Derivation of a poroelastic elliptic membrane shell model

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November 25, 2017

Abstract

A derivation of the model for a poroelastic elliptic membrane shell is undertaken. The flow and deformation in a three-dimensional shell domain is described by the quasi-static Biot equations of linear poroelasticity. We consider the limit when the shell thickness goes to zero and look for the limit equations. Using the technique developed in the seminal articles by Ciarlet, Lods, Miara et al and the recent results on the rigorous derivation of the equations for poroelastic plates and flexural poroelastic shells by Marciniak-Czochra, Mikelić and Tambača, we present a rigorous derivation of the linear poroelastic elliptic membrane shell model. After rescaling, the corresponding velocity and the pressure field are close in the $C([0,T];(H^1_x)^2\times(L^2_x)^2)$ norm and the stresses in $C([0,T];(L^2_x)^9)$ norm. We note the major difference with respect to the flexural case: (i) it is not anymore the rescaled total stress divided by the scaling parameter, but the rescaled total stress itself which converges ; (ii) the same comment applies to the pore fluid pressure and (iii) there is a deterioration of the convergence for the vertical component of the rescaled displacement. Consequence of the above differences is that the effective model remains of the 2nd order in space. In the case of a spherical membrane shell we confirm the results by Taber from the literature.

Keywords: Membrane poroelastic shell, Biot's quasi-static equations, elliptic-parabolic systems, asymptotic methods
AMS subject classification: 35B25, 74F10, 74K25, 74Q15, MSC 76S

Contents

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1 Introduction

The present work is devoted to the derivation of a model of the poroelastic elliptic membrane shell.

We follow the standpoint of Ciarlet et al, who derived the Kirchhoff-Love models for thin elastic bodies in the zero thickness limit (see [6, 7, 8]).

While this approach to the effective behavior of three-dimensional linearized elastic bodies is well-established, much less attention has been paid to the poroelastic thin bodies. As the recent example of the special issue of the journal Transport in Porous Media [34, 19] shows, this is likely to change. Also there is a related experimental work, see [17].

The poroelastic bodies are characterized by the simultaneous presence of the deformation and the filtration (flow). They are described by the quasi-static Biot's system of PDE's. It couples the Navier equations of linearized elasticity, containing the pressure gradient, with the mass conservation equation involving the fluid content change and divergence of the filtration velocity. The filtration velocity is the relative velocity for the upscaled fluid-structure problem and obeys Darcy's law. The fluid content change is proportional to the pressure and the elastic body compression. In the quasi-static Biot's system the mechanical part is elliptic in the displacement and the flow equation has a parabolic operator for the pressure. For more modeling details we refer to [10], [21] and [33] and for the mathematical theory to [28], [26] and [23].

The simplest relevant two dimensional poroelastic thin body is a poroelastic plate. A physically relevant choice of the time scale and the related coefficient size was set up by Marciniak-Czochra and Mikelić in [20]. They rigorously derived the effective equations for the Kirchhoff-Love-Biot poroelastic plate in the zero thickness limit of the 3D quasi-static Biot equations. The limiting zero thickness procedure is seriously affected by the presence of coupling structure-flow. As in the purely elastic case, the specificity of the poroelastic plate model from [20] is that the limit model contains simultaneously both flexural and membrane equations. This remarkable property does not transfer to the shells.

Following both Ciarlet et al zero thickness limit approach to flexural linearized shells and handling of the Biot quasi-static equations in a thin domain by Marciniak-Czochra and Mikelić, Mikelić and Tambača have undertaken the derivation of the equation for a linear flexural poroelastic shell in [22]. In this article we undertake derivation of a model for linear elliptic membrane poroelastic shell through the same type of the limit procedure.

The coupling elastic structure - flow is scaled as in [20]. It corresponds to the physical parameters leading to the quasi-static diphasic Biot's equations for the displacement and the pressure. As in [30] and [31], it means that the characteristic time scale is of Taber and Terzaghi and in the dimensionless form there will be the ratio between the width and length squared, multiplying Laplacean of the pressure. The flexural and the membrane poroelastic shells correspond to different regimes of the filtration, different sizes of the applied contact forces, different geometries of the shell and different boundary conditions. In our case they are applied at the top and the bottom boundaries. For the membrane shell case, we impose a given inflow/outflow velocity of order of the characteristic filtration velocity through a shell of width ℓ. The applied contact forces at the same top/bottom boundaries should be of order of the pressure drop between these boundaries. We recall that in the case of the flexural poroelastic shell, studied in [22], the contact forces at the top/bottom boundaries are an order of magnitude smaller and even smaller than the related inflow/outflow velocities.

The motivation for studying the flexural poroelastic shells comes from the industrial filters modeling. For instance, the results from [20] and [22] can be applied to the modeling of the air filters for the cars, while some oil car filters can be modeled as membrane poroelastic shells.

The motivation for studying the membrane poroelastic shells also comes from the biomechanics. An important example is the study of the mechanical behavior of fluid-saturated large living bone tissues. We recall that the bulk modulus of the bone is much larger than the bulk moduli of the soft tissues and the bone deformation is small. A full physiological understanding of the bone modeling would provide insight to important clinical problems which concern bones. For detailed review we refer to [27] and [11]. Many other living structures are fluid-saturated membrane shells, see e.g. [13].

Another modeling question, raised in [11], is of the modeling of the elastic wave propagation in a bone. As the Biot theory was originally developed to describe the wave propagation, the subject attracted attention. The reader can consult [3], [15] and [32] and references therein. With our scaling, our spatial operators do not coincide with models from these references, but rather with Taber's works [30] and [31]. Furthermore, the dynamic models of the diphasic Biot equations for a viscous fluid exhibit memory effects, as proposed by Biot through the introduction of the viscodynamic operator (see [33]). The homogenization derivation of the dynamic diphasic Biot's equations gives the memory terms (see [9]) for a viscous fluid. If the pores are filled by an ideal fluid, there are no memory effects (see [16]). The analysis of the relationship between the dynamic and the quasi-static diphasic Biot equation was undertaken in [24] and there are scalings when the memory effects are not important. But in general it is not possible just to add the acceleration to the quasi-static Biot system. Hence modeling of the elastic waves propagation in poroelastic plates/shell requires some future research.

Since in [23], the quasi-static Biot equations are obtained by homogenization of a pore scale fluid-structure problem, one can raise question why we do not study simultaneously homogenization of the fluid-structure problem and the zero thickness limit. In the applications we have in mind (industrial filters, living tissues…) the thickness is much bigger than the RVE size and such approach does not make much sense. For some other problems like the study of the overall behavior of curved layers of living cells, having a thickness of one cell, the simultaneous homogenization and singular perturbation would give new models.

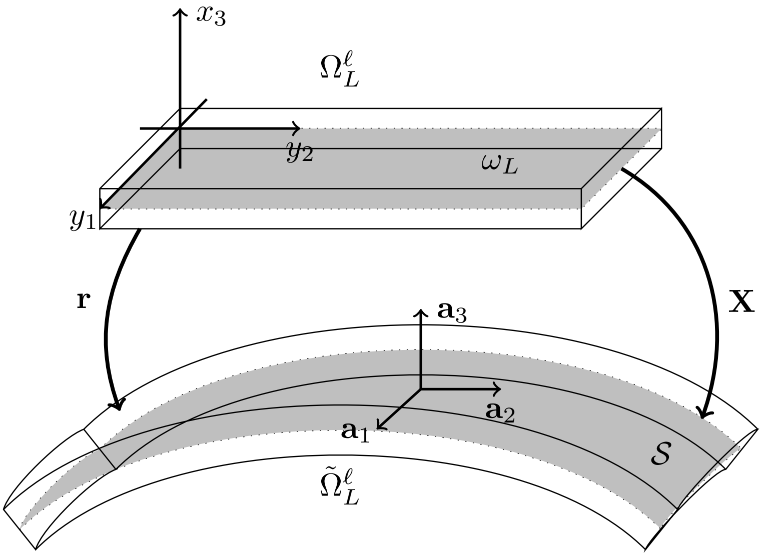
The flexural shell model is formulated on a subspace of infinitesimally inextensional displacements involving boundary conditions, usually denoted by V_F. However this function space for some geometries and boundary conditions turns to be trivial. In this case a model for extensional displacements is necessary. In this paper we focus on the shells with elliptic surfaces which are clamped at the whole boundary. The model in this case is called elliptic membrane shell model. In the case of the classical elasticity the membrane effects are measured by the change of the metric of the shell. This is different with the case of the flexural shell model where the potential energy is measured by the change in the curvature tensor. This difference results in a simpler model for the membrane case and lower order derivatives involved in the formulation. Remaining cases in which $V_F = \{0\}$ as well are covered by the generalized membrane shell model (an example is a tube clamped at ends). However, the formulation is given in abstract spaces, see [5].

