff}||_{L^2(P)} \le C\varepsilon^{3/2}, \\ ||v^\varepsilon - u^{eff}||_{H^1_{loc}(P)} \le C\varepsilon^{3/2}, \\ ||v^\varepsilon - u^{eff}||_{L^1(P)} \le C\varepsilon^2. \end{cases} \quad (47)$$

Note that the approximation on Σ is not good. In fact the boundary layer is concentrated around Σ and there is a price to pay for neglecting it.

STEP 4: Invariance of the wall law

It remains to prove that translation of the artificial boundary of order $O(\varepsilon)$ does not change our effective solution. We have established in Lemma 4 the formula (24), showing how the boundary tail changes with translation of the artificial interface for a. Next using the smoothness of u^{eff} we find out that $u^{eff}(\cdot, x_3 - a\varepsilon)$ satisfies the wall law at $x_3 = a$ with error $O(\varepsilon^2)$. Now if f does not depend on x_3, we see that *the translation of the artificial boundary* at $O(\varepsilon)$ changes the result at order $O(\varepsilon^2)$. Things are more complicated if f depends on x_3.

2.4 Some further questions: almost periodic rough boundaries and curved rough boundaries

In the above sections the roughness was *periodic*. This corresponds to uniformly distributed rough elements. This is acceptable for industrially produces surfaces. Natural rough surfaces contain random irregularly distributed roughness elements.

In applications it is important to derive wall laws for random surfaces. The natural question to be raised is if our construction still works in that case. In estimates we were using Poincaré's inequality and clearly one should impose that our roughness layer does not become of large size with positive probability. But the real difficulty is linked to construction of boundary layers without periodicity assumption.

In this direction there is a recent progress for flow problems (see e.g. [15]), but still there are open questions.

Let us discuss the question of decay at infinity of boundary layers which is crucial for our estimates. We will follow the results by Amar et al from [5].

For sake of simplicity, we shall work in $\mathbb{R}^2$. Our equation will be posed in the half space $\Pi = \{(x, y) \in \mathbb{R}^2 \,:\, y > 0\}$, whose boundary $\partial\Pi$ is the real axis $\{(x, y) \in \mathbb{R}^2 \,:\, y = 0\}$. Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function, which is almost-periodic in the sense of Bohr (simply, almost-periodic), which means that for every $\delta > 0$, there exists a strictly positive number $\ell_\delta > 0$ such that for every real interval of length ℓ_δ there exists a number τ_δ satisfying $\sup_{x\in\mathbb{R}} |h(x+\tau_\delta)-h(x)| \leq \delta$. A well known reference on almost-periodic functions is the book [19].

For any almost-periodic function h, the asymptotic average $M[h] = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} h(x)dx$ is well defined. Furthermore we can associate with h its generalized Fourier series, given by

$$h(x) \sim \sum_{\lambda\in\mathbb{R}} \tilde{h}(\lambda) e^{i\lambda x}, \qquad \tilde{h}(\lambda) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} h(x) e^{-i\lambda x} dx.$$

The number $\tilde{h}(\lambda)$ is the Fourier coefficient of h associated to the frequency λ. It is well known that there exists *at most a countable set* of frequencies for which the Fourier coefficients are different from zero. Also the Parseval identity holds.

Now, in analogy with the periodic case and with almost-periodic data on $\partial\Pi$, we expect to find solutions to Laplace equation that are almost-periodic in the x variable and decay to a certain constant, say d, as y tends to infinity. In the periodic case d was equal to the average of h. In the almost-periodic case, d is given by the asymptotic average $M[h]$, that we may fix to be zero without loss of generality. In analogy with the periodic case, we introduce the following space of weakly decaying functions

$$L^2_{ap}(\Pi) = \{\psi \ : \ x \to \psi(x,y) \quad \text{is almost–periodic } \ \forall y \geq 0, \ ||\psi||^2 = \int_0^{+\infty} \big[\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \psi^2(x,y)\ dx\big]\ dy = \int_0^{+\infty} M\big[\psi^2\big](y)\ dy < +\infty\}.$$

As noted in [5], a trouble with $L^2_{ap}(\Pi)$ is that it is not complete. This is a known disadvantage of Besicovitch's spaces.

Next we study our boundary layer problem. For a given smooth almost-periodic function h it reads

$$\begin{cases} \Delta\psi = 0 \quad \text{in} \quad \Pi, \\ \psi(x,0) = h(x) \quad \text{on} \quad \partial\Pi, \quad M[h] = 0. \end{cases} \tag{48}$$

It is well known that the unique smooth bounded solution for (48) is given by

$$\psi(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{yh(t)}{(x-t)^2+y^2}\ dt. \tag{49}$$

Then we have the following result

Theorem 7. *(see [5]) Let ψ be the unique bounded solution of (48). Then, for every fixed $y > 0$, the function $x \to \psi(x,y)$ is an almost-periodic function. Moreover, for any given $\gamma_0 > 0$, the following equivalence condition holds: $||\psi e^{\gamma y}|| < +\infty$ for every $0 < \gamma < \gamma_0$ if and only if $\tilde{h}(\lambda) = 0$ for every $|\lambda| < \gamma_0$.*

Further analysis in [5] lead to the conclusion that the necessary and sufficient condition for the exponential decay is that the frequencies λ of h are far from zero. It is worthwhile to point out that, in the purely periodic case, the frequencies are always far from zero and hence the exponential decay of the solution is in accordance with previous theorem. On the contrary, in the general almost-periodic case, the exponential decay property fails if the frequencies of h accumulate at zero. Difficulties are illustrated through the following explicit example from [5]:

Let $h(x) = \sum_{n=1}^{+\infty} \frac{1}{n^2}\sin(\frac{x}{n^3})$. Then the the series converges uniformly, the function h is well defined, almost-periodic and satisfies $M[h] = 0$. With this h, the problem (48) has a unique bounded solution

$$\psi(x,y) = \sum_{n=1}^{+\infty} \frac{1}{n^2}\sin(\frac{x}{n^3})e^{-y/n^2}, \quad \text{with} \quad ||\psi|| = +\infty.$$

In this case not only that we do not have an exponential decay, but ψ is even not in the space $L^2_{ap}(\Pi)$.