Derivation of the present model is more difficult than the derivation of the classical elliptic membrane shell model starting from the three-dimensional linearized [7]. Namely, we are dealing with an additional equation for the additional unknown (pressure). We use the results derived in the classical static case, see [5], as much as possible, but Biot's equations are quasi-static and, therefore, time dependent. Presence of the additional independent variable (time) requires special attention and careful analysis. Note also that in some parts this derivation is more demanding than the derivation of the poroelastic flexural shell model from [22] since here we have weaker a priori estimates in strains and we have to obtain the same convergences of the tangential displacements as in the flexural case. Finally, we recall that the flexural shell models are characterized by the presence of the 4th order differential operators and for the membrane shell models the differential operators are of the 2nd order.

2 Geometry of Shells and Setting of the Problem

We are starting by recalling the basic facts of geometry of shells. The text follows [22], but it is shorter since some terms are not needed in the membrane model. Namely, lower order derivatives are sufficient to express the membrane effects.

Throughout this paper we use boldfaced letters for vectors or matrices. The only exceptions are points in the Euclidean spaces (e.g., x, y,). $\mathbb{R}^{n\times m}$ denotes the space of all n by m matrices and the subscript sym denotes its subspace of symmetric matrices. By L^2 we denote the Lebesgue space of the square integrable functions, while H^1 stands for the Sobolev space.



Let the surface $\mathcal{S}$ is given as $\mathcal{S} = \mathbf{X}(\overline{\omega}_L)$, where $\omega \subset \mathbb{R}^2$ be an open bounded and simply connected set with Lipschitz-continuous boundary $\partial\omega_L$ and $\mathbf{X} : \overline{\omega}_L \to \mathbb{R}^3$ is a smooth injective immersion (that is $\mathbf{X} \in C^3$ and 3×2 matrix $\nabla\mathbf{X}$ is of rank two). Thus the vectors $\mathbf{a}_\alpha(y) = \partial_\alpha\mathbf{X}(y)$, $\alpha = 1, 2$, are linearly independent for all $y \in \overline{\omega}_L$ and form the covariant basis of the tangent plane to the 2-surface $\mathcal{S}$. Let $\Omega_L^\ell = \omega_L \times (-\ell/2, \ell/2)$. In this paper we study the deformation and the flow in a three-dimensional poroelastic shell $\tilde{\Omega}_L^\ell = \mathbf{r}(\Omega_L^\ell)$, $L, \ell > 0$, where the injective mapping $\mathbf{r}$ is given by

$$\mathbf{r} = \mathbf{r}(y, x_3) = \mathbf{X}(y) + x_3\mathbf{a}_3(y), \quad \mathbf{a}_3(y) = \frac{\mathbf{a}_1(y) \times \mathbf{a}_2(y)}{|\mathbf{a}_1(y) \times \mathbf{a}_2(y)|}, \tag{2.1}$$

for $x_3 \in (-\ell/2, \ell/2)$ and $(y_1, y_2) \in \omega_L$, diam $(\omega_L) = L$. The contravariant basis of the plane spanned by $\mathbf{a}_1(y), \mathbf{a}_2(y)$ is given by the vectors $\mathbf{a}^\alpha(y)$ defined by $\mathbf{a}^\alpha(y) \cdot \mathbf{a}_\beta(y) = \delta^\alpha_\beta$. By adding the vector $\mathbf{a}_3(y)$ defined in (2.1) we extend these two bases to the basis of $\mathbb{R}^3$ ($\mathbf{a}^3(y) = \mathbf{a}_3(y)$). The following matrix functions defined on ω_L will be extensively used to account for the geometry of the shell

$$\mathbf{Q} = \begin{bmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \end{bmatrix}, \qquad \mathbf{Q}^{-1} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{bmatrix}. \tag{2.2}$$

As usual, the first fundamental form of the surface $\mathcal{S}$, or the metric tensor, in covariant $\mathbf{A}_c = (a_{\alpha\beta})$ and contravariant $\mathbf{A}^c = (a^{\alpha\beta})$ components is given respectively by $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$, $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$, $\alpha, \beta = 1, 2$. Note here that, since $\mathbf{X}$ is smooth $\mathbf{A}_c$ and $\mathbf{A}^c$ are continuous on the compact set $\overline{\omega}_L$, there are constants $M^c \geq m^c > 0$ such that

$$m^c\mathbf{x} \cdot \mathbf{x} \leq \mathbf{A}^c(y)\mathbf{x} \cdot \mathbf{x}, \mathbf{A}_c(y)\mathbf{x} \cdot \mathbf{x} \leq M^c\mathbf{x} \cdot \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^3, y \in \overline{\omega}_L. \tag{2.3}$$

The second fundamental form of the surface $\mathcal{S}$, also known as the curvature tensor, in covariant $\mathbf{B}_c = (b_{\alpha\beta})$ and mixed components $\mathbf{B} = (b^\beta_\alpha)$ is given respectively by

$$b_{\alpha\beta} = \mathbf{a}^3 \cdot \partial_\beta\mathbf{a}_\alpha = -\partial_\beta\mathbf{a}^3 \cdot \mathbf{a}_\alpha, \quad b^\beta_\alpha = \sum_{\kappa=1}^{2} a^{\beta\kappa}b_{\kappa\alpha}, \quad \alpha, \beta = 1, 2.$$

The Christoffel symbols $\mathbf{\Gamma}^\kappa$ are defined by

$$\Gamma^\kappa_{\alpha\beta} = \mathbf{a}^\kappa \cdot \partial_\beta \mathbf{a}_\alpha = -\partial_\beta \mathbf{a}^\kappa \cdot \mathbf{a}_\alpha, \quad \alpha, \beta, \kappa = 1, 2.$$

The dual notation $\Gamma^3_{\alpha\beta}$ for $b_{\alpha\beta}$ will sometimes be used. The area element along $\mathcal{S}$ is $\sqrt{a}dy$, where $a := \det \mathbf{A}_c$. By (2.3) it is uniformly positive, i.e., there is $m_a > 0$ such that

$$0 < m_a \leq a(y), \qquad y \in \overline{\omega}_L. \tag{2.4}$$

In order to formulate the limit membrane model we define the notation:

$$\gamma_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \sum_{\kappa=1}^{2} \Gamma^\kappa_{\alpha\beta} v_\kappa - b_{\alpha\beta} v_3, \qquad \alpha, \beta = 1, 2, \tag{2.5}$$

$$n_{\alpha\beta}|_\beta = \partial_\beta n_{\alpha\beta} + \sum_{\kappa=1}^{2} \Gamma^\alpha_{\beta\kappa} n_{\beta\kappa} + \sum_{\kappa=1}^{2} \Gamma^\beta_{\beta\kappa} n_{\alpha\kappa}, \qquad \alpha, \beta = 1, 2,$$

for smooth vector fields $\mathbf{v}$ and tensor fields $\mathbf{n}$. $\boldsymbol{\gamma}(\mathbf{v})$ denotes linearization of the change of a metric tensor defined by displacement $\mathbf{v}$. Note here that in the membrane model case there is no need for the linearization of the change of curvature tensor, usually denoted by $\boldsymbol{\rho}(\mathbf{v})$.