We can only conclude that a reasonable theory would be possible in a correct setting and with well-prepared data.

Next difficulty is linked with the fact that in nature one has to handle *curved rough boundaries.* In the pioneering paper [1] the roughness was linked to a curved circular boundary. This work continued mainly with formal multiscale expansions and numerical simulations for flow problems (see [2], [3], [43] and references therein). Nevertheless, there is a recent article [41] by Madureira and Valentin, with analysis of the curvature influence on 2D effective wall laws. Their geometry is essentially annular and it was possible to describe the rough surface using just angular variable. Their boundary layer problems are posed in an open angle and the connection with known results is to be established. Also their Laplace's operator in polar coordinates systematically misses a term. The paper gives ideas but not really the complete construction of the approximation. Furthermore, we note that the two-dimensional case is very special because it allows for a global isometric parametrization of the boundary, while in the multidimensional case even the correct formulation of the problem setting is not obvious.

Derivation of the approximations and effective boundary conditions for solutions of the Poisson equation on a domain in $\mathbb{R}^n$ whose boundary differs from the smooth boundary of a domain $\mathbb{R}^n$ by rapid oscillations of size ε, was considered in [45]. More precisely, the Poisson equation was supposed in a bounded or unbounded domain Ω of $\mathbb{R}^n$, $n \geq 2$, with smooth compact boundary $\Gamma = \partial\Omega$, being an $(n-1)$-dimensional Riemannian manifold. Using the unit outer normal ν to Γ, the tubular neighborhood of Γ was defined by the mapping $\mathcal{T} : (x,t) \to x + t\nu(x)$, defined on $\Gamma \times (-\delta, \delta)$. Then, using a function γ^ε from Γ to $\mathbb{R}$ such that $|\gamma^\varepsilon(x)| \leq \varepsilon M < \delta/2$ on Γ, and that γ^ε is locally ε-periodic through an atlas of charts, it was possible to define a rough boundary $\Gamma^\varepsilon = \mathcal{T}(x, \gamma^\varepsilon(x)); x \in \Gamma$. For this fairly general geometric situation it was possible to accomplish the steps 1 to 3 from the above construction, for the flat rough boundary. The wall law (43) was obtained again. Nevertheless, it was found that the coefficient C^{bl} depends on position. The position was present as a parameter in the boundary layer construction. The construction from [45] is to be extended to systems, most notably to the Stokes system.

3 Wall laws for the Stokes and Navier-Stokes equations

In the text which follows we will try to give a brief resume of the results concerning the wall laws for the incompressible Stokes and Navier-Stokes equations. Also we will recall the basic steps of the construction of the boundary layer corrections, following the approach from [31].

Flow problems over rough surfaces were considered by O. Pironneau and collaborators in [43] , [2] and [3]. The paper [43] considers the flow over a rough surface and the flow over a wavy sea surface. It discusses a number of problems and announces a rigorous result for an approximation of the Stokes flow. Similarly, in the paper [2] numerical calculations are presented and rigorous results in [3] are announced. Finally, in the paper [3] the stationary incompressible flow at high Reynolds number $\mathbf{Re} \sim \frac{1}{\varepsilon}$ over a periodic rough boundary, with the roughness period ε, is considered. An asymptotic expansion is constructed and, with the help of boundary layer correctors defined in a semi-infinite cell, effective wall laws are obtained. A numerical validation is presented, but there are no mathematically rigorous convergence results. The error estimate for the approximation, announced in [2], was not proved in [3]. We mention also the article [14].

In this section we are going to present a sketch of the justification of the Navier slip law by the technique developed in [30] for Laplace's operator and then in [31] for the Stokes system. The result for a 2D laminar stationary incompressible viscous flow over a rough boundary is in [35]. It presents a generalization of the analogous results on the justification of the law by Beavers and Joseph [16] for a tangential viscous flow over a porous bed, obtained in [32], [33], [34] and [36]. For a review we refer to [42] and [38]. In the subsections which follow we consider a 3D Couette flow over a rough boundary. In §3.1 we introduce the corresponding boundary layer problem and in §3.2 we present the main steps in obtaining the Navier slip condition from [37].

3.1 Navier's boundary layer

As observed in hydrodynamics, the phenomena relevant to the boundary occur in a thin layer surrounding it. We are not interested in the boundary layers corresponding to the inviscid limit of the Navier-Stokes equations, but we undertake to construct the viscous boundary layer describing effects of the roughness. There is a similarity with boundary layers describing effects of interfaces between a perforated and a non-perforated domain. The corresponding theory for the Stokes system is in [31] and, in a more pedagogical way, in [42]. In this subsection we are going to present a sketch of construction of the main boundary layer, used for determining the coefficient in Navier's condition. It is natural to call it the *Navier's boundary layer.* In [35] the 2D boundary layer was constructed and the 3D case was studied in [37] .

We suppose the layer geometry from the beginning of the subsection 2.2.

Following the construction from [35], the crucial role is played by an auxiliary problem. It reads as follows:

For a given constant vector $\lambda \in \mathbb{R}^2$, find $\{\beta^\lambda, \omega^\lambda\}$ that solve

$$-\triangle_y \beta^\lambda + \nabla_y \omega^\lambda = 0 \qquad \text{in } Z^+ \cup (Y - b_3\vec{e}_3) \tag{50}$$

$$\operatorname{div}_y \beta^\lambda = 0 \qquad \text{in } Z_{bl} \tag{51}$$

$$\big[\beta^\lambda\big]_S(\cdot, 0) = 0 \quad \text{ on } S \tag{52}$$

$$\big[\{\nabla_y \beta^\lambda - \omega^\lambda I\}\vec{e}_3\big]_S(\cdot, 0) = \lambda \;\text{ on } S \tag{53}$$

$$\beta^\lambda = 0 \quad \text{ on } (\Upsilon - b_3\vec{e}_3), \qquad \{\beta^\lambda, \omega^\lambda\} \text{ is } y' = (y_1, y_2) - \text{periodic}, \tag{54}$$

where $S = (0, b_1) \times (0, b_2) \times \{0\}$, $Z^+ = (0, b_1) \times (0, b_2) \times (0, +\infty)$, and $Z_{bl} = Z^+ \cup S \cup (Y - b_3\vec{e}_3)$.

Let $V = \{z \in L^2_{loc}(Z_{bl})^3 : \nabla_y z \in L^2(Z_{bl})^9;\ z = 0 \text{ on } (\Upsilon - b_3\vec{e}_3);\ \operatorname{div}_y z = 0$ in Z_{bl} and z is $y' = (y_1, y_2)$-periodic $\}$. Then, by the Lax-Milgram lemma, there is a unique $\beta^\lambda \in V$ satisfying

$$\int_{Z_{bl}} \nabla \beta^\lambda \nabla \varphi \, dy = - \int_S \varphi \lambda \, dy_1 dy_2, \qquad \forall \varphi \in V. \tag{55}$$

Using De Rham's theorem we obtain a function $\omega^\lambda \in L^2_{loc}(Z_{bl})$, unique up to a constant and satisfying (50). By the elliptic theory, $\{\beta^\lambda, \omega^\lambda\} \in V \cap C^\infty(Z^+ \cup (Y - b_3\vec{e}_3))^3 \times C^\infty(Z^+ \cup (Y - b_3\vec{e}_3))$, for any solution to (50)-(54).

In the neighborhood of S we have $\beta^\lambda - (\lambda_1, \lambda_2, 0)(y_3 - y_3^2/2)e^{-y_3} H(y_3) \in W^{2,q}$ and $\omega^\lambda \in W^{1,q}$, $\forall q \in [1, \infty)$.

Then we have

Lemma 8. *([31], [32], [42]). For any positive a, a_1 and a_2, $a_1 > a_2$, the solution $\{\beta^\lambda, \omega^\lambda\}$ satisfies*

$$\begin{cases} \int_0^{b_1}\int_0^{b_2} \beta_2^\lambda(y_1, y_2, a)\ dy_1dy_2 = 0, \\ \int_0^{b_1}\int_0^{b_2} \omega^\lambda(y_1, y_2, a_1)\ dy_1dy_2 = \int_0^{b_1}\int_0^{b_2} \omega^\lambda(y_1, y_2, a_2)\ dy_1dy_2, \\ \int_0^{b_1}\int_0^{b_2} \beta_j^\lambda(y_1, y_2, a_1)dy_1dy_2 = \int_0^{b_1}\int_0^{b_2} \beta_j^\lambda(y_1, y_2, a_2)dy_1dy_2, j = 1, 2; \\ C_\lambda^{bl} = \sum_{j=1}^2 C_\lambda^{j,bl}\lambda_j = \int_S \beta^\lambda\lambda\ dy_1dy_2 = -\int_{Z_{bl}} |\nabla\beta^\lambda(y)|^2\ dy < 0. \end{cases} \tag{56}$$

Lemma 9. *Let $\lambda \in \mathbb{R}^2$ and let $\{\beta^\lambda, \omega^\lambda\}$ be the solution for (50)-(54) satisfying $\int_S \omega^\lambda\ dy_1dy_2 = 0$. Then $\beta^\lambda = \sum_{j=1}^2 \beta^j\lambda_j$ and $\omega^\lambda = \sum_{j=1}^2 \omega^j\lambda_j$, where $\{\beta^j, \omega^j\} \in V \times L^2_{loc}(Z_{bl})$, $\int_S \omega^j\ dy_1dy_2 = 0$, is the solution for (50)-(54) with $\lambda = \vec{e}_j$, $j = 1, 2$.*

Lemma 10. *Let $a > 0$ and let $\beta^{a,\lambda}$ be the solution for (50)-(54) with S replaced by $S_a = (0, b_1) \times (0, b_2) \times \{a\}$ and Z^+ by $Z_a^+ = (0, b_1) \times (0, b_2) \times (a, +\infty)$. Then we have*

$$C_\lambda^{a,bl} = \int_0^{b_1}\int_0^{b_2} \beta^{a,\lambda}(y_1, y_2, a)\lambda\ dy_1 = C_\lambda^{bl} - a \mid \lambda \mid^2 b_1b_2 \tag{57}$$

Proof. It goes along the same lines as Lemma 2 from [35] and we omit it. □

Lemma 11. *(see [37]) Let $\{\beta^j, \omega^j\}$ be as in Lemma 8 and let $M_{ij} = \frac{1}{b_1b_2}\int_S \beta_i^j\ dy_1dy_2$ be the Navier matrix. Then the matrix M is symmetric negatively definite.*

Lemma 12. *(see [37]) Let Y have the mirror symmetry with respect to y_j, where j is 1 or 2. The the matrix M is diagonal.*

Lemma 13. *(see [37]) Let us suppose that the shape of the boundary doesn't depend on y_2 . Then for $\lambda = \vec{e}_2$ the system (50)-(54) has the solution $\beta^2 =$*