The upper and lower face of the body $\tilde{\Omega}^\ell_L$ are given by

$$\tilde{\Sigma}^\ell_L = \mathbf{r}(\omega_L \times \{x_3 = \ell/2\}) = \mathbf{r}(\Sigma^\ell_L)$$
$$\tilde{\Sigma}^{-\ell}_L = \mathbf{r}(\omega_L \times \{x_3 = -\ell/2\}) = \mathbf{r}(\Sigma^{-\ell}_L)),$$

while $\tilde{\Gamma}^\ell_L = \mathbf{r}(\partial\omega_L \times (-\ell/2, \ell/2)) = \mathbf{r}(\Gamma^\ell_L)$ is the lateral boundary.

Behavior of the poroelastic bodies is described by the quasi-static Biot equations. For the poroelastic body $\tilde{\Omega}^\ell_L$ the equations take the following dimensional form:

$$\tilde{\sigma} = 2\mu \mathbf{e}(\tilde{\mathbf{u}}) + (\lambda \text{ div } \tilde{\mathbf{u}} - \alpha \tilde{p})\mathbf{I} \text{ in } \tilde{\Omega}^\ell_L, \tag{2.6}$$

$$-\text{ div } \tilde{\sigma} = -\mu \triangle \tilde{\mathbf{u}} - (\lambda + \mu) \bigtriangledown \text{div } \tilde{\mathbf{u}} + \alpha \bigtriangledown \tilde{p} = 0 \text{ in } \tilde{\Omega}^\ell_L, \tag{2.7}$$

$$\frac{\partial}{\partial t}(\beta_G \tilde{p} + \alpha \text{ div } \tilde{\mathbf{u}}) - \frac{k}{\eta} \triangle \tilde{p} = 0 \text{ in } \tilde{\Omega}^\ell_L. \tag{2.8}$$

Note that $\mathbf{e}(\mathbf{u}) = \text{sym} \bigtriangledown \mathbf{u}$ and $\tilde{\sigma}$ is the stress tensor. All other quantities are defined in Table 1. We prescribe the contact force $\tilde{\mathcal{P}}^{\pm\ell}_L$ and normal flux $\tilde{V}_L$ at upper and lower face while at the lateral

Table 1: Parameter and unknowns description

SYMBOL	*QUANTITY*
μ	shear modulus (Lamé's second parameter)
λ	Lamé's first parameter
β_G	inverse of Biot's modulus
α	effective stress coefficient
k	permeability
η	viscosity
L and ℓ	midsurface length and shell width, respectively
$\mathbf{u} = (u_1, u_2, u_3)$	solid phase displacement
p	pressure

boundary $\tilde{\Gamma}^\ell$ we impose a zero displacement and a zero normal flux. Thus the following boundary conditions are assumed

$$\tilde{\sigma}\boldsymbol{\nu} = \tilde{\mathcal{P}}_L^{\pm\ell} \text{ at } \tilde{\Sigma}_L^{\pm\ell}, \tag{2.9}$$

$$-\frac{k}{\eta}\frac{\partial \tilde{p}}{\partial x_3} = \tilde{V}_L \text{ at } \tilde{\Sigma}_L^{\pm\ell}, \tag{2.10}$$

$$\mathbf{u} = 0 \text{ at } \tilde{\Gamma}^\ell, \tag{2.11}$$

$$-\frac{k}{\eta}\frac{\partial \tilde{p}}{\partial \boldsymbol{\nu}} = 0 \text{ at } \tilde{\Gamma}^\ell. \tag{2.12}$$

Here $\boldsymbol{\nu}$ is the outer unit normal at the boundary. At initial time $t = 0$ the initial pressure $\tilde{p}^\ell_{L,\mathrm{in}}$ is given

$$\tilde{p} = \tilde{p}^\ell_{L,\mathrm{in}} \text{ for } t = 0. \tag{2.13}$$

The equations (2.6), (2.7) and (2.8) together with the boundary conditions (2.9)–(2.12) and initial condition (2.13) constitute the model that we analyze.

Our goal is to extend the elliptic membrane shell justification by Ciarlet, Lods et al and by Dauge et al to the poroelastic case. Thus in the sequel we assume that the middle surface is elliptic (Gaussian curvature (product of principal curvatures) is positive at all points) and that is the reason that we assumed the shell is clamped at its entire boundary. Further we assume that the body $\tilde{\Omega}^\ell_L$ is thin in one direction comparing to two others, i.e. that the ratio between the shell thickness and the characteristic horizontal length $\varepsilon = \ell/L \ll 1$. This ε is the small parameter in the problem and we do the asymptotic analysis with respect to it.

We announce briefly the differential equations of the membrane poroelastic shell in dimensional form. These equations follow from the weak formulation of the limit model (3.10)–(3.12) that we derive in this paper.

Effective dimensional equations:

The model is given in terms of $\mathbf{u}^{\mathrm{eff}} : \omega_L \to \mathbb{R}^3$ which is the vector of components of the displacement of the middle surface of the shell in the contravariant basis and $p^{\mathrm{eff}} : \Omega^\ell_L \to \mathbb{R}$ which is the pressure in the 3D shell. Let us denote the stress tensor due to the variation in pore pressure across the shell thickness by

$$\mathbf{n} = \ell\tilde{\mathcal{C}}^c(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u}^{\mathrm{eff}}))\mathbf{A}^c - \frac{2\mu\alpha}{\lambda+2\mu}\int_{-\ell/2}^{\ell/2} p^{\mathrm{eff}}\, dy_3 \mathbf{A}^c, \tag{2.14}$$

where $\boldsymbol{\gamma}(\cdot)$ is given by (2.5) and $\tilde{\mathcal{C}}^c$ is the elasticity tensor, usually appearing in the classical shell theories, given by

$$\tilde{\mathcal{C}}^c\mathbf{E} = 2\mu\frac{\lambda}{\lambda+2\mu}\,\mathrm{tr}\,(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}, \qquad \mathbf{E} \in \mathbb{R}^{2\times 2}_{sym}.$$

The stress tensor $\mathbf{n}$ is defined in order to formulate the differential equations as it is done in the classical elastic elliptic membrane model (see [5, Chapter 4]). Then the model in the differential formulation reads as follows:

$$\begin{aligned} &-\sum_{\beta=1}^{2} n_{\alpha\beta}|_\beta = (\mathcal{P}_L^{+\ell})_\alpha + (\mathcal{P}_L^{-\ell})_\alpha \text{ in } \omega_L, \quad \alpha = 1,2, \\ &-\sum_{\alpha,\beta=1}^{2} b_{\alpha\beta}n_{\alpha\beta} = (\mathcal{P}_L^{+\ell})_3 + (\mathcal{P}_L^{-\ell})_3 \text{ in } \omega_L, \\ &u^{\mathrm{eff}}_\alpha = 0, \alpha = 1,2, \quad \text{ on } \partial\omega_L, \qquad \text{for every } \; t \in (0,T), \end{aligned} \tag{2.15}$$

$$
\begin{aligned}
&\left(\beta_G + \frac{\alpha^2}{\lambda+2\mu}\right)\frac{\partial p^{\rm eff}}{\partial t} + \alpha\frac{2\mu}{\lambda+2\mu}\mathbf{A}^c : \boldsymbol{\gamma}(\frac{\partial \mathbf{u}^{\rm eff}}{\partial t}) - \frac{k}{\eta}\frac{\partial^2 p^{\rm eff}}{\partial (y_3)^2} = 0\\
&\quad \text{in } \ (0,T)\times\omega_L\times(-\ell/2,\ell/2),\\
&\frac{k}{\eta}\frac{\partial p^{\rm eff}}{\partial y_3} = -V_L, \qquad \text{on } (0,T) \ \times\omega_L\times(\{-\ell/2\}\cup\{\ell/2\}),\\
&p^{\rm eff} = p^{\ell}_{L,\rm in} \quad \text{given at } \ t=0.
\end{aligned}
\tag{2.16}
$$