$(0,\beta_2^2(y_1,y_3),0)$ and $\omega^2=0$, where β_2^2 is determined by

$$-\frac{\partial^2\beta_2^2}{\partial y_1^2}-\frac{\partial^2\beta_2^2}{\partial y_3^2}=0 \qquad in\ (0,b_1)\times(0,+\infty)\cup(Y\cap\{y_2=0\}-b_3\vec{e}_3) \quad (58)$$

$$\big[\beta_2^2\big](\cdot,0)=0 \quad on\ (0,b_1)\times\{0\} \quad (59)$$

$$\big[\frac{\partial\beta_2^2}{\partial y_3}\big](\cdot,0)=1 \quad on\ (0,b_1)\times\{0\} \quad (60)$$

$$\beta_2^2=0 \quad on\ (\Upsilon\cap\{y_2=0\}-b_3\vec{e}_3),\quad \beta_2^2\ is\ y_1-periodic, \quad (61)$$

Furthermore, for $\lambda=\vec{e}_1$, the system (50)-(54) has the solution $\beta^1=(\beta_1^1(y_1,y_3),$ $0,\beta_3^1(y_1,y_3))$ and $\omega^1=\omega(y_1,y_3)$ satisfying

$$-\frac{\partial\beta_j^1}{\partial y_1^2}-\frac{\partial\beta_j^1}{\partial y_3^2}+\frac{\partial\omega}{\partial y_j}=0\ \ in\ (0,b_1)\times(0,+\infty)\cup(Y\cap\{y_2=0\}-b_3\vec{e}_3),$$

$$j=1\ and\ j=3 \quad (62)$$

$$\frac{\partial\beta_1^1}{\partial y_1}+\frac{\partial\beta_3^1}{\partial y_3}=0 \qquad in\ Z_{bl}\cap\{y_2=0\} \quad (63)$$

$$\big[\beta_j^1\big](\cdot,0)=0\ \ on\ (0,b_1)\times\{0\},\ j=1\ \ and\ \ j=3 \quad (64)$$

$$\big[\omega\big]=0\ \ and\ \ \big[\frac{\partial\beta_1^1}{\partial y_3}\big](\cdot,0)=1,\ \big[\frac{\partial\beta_3^1}{\partial y_3}\big](\cdot,0)=1 \quad on\ (0,b_1)\times\{0\} \quad (65)$$

$$\beta_1^1=\beta_3^1=0 \quad on\ (\Upsilon\cap\{y_2=0\}-b_3\vec{e}_3),\ \{\beta_1^1,\beta_3^1,\omega\}\ \ is\ y_1-periodic. \quad (66)$$

Finally,

$$\begin{cases} M_{11}=\dfrac{1}{b_1}\displaystyle\int_0^{b_1}\beta_1^1(y_1,0)\ dy_1\\ M_{12}=M_{21}=0\\ M_{22}=\dfrac{1}{b_1}\displaystyle\int_0^{b_1}\beta_2^2(y_1,0)\ dy_1\end{cases} \quad (67)$$

and $\mid M_{11}\mid\le\mid M_{22}\mid$.

Lemma 14. *Let $\{\beta^j,\omega^j\}, j=1$ and $j=3$, be as in Lemma 8. Then we have*

$$\begin{cases}\mid D^\alpha\ curl_y\beta^j(y)\mid\le Ce^{-2\pi y_3\min\{1/b_1,1/b_2\}}, \quad y_3>0,\ \alpha\in I\!N^2\cup(0,0)\\ \mid\beta^j(y)-(M_{1j},M_{2j},0)\mid\le C(\delta)e^{-\delta y_3},\ y_3>0,\ \forall\delta<2\pi\min\{1/b_1,1/b_2\}\\ \mid D^\alpha\beta^j(y)\mid\le C(\delta)e^{-\delta y_3},\ y_3>0,\ \alpha\in I\!N^2,\ \forall\delta<2\pi\min\{1/b_1,1/b_2\}\\ \mid\omega^j(y)\mid\le Ce^{-2\pi y_3\min\{1/b_1,1/b_2\}}, \qquad y_3>0.\end{cases}$$

$$(68)$$

Proof. As in [32] we take the curl of the equation (50) and obtain the following problem for $\xi^j_m = \left(\text{curl } \beta^j\right)_m$, $m = 1, 2, 3$

$$\begin{cases} \Delta \xi^j_m = 0 \qquad \text{in } Z^+ \\ \xi^j_m \in W^{1-1/q,q}(S), \ \forall q < +\infty \quad \xi^j_m \text{ is periodic in } y' = (y_1, y_2) \end{cases} \tag{69}$$

Now Tartar's lemma from [39] (see Lemma 5) implies an exponential decay of $\nabla \xi^j_m$ to zero and of ξ^j_m. Since $\xi^j_m \in L^2(Z^+)$, this constant equals to zero. Furthermore, having established an exponential decay, we are in situation to apply the separation of variables. Then explicit calculations, analogous to those in [36], give the first estimate in (68).

In the next step we use the following identity, holding for the divergence free fields:

$$-\Delta \beta^j = \text{ curl } \text{ curl } \beta^j = \text{ curl } \xi^j$$

and the same arguing as above leads to the second and the third estimate.

After taking the divergence of the equation (50) we find out that the pressure is harmonic in Z^+. Since the averages of the pressure over the sections $\{y_3 = a\}$ are zero, we obtain the last estimate in (69). □

Corollary 15. *The system (50)-(54) defines a boundary layer.*

3.2 Justification of the Navier slip condition for the laminar 3D Couette flow

A mathematically rigorous justification of the Navier slip condition for the 2D Poiseuille flow over a rough boundary is in [35]. Rough boundary was the periodic repetition of a basic cell of roughness, with characteristic heights and lengths of the impurities equal to a small parameter ε. Then the flow domain was decomposed to a rough layer and its complement.