Here $(\mathcal{P}_L^{\pm\ell})_i, i=1,2,3$ are components of the contact force $\tilde{\mathcal{P}}_L^{\pm\ell}\circ\boldsymbol{r}$ at $\Sigma_L^{\pm\ell}$ in the covariant basis, $V_L = \tilde{V}_L\circ\boldsymbol{X}$, $p^{\ell}_{L,\rm in} = \tilde{p}^{\ell}_{L,\rm in}\circ\boldsymbol{r}$. Thus, the poroelastic elliptic membrane shell model in the differential formulation is given by equations (2.14), (2.15) and (2.16). The unknowns in the problem are contact forces as components of $\mathbf{n}$, the (vector) effective displacement of the middle surface $\mathbf{u}^{\rm eff}$ and effective pressure $p^{\rm eff}$. The first two equations in (2.15) are the same as in the differential equation of the elliptic membrane shell model (see [5, Theorem 4.5-1]). The evolution equation for the effective pressure with associated boundary and initial conditions is given in (2.16).

In the case of the classical theory of the purely elastic shell, we recall that, in addition to already quoted articles and books by Ciarlet and al, there is a huge literature, with both mathematical and engineering approaches (see e.g. [1], [12], [18], [25] and references therein).

In Section 3, with the help of curvilinear coordinates, we formulate the problem in the dimensionless form on the domain $\Omega = \omega\times(-1/2,1/2)$ independent of ε suitable for asymptotic analysis. We recall existence and uniqueness result of the smooth solution for the starting problem and formulate the limit model and the main convergence results. In further sections we prove the main results. First, in Section 4, we derive a priori estimates for the family of solutions. Then, in Section 5, the convergence (including strong) of the solutions to the rescaled problem, is studied as $\varepsilon\to 0$. In Appendix we give the limit model written for a part of the spherical surface. Further, the radially symmetric effective equations of the problem on the whole sphere are derived and the result is compared with the one in [31].

3 Problem setting in curvilinear coordinates and the main results

3.1 Dimensionless equations

The Biot's diphasic equations describe behavior of the system at so called Terzaghi's time scale $T = \eta L_c^2/(k\mu)$, where L_c is the characteristic domain size, η is dynamic viscosity, k is permeability and μ is the shear modulus. Thus we have two possible characteristic length. Similarly as in [20] and [22] we chose as the characteristic length $L_c = \ell$, which leads to the Taber-Terzaghi transversal time $T_{tab} = \eta\ell^2/(k\mu)$. Our scaling corresponds to the one chosen by Blanchard and Francfort in [2] for the derivation of a thermoelastic plate model.

Thus in order to obtain the dimensionless equations to be able to perform the asymptotic analysis we introduce the dimensionless unknowns and variable by setting

$$
\begin{aligned}
&\varepsilon = \frac{\ell}{L}; \quad \beta = \beta_G\mu; \quad P = \frac{\mu U}{L}; \quad U\tilde{\mathbf{u}}^\varepsilon = \tilde{\mathbf{u}}; \quad T = \frac{\eta\ell^2}{k\mu}; \quad \tilde{\lambda} = \frac{\lambda}{\mu};\\
&P\tilde{p}^\varepsilon = \tilde{p}; \quad \tilde{y}L = y; \quad \tilde{x}_3 L = x_3; \quad \tilde{\mathbf{r}}L = \mathbf{r}; \quad \tilde{\mathbf{X}}L = \mathbf{X}; \quad \tilde{t}T = t; \quad \tilde{\sigma}^\varepsilon\frac{\mu U}{L} = \tilde{\sigma}.
\end{aligned}
$$

Then we rewrite the problem in terms of the dimensionless quantities. It is then given on a the domain

$$
\tilde{\Omega}^\varepsilon = \tilde{\Omega}^\ell_L/L = \mathbf{r}(\Omega^\varepsilon), \qquad \Omega^\varepsilon = \omega\times(-\varepsilon/2,\varepsilon/2), \qquad \omega = \omega_L/L.
$$

The upper and lower face of the shell-like body $\tilde{\Omega}^\varepsilon$ are denoted by $\tilde{\Sigma}^\varepsilon_+$ and $\tilde{\Sigma}^\varepsilon_-$, while $\tilde{\Gamma}^\varepsilon = \tilde{\Gamma}^\ell_L/L$is lateral boundary. Let $\mathcal{V}(\tilde{\Omega}^\varepsilon) = \{\tilde{\mathbf{v}}\in H^1(\tilde{\Omega}^\varepsilon;\mathbb{R}^3) : \tilde{\mathbf{v}}|_{\tilde{\Gamma}^\varepsilon} = 0\}$. The weak formulation of the dimensionless problem associated to the system (2.6)–(2.13) is given by (see [22] for more details):

find $\tilde{\mathbf{u}}^\varepsilon \in H^1(0,T,\mathcal{V}(\tilde{\Omega}^\varepsilon))$, $\tilde{p}^\varepsilon \in H^1(0,T;H^1(\tilde{\Omega}^\varepsilon))$ such that it holds

$$\int_{\tilde{\Omega}^\varepsilon} 2\,\mathbf{e}(\tilde{\mathbf{u}}^\varepsilon):\mathbf{e}(\tilde{\mathbf{v}})\,dx + \tilde{\lambda}\int_{\tilde{\Omega}^\varepsilon} \text{div}\,\tilde{\mathbf{u}}^\varepsilon\,\text{div}\,\tilde{\mathbf{v}}\,dx - \alpha\int_{\tilde{\Omega}^\varepsilon}\tilde{p}^\varepsilon\,\text{div}\,\tilde{\mathbf{v}}\,dx$$
$$= \int_{\tilde{\Sigma}^\varepsilon_+}\varepsilon\tilde{\mathcal{P}}_+\cdot\tilde{\mathbf{v}}\,ds + \int_{\tilde{\Sigma}^\varepsilon_-}\varepsilon\tilde{\mathcal{P}}_-\cdot\tilde{\mathbf{v}}\,ds, \quad \text{for every } \tilde{\mathbf{v}}\in\mathcal{V}(\tilde{\Omega}^\varepsilon) \text{ and } t\in(0,T), \tag{3.1}$$

$$\beta\int_{\tilde{\Omega}^\varepsilon}\partial_t\tilde{p}^\varepsilon\tilde{q}\,dx + \int_{\tilde{\Omega}^\varepsilon}\alpha\,\text{div}\,\partial_t\tilde{\mathbf{u}}^\varepsilon\tilde{q}\,dx + \varepsilon^2\int_{\tilde{\Omega}^\varepsilon}\nabla\tilde{p}^\varepsilon\cdot\nabla\tilde{q}\,dx$$
$$= \varepsilon\int_{\tilde{\Sigma}^\varepsilon_-}\tilde{V}\tilde{q}\,ds - \varepsilon\int_{\tilde{\Sigma}^\varepsilon_+}\tilde{V}\tilde{q}\,ds, \quad \text{for every } \tilde{q}\in H^1(\tilde{\Omega}^\varepsilon) \text{ and } t\in(0,T), \tag{3.2}$$

$$\tilde{p}^\varepsilon|_{\{t=0\}} = \tilde{p}_{\text{in}}, \quad \text{in } \tilde{\Omega}^\varepsilon, \tag{3.3}$$