The no-slip condition was imposed on the rough boundary and there were inflow and outflow boundaries, not interacting with the humps. The flow was governed by a given constant pressure drop. The mathematical model were the stationary Navier-Stokes equations. In [35] the flow under moderate Reynolds numbers was considered and the following results were proved:

a) A non-linear stability result with respect to small perturbations of the smooth boundary with a rough one;

b) An approximation result of order $\varepsilon^{3/2}$;

c) Navier's slip condition was justified.

In this review we are going to present analogous results for a 3D Couette flow from [37].

We consider a viscous incompressible fluid flow in a domain Ω^ε defined in Subsection §2.1

Then, for a fixed $\varepsilon > 0$ and a given constant velocity $\vec{U} = (U_1, U_2, 0)$, the Couette flow is described by the following system

$$-\nu \triangle \mathbf{v}^\varepsilon + (\mathbf{v}^\varepsilon \nabla)\mathbf{v}^\varepsilon + \nabla p^\varepsilon = 0 \qquad \text{in } \Omega^\varepsilon, \tag{70}$$

$$\text{div } \mathbf{v}^\varepsilon = 0 \qquad \text{in } \Omega^\varepsilon, \tag{71}$$

$$\mathbf{v}^\varepsilon = 0 \qquad \text{on } \mathcal{B}^\varepsilon, \tag{72}$$

$$\mathbf{v}^\varepsilon = \vec{U} \qquad \text{on } \Sigma_2 = (0, L_1) \times (0, L_2) \times \{L_3\} \tag{73}$$

$$\{\mathbf{v}^\varepsilon, p^\varepsilon\} \quad \text{is periodic in } (x_1, x_2) \text{ with period } (L_1, L_2) \tag{74}$$

where $\nu > 0$ is the kinematic viscosity and $\int_\Omega^\varepsilon p^\varepsilon \, dx = 0$.

Let us note that a similar problem was considered in [9], but in an infinite strip with a rough boundary. In [9] the authors were looking for solutions periodic in (x_1, x_2), with the period $\varepsilon(b_1, b_2)$.

Since we need not only existence for a given ε, but also the a priori estimates independent of ε, we give a non-linear stability result with respect to rough perturbations of the boundary, leading to uniform a priori estimates.

First, we observe that the Couette flow in P, satisfying the no-slip conditions at Σ, is given by

$$\mathbf{v}^0 = \frac{U_1 x_3}{L_3}\vec{e}_1 + \frac{U_2 x_3}{L_3}\vec{e}_2 = \vec{U}\frac{x_3}{L_3}, \qquad p^0 = 0. \tag{75}$$

Let $|U| = \sqrt{U_1^2 + U_2^2}$. Then it is easy to see that $\mathbf{v}^0$ is the unique solution for the Couette flow in P if $|U| L_3 < 2\nu$, i.e. if the Reynolds number is moderate.

We extend the velocity field to $\Omega^\varepsilon \setminus P$ by zero.

The idea is to construct the solution to (70)-(74) as a small perturbation to the Couette flow (75). Before the existence result, we prove an auxiliary lemma:

Lemma 16. *([35]). Let* $\varphi \in H^1(\Omega^\varepsilon \setminus P)$ *be such that* $\varphi = 0$ *on* $\mathcal{B}^\varepsilon$*. Then we have*

$$\|\varphi\|_{L^2(\Omega^\varepsilon\setminus P)} \le C\varepsilon \|\nabla\varphi\|_{L^2(\Omega^\varepsilon\setminus P)^3}, \tag{76}$$

$$\|\varphi\|_{L^2(\Sigma)} \le C\varepsilon^{1/2} \|\nabla\varphi\|_{L^2(\Omega^\varepsilon\setminus P)^3}. \tag{77}$$

Now we are in position to prove the desired non-linear stability result:

Theorem 17. *([37]). Let* $|U|L_3 \le \nu$*. Then there exists a constant* $C_0 = C_0(b_1, b_2, b_3,\ L_1, L_2)$ *such that for* $\varepsilon \le C_0(\frac{L_3}{|U|})^{3/4}\nu^{3/4}$ *the problem (70)-(74) has a unique solution* $\{\mathbf{v}^\varepsilon, p^\varepsilon\} \in H^2(\Omega^\varepsilon)^3 \times H^1(\Omega^\varepsilon)$*,* $\int_\Omega^\varepsilon p^\varepsilon \, dx = 0$*, satisfying*

$$\|\nabla(\mathbf{v}^\varepsilon - \mathbf{v}^0)\|_{L^2(\Omega^\varepsilon)^9} \le C\sqrt{\varepsilon}\frac{|U|}{L_3}. \tag{78}$$

Moreover,

$$\|\mathbf{v}^\varepsilon\|_{L^2(\Omega^\varepsilon\setminus P)^3} \le C\varepsilon\sqrt{\varepsilon}\frac{|U|}{L_3}, \tag{79}$$

$$\|\mathbf{v}^\varepsilon\|_{L^2(\Sigma)^3} + \|\mathbf{v}^\varepsilon - \mathbf{v}^0\|_{L^2(P)^3} \le C\varepsilon\frac{|U|}{L_3}, \tag{80}$$

$$\|p^\varepsilon - p^0\|_{L^2(P)} \le C\frac{|U|}{L_3}\sqrt{\varepsilon}, \tag{81}$$

where $C = C(b_1, b_2, b_3, L_1, L_2)$*.*

Therefore, we have obtained the uniform a priori estimates for $\{\mathbf{v}^\varepsilon, p^\varepsilon\}$. Moreover, we have found that Couette's flow in P is an $O(\varepsilon)$ L^2-approximation for $\mathbf{v}^\varepsilon$.

Following the approach from [35], the Navier slip condition should correspond to taking into the account the next order corrections for the velocity. Then formally we get

$$\mathbf{v}^\varepsilon = \mathbf{v}^0 - \frac{\varepsilon}{L_3}\sum_{j=1}^{2} U_j\big(\beta^j(\frac{x}{\varepsilon}) - (M_{j1}, M_{j2}, 0)H(x_3)\big) - \frac{\varepsilon}{L_3}\sum_{j=1}^{2} U_j(1 - \frac{x_3}{L_3})(M_{j1}, M_{j2}, 0)H(x_3) + O(\varepsilon^2)$$

where $\mathbf{v}^0$ is the Couette velocity in P and the last term corresponds to the counterflow generated by the motion of Σ. Then on the interface Σ

$$\frac{\partial \mathbf{v}^\varepsilon{}_j}{\partial x_3} = \frac{U_j}{L_3} - \frac{1}{L_3}\sum_{i=1}^{2} U_i \frac{\partial \beta^i_j}{\partial y_3} + O(\varepsilon) \quad \text{and} \quad \frac{1}{\varepsilon}\mathbf{v}^\varepsilon{}_j = -\frac{1}{L_3}\sum_{i=1}^{2} U_i \beta^i_j(\frac{x}{\varepsilon}) + O(\varepsilon).$$

After averaging we obtain the familiar form of the Navier slip condition

$$u_j^{eff} = -\varepsilon \sum_{i=1}^{2} M_{ji} \frac{\partial u_i^{eff}}{\partial x_3} \quad \text{on} \quad \Sigma, \tag{82}$$

where u^{eff} is the average over the impurities and the matrix M is defined in Lemma 11. The higher order terms are neglected.

Now let us make this formal asymptotic expansion rigorous.

It is clear that in P the flow continues to be governed by the Navier-Stokes system. The presence of the irregularities would only contribute to the effective boundary conditions at the lateral boundary. The leading contribution for the estimate (78) were the interface integral terms $\int_\Sigma \varphi_j$. Following the approach from [35], we eliminate it by using the boundary layer-type functions

$$\beta^{j,\varepsilon}(x) = \varepsilon \beta^j(\frac{x}{\varepsilon}) \quad \text{and} \quad \omega^{j,\varepsilon}(x) = \omega^j(\frac{x}{\varepsilon}), \quad x \in \Omega^\varepsilon, \ j = 1, 2, \tag{83}$$

where $\{\beta^j, \omega^j\}$ is defined in Lemma 8. We have, for all $q \geq 1$ and $j = 1, 2$,

$$\frac{1}{\varepsilon}\|\beta^{j,\varepsilon} - \varepsilon(M_{1j}, M_{2j}, 0)\|_{L^q(P)^3} + \|\omega^{j,\varepsilon}\|_{L^q(P)} + \|\nabla \beta^{j,\varepsilon}\|_{L^q(\Omega)^9} = C\varepsilon^{1/q} \tag{84}$$

and

$$-\triangle \beta^{j,\varepsilon} + \nabla \omega^{j,\varepsilon} = 0 \qquad \text{in } \Omega^\varepsilon \setminus \Sigma, \tag{85}$$

$$\text{div } \beta^{j,\varepsilon} = 0 \qquad \text{in } \Omega^\varepsilon, \tag{86}$$

$$\big[\beta^{j,\varepsilon}\big]_\Sigma(\cdot, 0) = 0 \quad \text{on } \Sigma, \tag{87}$$

$$\big[\{\nabla \beta^{j,\varepsilon} - \omega^{j,\varepsilon} I\} e_3\big]_\Sigma(\cdot, 0) = e_j \quad \text{on } \Sigma. \tag{88}$$

As in [35] stabilization of $\beta^{j,\varepsilon}$ towards a nonzero constant velocity $\varepsilon\big(M_{1j}, M_{2j}, 0\big)$, at the upper boundary, generates a counterflow. It is given by the 3D Couette flow $d^i = (1 - \dfrac{x_3}{L_3})\vec{e}_i$ and $g^i = 0$.

Now, we would like to prove that the following quantities are $o(\varepsilon)$ for the velocity and $O(\varepsilon)$ for the pressure:

$$\mathcal{U}^\varepsilon(x) = \mathbf{v}^\varepsilon - \frac{1}{L_3}\Big(x_3^+\vec{U} - \varepsilon\sum_{j=1}^{2} U_j\beta^j(\frac{x}{\varepsilon}) + \varepsilon\frac{x_3^+}{L_3}M\vec{U}\Big), \tag{89}$$

$$\mathcal{P}^\varepsilon = p^\varepsilon + \frac{\nu}{L_3}\sum_{j=1}^{2} U_j\omega^{j,\varepsilon}. \tag{90}$$

Then we have the following result:

Theorem 18. *([37]). Let $\mathcal{U}^\varepsilon$ be given by (89) and $\mathcal{P}^\varepsilon$ by (90). Then $\mathcal{U}^\varepsilon \in H^1(\Omega^\varepsilon)^3$, $\mathcal{U}^\varepsilon = 0$ on Σ, it is periodic in (x_1, x_2), exponentially small on Σ_2 and div $\mathcal{U}^\varepsilon = 0$ in Ω^ε. Furthermore, $\forall\varphi$ satisfying the same boundary conditions, we have the following estimate*

$$|\nu\int_{\Omega^\varepsilon}\nabla\mathcal{U}^\varepsilon\nabla\varphi - \int_{\Omega^\varepsilon}\mathcal{P}^\varepsilon \text{div}\varphi + \int_{\Omega^\varepsilon}\frac{x_3^+}{L_3}\sum_{j=1}^{2}U_j\frac{\partial\mathcal{U}^\varepsilon}{\partial x_j}\varphi + \int_{\Omega^\varepsilon}\mathcal{U}_3^\varepsilon\frac{\vec{U}}{L_3}\varphi$$
$$+\int_{\Omega^\varepsilon}\big((\mathbf{v}^\varepsilon - \mathbf{v}^0)\nabla\big)(\mathbf{v}^\varepsilon - \mathbf{v}^0)\varphi| \le C\varepsilon^{3/2}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)^9}\frac{|U|^2}{L_3}. \tag{91}$$