here $\tilde{\mathbf{u}}^\varepsilon = (\tilde{u}^\varepsilon_1, \tilde{u}^\varepsilon_2, \tilde{u}^\varepsilon_3)$ denotes the dimensionless displacement field and $\tilde{p}^\varepsilon$ the dimensionless pressure. Note that for two 3×3 matrices A and B the Frobenius scalar product is denoted by $A:B = \text{tr}\,(AB^T)$. The coefficient ε^2 in (3.2) is a consequence of the choice of the characteristic length in the Terzaghi time. There are also three important assumptions on the order of prescribed quantities $\mathcal{P}^{\pm\ell}_L, V_L, p^\ell_{L,\text{in}}$ made here:

$$\tilde{\sigma}^\varepsilon\boldsymbol{\nu} = (2\mathbf{e}(\tilde{\mathbf{u}}^\varepsilon) - \alpha\tilde{p}^\varepsilon I + \tilde{\lambda}(\text{div}\,\tilde{\mathbf{u}}^\varepsilon)I)\boldsymbol{\nu} = \varepsilon\tilde{\mathcal{P}}_\pm \ \text{ on } \tilde{\Sigma}^\varepsilon_\pm, \tag{3.4}$$
$$-\varepsilon^2\bigtriangledown\tilde{p}^\varepsilon\cdot\boldsymbol{\nu} = \pm\varepsilon\tilde{V}, \ \text{ on } \tilde{\Sigma}^\varepsilon_\pm, \tag{3.5}$$
$$\tilde{p}^\varepsilon(x_1,x_2,x_3,0) = \tilde{p}_{\text{in}}(x_1,x_2) \quad \text{in } \tilde{\Omega}^\varepsilon. \tag{3.6}$$

Remark 1. The difference here, with respect to flexural shell case ([22]), is that the contact loads in (3.4) and filtration velocity in (3.5) are differently scaled. More precisely, we recall that in the flexural shell case contact loads were assumed to behave like $\varepsilon^3\tilde{\mathcal{P}}_\pm$ and the normal boundary filtration velocity by $-\varepsilon^2\bigtriangledown\tilde{p}^\varepsilon\cdot\boldsymbol{\nu} = \pm\varepsilon^2\tilde{V}$. Furthermore the initial pressure was of order ε in the flexural shell case.

We follow [20] and get the existence and uniqueness result for (3.1)-(3.3).

Proposition 2. *Let us suppose*

$$\tilde{p}_{in}\in H^2_0(\tilde{\Omega}^\varepsilon),\ \mathcal{P}_\pm\in H^2(0,T;L^2(\omega;\mathbb{R}^3)) \text{ and } \tilde{V}\in H^1(0,T;L^2(\omega)),\ \tilde{V}|_{\{t=0\}} = 0. \tag{3.7}$$

Then problem (3.1)–(3.3) has a unique solution $\{\tilde{\mathbf{u}}^\varepsilon, \tilde{p}^\varepsilon\}\in H^1(0,T;\mathcal{V}(\tilde{\Omega}^\varepsilon)))\times H^1(0,T;H^1(\tilde{\Omega}^\varepsilon))$.

3.2 Problem in Curvilinear Coordinates and the Scaled Problem

In this section we introduce the formulation of the problem in curvilinear coordinates. The formulation is similar as in Subsection 3.3 in [22, pages 371–374] but with some prescribed quantities differently scaled which will at the end lead to a different model. For completeness and for the comfort of the reader we point out the main steps and define only necessary quantities, for details see [22].

Our goal is to find the limits of the solutions of problem (3.1)–(3.3) when ε tends to zero. Since the expected behavior of local components of the displacement is different we turn to the formulation in curvilinear coordinates given by $\mathbf{r}$. Further, we rescale the problem on a domain independent of ε.

To introduce the curvilinear coordinates we still need to define some geometry. The covariant basis of the the three-dimensional manifold $\overline{\tilde{\Omega}^\varepsilon}$, parameterized by $\mathbf{r}$, is defined by $\mathbf{g}^\varepsilon_i = \partial_i\mathbf{r} : \Omega^\varepsilon \to \mathbb{R}^3, i = 1,2,3$. Similarly as before the contravariant basis $\{\mathbf{g}^{1,\varepsilon}, \mathbf{g}^{2,\varepsilon}, \mathbf{g}^{3,\varepsilon}\}$ is biorthogonal one, defined by $\mathbf{g}^{j,\varepsilon}\cdot\mathbf{g}^\varepsilon_i = \delta_{ij}$ on $\overline{\Omega}^\varepsilon, i,j = 1,2,3$, where δ_{ij} is the Kronecker symbol. The covariant metric tensor $\mathbf{G}^\varepsilon_c = (g^\varepsilon_{ij})$ and the Christoffel symbols $\Gamma^{i,\varepsilon}_{jk}$ of the body $\overline{\Omega}^\varepsilon$ are defined by

$$g^\varepsilon_{ij} = \mathbf{g}^\varepsilon_i\cdot\mathbf{g}^\varepsilon_j, \quad \Gamma^{i,\varepsilon}_{jk} = \mathbf{g}^{i,\varepsilon}\cdot\partial_j\mathbf{g}^\varepsilon_k \text{ on } \overline{\Omega}^\varepsilon, \quad i,j,k = 1,2,3.$$

We set $\boldsymbol{\Gamma}^{i,\varepsilon} = (\Gamma^{i,\varepsilon}_{jk})_{j,k=1,\dots,3}$, $g^\varepsilon = \det \mathbf{G}^\varepsilon_c$ and $\mathbf{Q}^\varepsilon = (\nabla \mathbf{r})^{-T} = (\mathbf{g}^\varepsilon_1\ \mathbf{g}^\varepsilon_2\ \mathbf{g}^\varepsilon_3)^{-T} = (\mathbf{g}^{1,\varepsilon}\ \mathbf{g}^{2,\varepsilon}\ \mathbf{g}^{3,\varepsilon})$. Now the displacement is rewritten in the contravariant basis while the forces are rewritten in the covariant basis. New functions $\mathbf{u}$, $\mathbf{v}$ and $\mathcal{P}_\pm$ are defined by $\tilde{\mathbf{u}}^\varepsilon \circ \mathbf{r} = \mathbf{Q}^\varepsilon \mathbf{u}$, $\tilde{\mathbf{v}}^\varepsilon \circ \mathbf{r} = \mathbf{Q}^\varepsilon \mathbf{v}$, $\mathcal{P}_\pm = (\mathbf{Q}^\varepsilon)^T \tilde{\mathcal{P}}_\pm \circ \mathbf{r}$. For scalar functions we just change the coordinates $\tilde{p}^\varepsilon \circ \mathbf{r} = p^\varepsilon, \tilde{q} \circ \mathbf{r} = q, \tilde{V} \circ \mathbf{r} = V, \tilde{p}_{\text{in}} \circ \mathbf{r} = p_{\text{in}}$, on $\overline{\Omega}^\varepsilon$.

Still, problem for $\tilde{\mathbf{u}}^\varepsilon, \tilde{p}^\varepsilon$ and $\mathbf{u}^\varepsilon, p^\varepsilon$ is posed on ε–dependent domains. Therefore we follow the idea from Ciarlet, Destuynder [6] and rewrite (3.1)–(3.3) on the canonical domain independent of ε. As a consequence, the coefficients of the resulting weak formulation will depend on ε explicitly. The rescaling we use is given by

$$\mathbf{R}^\varepsilon(z) = (z^1, z^2, \varepsilon z^3), \quad z \in \Omega,\ \varepsilon \in (0, \varepsilon_0), \qquad \Omega = \omega \times (-1/2, 1/2).$$

By $\Sigma_\pm = \omega \times \{\pm 1/2\}$ we denote the upper and lower face of Ω and $\Gamma = \partial\omega \times (-1/2, 1/2)$. To the functions $\mathbf{u}^\varepsilon$, p^ε, g^ε, $\mathbf{g}^\varepsilon_i$, $\mathbf{g}^{\varepsilon,i}$, $\mathbf{Q}^\varepsilon$, $\Gamma^{i,\varepsilon}_{jk}$, $i,j,k = 1,2,3$ defined on $\overline{\Omega}^\varepsilon$ we associate the functions $\mathbf{u}(\varepsilon)$, $p(\varepsilon)$, $g(\varepsilon)$, $\mathbf{g}_i(\varepsilon)$, $\mathbf{g}^i(\varepsilon)$, $\mathbf{Q}(\varepsilon)$, $\Gamma^i_{ij}(\varepsilon)$, $i,j,k = 1,2,3$ defined on $\overline{\Omega}$ by composition with $\mathbf{R}^\varepsilon$. Let us also define

$$\mathcal{V}(\Omega) = \{\mathbf{v} = (v_1, v_2, v_3) \in H^1(\Omega; \mathbb{R}^3)\ :\ \mathbf{v}|_\Gamma = 0\}.$$

Then the problem (3.1)–(3.3) can be written as

$$\begin{aligned}
&\varepsilon \int_\Omega \mathcal{C}\left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))\mathbf{Q}(\varepsilon)^T\right) : \left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\mathbf{v})\mathbf{Q}(\varepsilon)^T\right)\sqrt{g(\varepsilon)}dz \\
&\qquad - \varepsilon\alpha \int_\Omega p(\varepsilon)\operatorname{tr}\left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\mathbf{v})\mathbf{Q}(\varepsilon)^T\right)\sqrt{g(\varepsilon)}dz \\
&= \varepsilon \int_{\Sigma_\pm} \mathcal{P}_\pm \cdot \mathbf{v}\sqrt{g(\varepsilon)}ds, \qquad \mathbf{v} \in \mathcal{V}(\Omega),\ \text{a.e. } t \in [0,T], \\
&\varepsilon \int_\Omega \beta \frac{\partial p(\varepsilon)}{\partial t} q\sqrt{g(\varepsilon)}dz + \varepsilon \int_\Omega \alpha \frac{\partial}{\partial t}\operatorname{tr}\left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))\mathbf{Q}(\varepsilon)^T\right) q\sqrt{g(\varepsilon)}dz \\
&\qquad + \varepsilon^3 \int_\Omega \mathbf{Q}(\varepsilon)\nabla^\varepsilon p(\varepsilon) \cdot \mathbf{Q}(\varepsilon)\nabla^\varepsilon q\sqrt{g(\varepsilon)}dz \\
&= \mp\varepsilon \int_{\Sigma_\pm} V q\sqrt{g(\varepsilon)}ds, \qquad q \in H^1(\Omega),\ \text{a.e. } t \in [0,T], \\
&p(\varepsilon) = p_{\text{in}}, \qquad \text{for } t = 0.
\end{aligned} \tag{3.8}$$

Here $\mathcal{C}\mathbf{E} = \tilde{\lambda}(\operatorname{tr}\mathbf{E})\mathbf{I} + 2\mathbf{E}$, for all $\mathbf{E} \in \mathbb{R}^{3\times3}_{sym}$,

$$\boldsymbol{\gamma}^\varepsilon(\mathbf{v}) = \frac{1}{\varepsilon}\boldsymbol{\gamma}_z(\mathbf{v}) + \boldsymbol{\gamma}_y(\mathbf{v}) - \sum_{i=1}^3 v_i \boldsymbol{\Gamma}^i(\varepsilon), \tag{3.9}$$

$$\boldsymbol{\gamma}_z(\mathbf{v}) = \begin{bmatrix} 0 & 0 & \frac12\partial_3 v_1 \\ 0 & 0 & \frac12\partial_3 v_2 \\ \frac12\partial_3 v_1 & \frac12\partial_3 v_2 & \partial_3 v_3 \end{bmatrix}, \ \boldsymbol{\gamma}_y(\mathbf{v}) = \begin{bmatrix} \partial_1 v_1 & \frac12(\partial_2 v_1 + \partial_1 v_2) & \frac12\partial_1 v_3 \\ \frac12(\partial_2 v_1 + \partial_1 v_2) & \partial_2 v_2 & \frac12\partial_2 v_3 \\ \frac12\partial_1 v_3 & \frac12\partial_2 v_3 & 0 \end{bmatrix},$$

$$\nabla^\varepsilon q = \frac{1}{\varepsilon}\nabla_z q + \nabla_y q, \qquad \nabla_z q = \begin{bmatrix} 0 & 0 & \partial_3 q \end{bmatrix}, \qquad \nabla_y q = \begin{bmatrix} \partial_1 q & \partial_2 q & 0 \end{bmatrix}$$

and we have also used the notation

$$\begin{aligned}
\mp \int_{\Sigma_\pm} V q\sqrt{g(\varepsilon)}ds &= \int_{\Sigma_-} V q\sqrt{g(\varepsilon)}ds - \int_{\Sigma_+} V q\sqrt{g(\varepsilon)}ds, \\
\int_{\Sigma_\pm} \mathcal{P}_\pm \cdot \mathbf{v}\sqrt{g(\varepsilon)}ds &= \int_{\Sigma_+} \mathcal{P}_+ \cdot \mathbf{v}\sqrt{g(\varepsilon)}ds + \int_{\Sigma_-} \mathcal{P}_- \cdot \mathbf{v}\sqrt{g(\varepsilon)}ds.
\end{aligned}$$

A consequence of Proposition 2 and the smoothness of the curvilinear coordinates transformation is the existence and uniqueness of the solution of (3.8). Also note that in the present (elliptic membrane) case there is no rescaling of the pressure. It will appear to be of order one which is in contrast to the flexural shell case where it was of order ε.

3.3 Convergence results

In the remainder of the paper we make the following assumptions.

Assumption 3. *For simplicity, we assume that $p_{in} = 0$, that $V \in H^1(0,T;L^2(\omega))$, $V|_{\{t=0\}} = 0$ and that $\mathcal{P}_\pm \in H^2(0,T;L^2(\omega;\mathbb{R}^3))$, with $\mathcal{P}_\pm|_{\{t=0\}} = 0$.*

To describe the limit problem, which we presented in the differential formulation in (2.14)–(2.16), we introduce the function space $\mathcal{V}_M(\omega) = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$. Contrary to $\mathcal{V}_F(\omega)$, which is the function space for the flexural shell model, it is always non-trivial. The boundary value problem in $\Omega = \omega \times (-1/2, 1/2)$ for the effective displacement and the effective pressure is given by:

find $\{\mathbf{u}, p^0\} \in C([0,T]; \mathcal{V}_M(\omega) \times L^2(\Omega))$, $\partial_{z_3} p^0 \in L^2((0,T)\times\Omega)$ satisfying the system

$$\begin{aligned}
&\int_\omega \tilde{\mathcal{C}}(\mathbf{A}^c \boldsymbol{\gamma}(\mathbf{u})) : \boldsymbol{\gamma}(\mathbf{v})\mathbf{A}^c \sqrt{a} dz_1 dz_2 - \frac{2\alpha}{\tilde{\lambda}+2} \int_\omega \int_{-1/2}^{1/2} p^0 dz_3 \mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{v})\sqrt{a} dz_1 dz_2 \\
&\qquad = \int_\omega (\mathcal{P}_+ + \mathcal{P}_-) \cdot \mathbf{v}\sqrt{a} dz_1 dz_2., \qquad \mathbf{v} \in \mathcal{V}_M(\omega),
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\int_\Omega \left(\beta + \frac{\alpha^2}{\tilde{\lambda}+2}\right) \frac{\partial p^0}{\partial t} q\sqrt{a} dz + \int_\Omega \alpha \frac{\partial}{\partial t}\left(\frac{2}{\tilde{\lambda}+2}\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u})\right) q\sqrt{a} dz + \int_\Omega \frac{\partial p^0}{\partial z_3}\frac{\partial q}{\partial z_3}\sqrt{a} dz \\
&\qquad = \mp \int_{\Sigma_\pm} V q \sqrt{a} ds, \qquad q \in H^1(\Omega).
\end{aligned} \tag{3.11}$$

$$p^0 = 0 \qquad \text{at} \quad t = 0, \tag{3.12}$$

where $\boldsymbol{\gamma}(\cdot)$ is given by (2.5) and

$$\tilde{\mathcal{C}}\mathbf{E} = 2\frac{\tilde{\lambda}}{\tilde{\lambda}+2} \operatorname{tr}(\mathbf{E})\mathbf{I} + 2\mathbf{E}, \qquad \mathbf{E} \in \mathbb{R}^{2\times 2}_{\text{sym}}. \tag{3.13}$$