Corollary 19. *([37]). Let $\mathcal{U}^\varepsilon(x)$ and $\mathcal{P}^\varepsilon$ be defined by (89)-(90) and let*

$$\varepsilon \le \frac{\nu^{6/7}}{|U|}\min\big\{\frac{\nu^{1/7}}{4(|M| + \|\beta\|_{L^\infty})}, C(b_1, b_2, b_3, L_1, L_2)L_3^{3/7}|U|^{1/7}\big\}. \tag{92}$$

Then $\mathbf{v}^\varepsilon$, constructed in Theorem 17, is a unique solution to (70)-(74) and

$$\|\nabla\mathcal{U}^\varepsilon\|_{L^2(\Omega^\varepsilon)^9} + \mathcal{P}^\varepsilon\|_{L^2(P)} \le C\varepsilon^{3/2}\frac{|U|^2}{\nu L_3}, \tag{93}$$

$$\|\mathcal{U}^\varepsilon\|_{L^2(P)^3} + \|\mathcal{U}^\varepsilon\|_{L^2(\Sigma)^3} \le C\varepsilon^2\frac{|U|^2}{\nu L_3}. \tag{94}$$

The estimates (93)-(94) allow to justify Navier's slip condition.

Remark 20. *It is possible to add further correctors and then our problem would contain an exponentially decreasing forcing term. This is in accordance with [9] for the Navier-Stokes system and with [7], [8] and [13] for the*

Stokes system. For the case of rough boundaries with different characteristic heights and lengths we refer to the doctoral dissertation of I. Cotoi [28]. The estimate (92) is of the same order in ε as the H^1-estimate in [4], obtained for the Laplace operator. The advantage of our approach is that we are going to obtain the Navier slip condition with a negatively definite matricial coefficient.

Now we introduce the effective Couette-Navier flow through the following boundary value problem:

Find a velocity field u^{eff} and a pressure field p^{eff} such that

$$-\nu\triangle\mathbf{u}^{eff} + (\mathbf{u}^{eff}\nabla)\mathbf{u}^{eff} + \nabla p^{eff} = 0 \qquad \text{in } P, \tag{95}$$

$$\text{div } \mathbf{u}^{eff} = 0 \quad \text{in } P, \quad \mathbf{u}^{eff} = (U_1, U_2, 0) \text{ on } \Sigma_2, \quad \mathbf{u}_3^{eff} = 0 \text{ on } \Sigma, \tag{96}$$

$$\mathbf{u}_j^{eff} = -\varepsilon\sum_{i=1}^{2} M_{ji}\frac{\partial \mathbf{u}_i^{eff}}{\partial x_3}, \ j = 1, 2 \text{ on } \Sigma, \tag{97}$$

$$\{\mathbf{u}^{eff}, p^{eff}\} \text{ is periodic in } (x_1, x_2) \text{ with period } (L_1, L_2). \tag{98}$$

If $|U|L_3 \leq \nu$, the problem (95)-(98) has a unique solution

$$\begin{cases} \mathbf{u}^{eff} = (\tilde{u}^{eff}, 0), \tilde{u}^{eff} = \vec{U} + (\dfrac{x_3}{L_3} - 1)\Big(I - \dfrac{\varepsilon}{L_3}M\Big)^{-1}\vec{U}, & x \in P; \\ p^{eff} = 0, & x \in P. \end{cases} \tag{99}$$

Let us estimate the error made when replacing $\{\mathbf{v}^\varepsilon, p^\varepsilon, \mathcal{M}^\varepsilon\}$ by $\{\mathbf{u}^{eff}, p^{eff}, \mathcal{M}^{eff}\}$.

Theorem 21. *([37]). Under the assumptions of Theorem 17 we have*

$$\|\nabla(\mathbf{v}^\varepsilon - \mathbf{u}^{eff})\|_{L^1(P)^9} \leq C\varepsilon, \tag{100}$$

$$\sqrt{\varepsilon}\|\mathbf{v}^\varepsilon - \mathbf{u}^{eff}\|_{L^2(P)^3} + \|\mathbf{v}^\varepsilon - \mathbf{u}^{eff}\|_{L^1(P)^3} \leq C\varepsilon^2\frac{|U|}{L_3}. \tag{101}$$

Our next step is to calculate the *tangential drag force* or the *skin friction*

$$\mathcal{F}^\varepsilon_{t,j} = \frac{1}{L_1L_2}\int_\Sigma \nu\frac{\partial \mathbf{v}^\varepsilon{}_j}{\partial x_3}(x_1, x_2, 0)\, dx_1dx_2, \ j = 1, 2. \tag{102}$$

Theorem 22. *([37]). Let the skin friction $\mathcal{F}^\varepsilon_t$ be defined by (102). Then we have*

$$|\mathcal{F}^\varepsilon_t - \nu\frac{1}{L_3}\big(\vec{U} + \frac{\varepsilon}{L_3}M\vec{U}\big)| \leq C\varepsilon^2\frac{|U|^2}{\nu L_3}(1 + \frac{\nu}{L_3|U|}). \tag{103}$$

Corollary 23. . *Let* $\mathcal{F}_t^{eff} = \nu \frac{1}{L_3}\big(I - \frac{\varepsilon}{L_3} M\big)^{-1}\vec{U}$ *be the tangential drag force corresponding to the effective velocity* $\mathbf{u}^{eff}$. *Then we have*

$$|\mathcal{F}_t^{eff} - \mathcal{F}_t^{\varepsilon}| \leq C\varepsilon^2 \frac{|U|^2}{\nu L_3}(1 + \frac{\nu}{L_3|U|}) \tag{104}$$

Remark 24. *We see that the presence of the periodic roughness diminishes the tangential drag. The contribution is linear in* ε, *and consequently rather small. It coincides with the conclusion from [8] that for laminar flows there is no palpable drag reduction. Nevertheless, we are going to see in the next subsection that the calculations from the laminar case could be useful for turbulent Couette flow.*

3.3 Wall laws for fluids obeying Fourier's boundary conditions at the rough boundary

In number of situations, the adherence conditions, that are used to describe fluid behavior when moderate pressure and low surface stresses are involved, are no longer valid. Physical considerations lead to slip boundary conditions. These conditions are of particular interest in the study of polymers, blood flow, and flow through filters. We mention also the near wall models from turbulence theories.