We observe that, contrary to the effective flexural shell system from [22], problem (3.10)–(3.12) is of the second order. Also, pressure enters the shell equation (3.10) with the average over the cross–section in difference with the first moment over the cross–section in the flexural shell model.

Fundamental for the analysis of this model is the inequality of Korn's type on an elliptic surface, see [5, Theorem 2.7-3] or [7, Theorem 4.2].

Lemma 4. *Let ω be a domain in R^2 and let $\boldsymbol{X} \in C^{2,1}(\overline{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\mathbf{a}_\alpha = \partial_\alpha \boldsymbol{X}$ are linearly independent at all points of $\overline{\omega}$ and such that the surface $\boldsymbol{X}(\overline{\omega})$ is elliptic. Then there is $C_M > 0$ such that*

$$\|v_1\|^2_{H^1(\omega)} + \|v_2\|^2_{H^1(\omega)} + \|v_3\|^2_{L^2(\omega)} \le C_M \|\boldsymbol{\gamma}(\mathbf{v})\|^2_{L^2(\omega;\mathbb{R}^{3\times 3})}, \qquad \mathbf{v} \in \mathcal{V}_M(\omega).$$

Proposition 5. *Under Assumption 3, problem (3.10)–(3.12) has a unique solution $\{\mathbf{u}, p^0\}$ in the space $C([0,T]; \mathcal{V}_M(\omega) \times L^2(\Omega))$, $\partial_{z_3} p^0 \in L^2((0,T)\times\Omega)$ Furthermore, $\partial_t p^0 \in L^2((0,T)\times\Omega)$ and $\partial_t \mathbf{u} \in L^2(0,T;\mathcal{V}_M(\omega))$.*

Proof. We follow the proof of Proposition 4 from [22] and first prove that $\{\mathbf{u}, p^0\} \in C([0,T]; \mathcal{V}_M(\omega) \times L^2(\Omega))$ and $\partial_{z_3} p^0 \in L^2((0,T)\times\Omega)$ imply a higher regularity in time. Ideas are analogous but details of the calculations are different.

We take $q = \overline{q}(z_1, z_2)$, $\overline{q} \in C^\infty(\overline{\omega})$ as a test function in (3.11). The time continuity and (3.11) yield

$$\left(\beta + \frac{\alpha^2}{\tilde{\lambda}+2}\right) \int_{-1/2}^{1/2} p^0 \, dz_3 + \frac{2\alpha}{\tilde{\lambda}+2}\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}) = 0. \tag{3.14}$$

We insert (3.14) into (3.10) and obtain

$$\begin{aligned}
&\int_\omega \tilde{\mathcal{C}}(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})):\boldsymbol{\gamma}(\mathbf{v})\mathbf{A}^c\sqrt{a}dz_1dz_2+\frac{4\alpha^2}{(\tilde\lambda+2)\big(\beta(\tilde\lambda+2)+\alpha^2\big)}\int_\omega \mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{u})\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{v})\sqrt{a}dz_1dz_2\\
&\quad=\int_\omega(\mathcal{P}_++\mathcal{P}_-)\cdot\mathbf{v}\sqrt{a}dz_1dz_2,\qquad \mathbf{v}\in\mathcal{V}_M(\omega).
\end{aligned}\tag{3.15}$$

The H^2-regularity in time of $\mathcal{P}_\pm$ allows taking time derivatives of equation (3.15) up to order 2. It yields $\partial_t\mathbf{u}\in L^2(0,T;\mathcal{V}_M(\omega))$ and $\partial_{tt}\mathbf{u}\in L^2(0,T;\mathcal{V}_M(\omega))$. Hence $\partial_t\mathbf{u}\in H^1(0,T;\mathcal{V}_M(\omega))$. Now the classical parabolic regularity theory applied at (3.11) implies $\partial_t p^0\in L^2((0,T)\times\Omega)$.

We insert $\mathbf{v}=\dfrac{\partial\mathbf{u}}{\partial t}$ as a test function in (3.10) and p^0 as a test function in (3.11) and sum up the equations. It follows

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\int_\omega\tilde{\mathcal{C}}(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})):\boldsymbol{\gamma}(\mathbf{u})\mathbf{A}^c\sqrt{a}dz_1dz_2+\int_\Omega\Big(\beta+\frac{\alpha^2}{\tilde\lambda+2}\Big)(p^0)^2\sqrt{a}dz\\
&\qquad-2\int_\omega(\mathcal{P}_++\mathcal{P}_-)\cdot\mathbf{u}\sqrt{a}dz_1dz_2\Big\}+\int_\Omega\Big(\frac{\partial p^0}{\partial z_3}\Big)^2\sqrt{a}dzdt\\
&\qquad=-\int_\omega\partial_t(\mathcal{P}_++\mathcal{P}_-)\cdot\mathbf{u}\sqrt{a}dz_1dz_2\mp\int_{\Sigma_\pm}Vp^0\sqrt{a}\,dz_1dz_2.
\end{aligned}\tag{3.16}$$

Equality (3.16) implies uniqueness of solutions to problem (3.10)–(3.12). Equality (3.16) allows to obtain the uniform bounds for $\boldsymbol{\gamma}(\mathbf{u})$ in $L^\infty(0,T;\mathcal{V}_M(\omega))$, for p^0 in $L^\infty(0,T;L^2(\Omega))$ and for $\partial_{z_3}p^0$ in $L^2(0,T;L^2(\Omega))$. Using Lemma 4 and the classical weak compactness reasoning, we conclude the existence of the solution. □

Remark 6. Note that the equation (3.15) can be used to decouple the problem. Thus we first can solve the membrane problem with slightly changed coefficients and with time as a parameter in the equation. In the second step we plug this solution into (3.11). This approach can also lead to alternative existence proof. Namely, standard existence theory for membrane shell model applied on (3.15) yields that there is a unique $\mathbf{u}\in H^2(0,T;\mathcal{V}_M(\omega))$ solving (3.15). Then the standard parabolic theory for (3.11) implies the existence of p^0 in $C([0,T];L^2(\Omega))$ and such $\partial_{z_3}p^0$ in $L^2(0,T;L^2(\Omega))$. Further regularity is classical.

Standard computations give

$$p^0-\int_{-1/2}^{1/2}p^0dz_3=-V(t)y_3+\frac{4\overline{\beta}}{\pi^2}\sum_{m=0}^\infty\int_0^t e^{-\frac{(2m+1)^2\pi^2}{\overline{\beta}}(t-s)}\partial_tV ds\frac{(-1)^m}{(2m+1)^2}\sin\big((2m+1)\pi y_3\big),\tag{3.17}$$

with $\overline{\beta}=\beta+\dfrac{\alpha^2}{\tilde\lambda+2}$. After inserting (3.14) into (3.10) for displacement we get a standard elastic membrane shell equations with modified coefficients. Then we use (3.14) to compute the mean pressure and finally (3.17) to reconstruct the pressure fluctuation.