These conditions are of Fourier's type and in number of recent publications, authors undertook the homogenization of Stokes and Navier-Stokes equations in such setting. An early reference is [10], but it was the work of Simon et al [26] which attracted lot of interest. This is a fast developing research area and we mention only the articles [22] and [23]. In most cases the effective boundary condition is the no-slip condition. Consequently, the boundary layers do not enter into the wall law and the effective models are valid for much larger class of the rough boundaries than the wall law derived in the previous section.

Finally, we mention that there is a work on roughness induced wall laws for geostrophic flows. For more information see the article [20] and references therein.

4 Rough boundaries and drag minimization

Drag reduction for planes, ships and cars reduces significantly the spending of the energy, and consequently the cost for all type of land, sea and air transportation.

Drag-reduction adaptations were important for the survival of Avians and Nektons, since their efficiency or speed, or both, have improved.

Essentially, there are three forms of drag. The largest drag component is pressure or form drag. It is particularly troublesome when flow separation occurs. The two remaining drag components are skin-friction drag and drag due to lift. Skin-friction drag is the result of the no-slip condition on the surface. Those components are present for both laminar (low Reynolds number) or turbulent (high Reynolds number) flows.

There are several drag-reduction methods and here we discuss only the use of drag-reducing surfaces. For an overview of other techniques we refer to Bushnell, Moore [24].

The inspiration comes from morphological observations. It is known that the skin of fast sharks is covered with tiny scales having little longitudinal ribs on their surface (shark dermal denticles). These are tiny ridges, closely spaced (less than 100 μm apart and still less in height). We note that the considered sharks have a length of approximately 2 m and swim at Reynolds numbers $\mathbf{Re} \approx 3 \cdot 10^7$ (see e.g. Vogel [50]). Such grooves are similar to ones used on the yacht " Stars and Strips " in America's Cup finals and seem to reduce the skin-friction for $\mathcal{O}(10\%)$ (see [24]).

In the applications, the main interest is in the turbulent case. Mathematical modeling of the turbulent flows in the presence of solid walls is still out of reach. However the turbulent boundary layers on surfaces with fine roughness contain a viscous sublayer. It was found that the viscous sublayer exhibits a streaky structure. Those " low-speed streaks " are believed to be produced by slowly rotating longitudinal vortices. For a streaky structure, with a preferred lateral wavelength, a turbulent shear stress reduction was observed.

The experimental facts were theoretically explained in the papers by Bechert and Bartenwerfer [17] and Luchini, Manzo and Pozzi [40] (see also [18] and references in mentioned articles).

In [37] the theory developed in the laminar situation was applied to the turbulent flow. It is known that the turbulent Couette flow has a 2-layer structure. There is a large core layer where the molecular momentum transfer can be neglected and a thin wall layer (or sublayer) where both turbulent and molecular momentum transfer are important. The flow in the wall layer is governed by the turbulent viscous shear stress τ_w, supposed to depend only on time. In connection with τ_w authors use the friction velocity $v = \sqrt{\frac{\tau_w}{\rho}}$, where ρ is the density. Then the wall layer thickness is $\delta_v = \frac{\nu}{v}$, we suppose that our riblets remain all the time in the pure viscous sublayer and try to apply the analysis from the subsection §3.2.

The corresponding equations are (70)-(74) with $L_3 = \delta_v$ and velocity $v = \sqrt{\frac{\tau_w}{\rho}} = (v_1, v_2, 0)$ at $x_3 = \delta_v$. Since $\delta_v \sqrt{\frac{\tau_w}{\rho}} = \nu < 2\nu$, our results from §3.2 are applicable and we get

$$|\mathcal{F}_t^\varepsilon - \frac{\nu}{\delta_v}\big(v + \frac{\varepsilon}{\delta_v} Mv\big)| \leq C(\frac{\varepsilon |U|}{\delta_v})^2. \tag{105}$$

Since $\delta_v = \nu \sqrt{\frac{\rho}{\tau_w}}$, we see that the effects of roughness are significant.

For the shark skin $\varepsilon/\delta_v = 0.1, L_3 = \delta_v = 10^{-3} = \sqrt{\nu}$ and $|U| = \sqrt{\nu} = 10^{-3}$. The uniqueness condition from Corollary 19 applies if $\varepsilon \leq C\nu^{9/4}$. Since $\varepsilon \approx 10^{-4}$ and $\nu^{9/14} \approx 1.389 \cdot 10^{-4}$. We see that our theory is applicable to the swimming of Nektons. For more details we refer to [36].

Furthermore, let us suppose the geometry of the rough boundary from [17] and [40]. Then M is diagonal and the origins of the cross and longitudinal flows are at the characteristic walls coordinates (see [49]) $y^+ = \frac{\varepsilon}{\delta_v} M_{11}$ and $y^+ = \frac{\varepsilon}{\delta_v} M_{22}$, respectively. Hence the proposition is to model the flow in the viscous sublayer in the presence of the rough boundary by the Couette-Navier profile (98) instead of the simple Couette profile in the smooth case.

We note that these observations were implemented numerically into a shape optimization procedure in [29]. The numerically obtained drag reduction confirmed the theoretical predictions from [36].

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