Next we formulate the main result of the paper.

Theorem 7. *Let us suppose Assumption 3. Let $\{\mathbf{u}(\varepsilon),p(\varepsilon)\}\in H^1(0,T;\mathcal{V}(\Omega))\times H^1(0,T;H^1(\Omega))$ be the unique solution of (3.8) and let $\{\mathbf{u},p^0\}$ be the unique solution for (3.10)–(3.12). Then we obtain*

$$\begin{aligned}
&\mathbf{u}(\varepsilon)\to\mathbf{u}\qquad \textit{strongly in } C([0,T];H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)),\\
&\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))\to\boldsymbol{\gamma}^0\qquad \textit{strongly in } C([0,T];L^2(\Omega;\mathbb{R}^{3\times3})),\\
&p(\varepsilon)\to p^0\qquad \textit{strongly in } C([0,T];L^2(\Omega)),\\
&\frac{\partial p(\varepsilon)}{\partial z_3}\to\frac{\partial p^0}{\partial z_3}\qquad \textit{strongly in } L^2(0,T;L^2(\Omega)),
\end{aligned}$$

where

$$\boldsymbol{\gamma}^0 = \begin{bmatrix} \boldsymbol{\gamma}(\mathbf{u}) & & 0 \\ & & 0 \\ 0 & 0 & \frac{\alpha}{\tilde{\lambda}+2}p^0 - \frac{\tilde{\lambda}}{\tilde{\lambda}+2}\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}) \end{bmatrix}. \tag{3.18}$$

Remark 8. We observe differences between convergence theorem (Theorem 6) from [22] and this result: the scaling of symmetrized gradient in curvilinear coordinates $\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))$, the function space for the transversal displacement and the structure of $\boldsymbol{\gamma}^0$.

Convergence of the symmetrized gradient in curvilinear coordinates $\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))$ implies convergence of the stress tensor.

Corollary 9. *For the stress tensor* $\sigma(\varepsilon) = \mathcal{C}(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))\mathbf{Q}(\varepsilon)^T) - \alpha p(\varepsilon)\mathbf{I}$ *one has*

$$\sigma(\varepsilon) \to \sigma = \mathcal{C}(\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T) - \alpha p^0\mathbf{I} \qquad \textit{strongly in } C([0,T];L^2(\Omega;\mathbb{R}^{3\times 3})). \tag{3.19}$$

The limit stress in the local contravariant basis $\mathbf{Q} = (\mathbf{a}^1\ \mathbf{a}^2\ \mathbf{a}^3)$ *is given by*

$$\mathbf{Q}^T\sigma\mathbf{Q} = \begin{bmatrix} \left(-\frac{2\alpha}{\tilde{\lambda}+2}p^0\mathbf{I} + \frac{2\tilde{\lambda}}{\tilde{\lambda}+2}(\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}))\mathbf{I} + 2\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})\right)\mathbf{A}^c & 0 \\ 0 & 0 \end{bmatrix}.$$

4 A priori estimates and consequences

The following three-dimensional inequality of Korn's type for a family of linearly elastic elliptic membrane shells is essential for derivation of a priori estimates. It is different from the inequality used in derivation of flexural shell models.

Theorem 10 ([5, Theorem 4.3-1], [7, Theorem 4.1])**.** *Assume that* $\mathbf{X} \in C^3(\overline{\omega};\mathbb{R}^3)$ *parameterizes an elliptic surface. Then there exist constants* $\varepsilon_0 > 0, C > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_0)$ *one has*

$$\|v_1\|^2_{H^1(\Omega)} + \|v_2\|^2_{H^1(\Omega)} + \|v_3\|^2_{L^2(\Omega)} \le C\|\boldsymbol{\gamma}^\varepsilon(\mathbf{v})\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}, \qquad \mathbf{v} \in \mathcal{V}(\Omega).$$

Remark 11. Note that the above estimate applies to the functions on the three-dimensional domain Ω and will be the basis for the a priori estimates for the solution of (3.8), while in Lemma 4 the estimate was for functions on ω and is the basis for the existence and uniqueness of the solution of the limit model (3.10)–(3.12).

Next we state the asymptotic properties of the geometry coefficients in the equation (3.8). Direct calculation shows that there are constants $0 < m_g \le M_g$, independent of $\varepsilon \in (0,\varepsilon_0)$, such that $m_g \le \sqrt{g(\varepsilon)} \le M_g$. The functions $\mathbf{g}^i(\varepsilon), \mathbf{g}_i(\varepsilon), g^{ij}(\varepsilon), g(\varepsilon), \Gamma^i_{jk}(\varepsilon), \mathbf{Q}(\varepsilon)$ are in $C(\overline{\Omega})$ by assumptions. Further, there is $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$,

$$\begin{aligned} &\|\mathbf{g}^i(\varepsilon) - \mathbf{a}^i\|_\infty + \|\mathbf{g}_i(\varepsilon) - \mathbf{a}_i\|_\infty \le C\varepsilon, \\ &\|\frac{\partial}{\partial z_3}\sqrt{g(\varepsilon)}\|_\infty + \|\sqrt{g(\varepsilon)} - \sqrt{a}\|_\infty \le C\varepsilon, \\ &\|\mathbf{Q}(\varepsilon) - \mathbf{Q}\|_\infty \le C\varepsilon, \end{aligned} \tag{4.1}$$

where $\|\cdot\|_\infty$ is the norm in $C(\overline{\Omega})$. See [7, 8] for details. In addition, in [5, Theorem 3.3-1] and [8, Lemma 3.1] the asymptotic of the Christoffel symbols is given by

$$\boldsymbol{\Gamma}^\kappa(\varepsilon) = \begin{bmatrix} \Gamma^\kappa_{11} & \Gamma^\kappa_{12} & -b^\kappa_1 \\ \Gamma^\kappa_{21} & \Gamma^\kappa_{22} & -b^\kappa_2 \\ -b^\kappa_1 & -b^\kappa_2 & 0 \end{bmatrix} + O(\varepsilon), \quad \kappa = 1,2, \qquad \boldsymbol{\Gamma}^3(\varepsilon) = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} + O(\varepsilon). \tag{4.2}$$

A priori estimates are given in the following two lemmas. The estimates are similar, but different from the flexural case. Namely, the scaling of $\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))$ is different. The proof is analogous to the proof of Lemma 10 from [22] and we omit it.

Lemma 12. *There is $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ one has*

$$\|\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))}, \|p(\varepsilon)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}))}, \|\varepsilon\nabla^\varepsilon p(\varepsilon)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \le C.$$

We now first take the time derivative of the first equation in (3.8) and then insert $\mathbf{v} = \frac{\partial \mathbf{u}(\varepsilon)}{\partial t}$ as a test functions and sum it with (3.8) for the test function $q = \frac{\partial p(\varepsilon)}{\partial t}$. We obtain

$$\begin{aligned}
&\int_\Omega \mathcal{C}\left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\frac{\partial \mathbf{u}(\varepsilon)}{\partial t})\mathbf{Q}(\varepsilon)^T\right) : \left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\frac{\partial \mathbf{u}(\varepsilon)}{\partial t})\mathbf{Q}(\varepsilon)^T\right)\sqrt{g(\varepsilon)}dz \\
&\qquad + \beta\int_\Omega \frac{\partial p(\varepsilon)}{\partial t}\frac{\partial p(\varepsilon)}{\partial t}\sqrt{g(\varepsilon)}dz + \frac{1}{2}\varepsilon^2\frac{d}{dt}\int_\Omega \mathbf{Q}(\varepsilon)\nabla^\varepsilon p(\varepsilon)\cdot\mathbf{Q}(\varepsilon)\nabla^\varepsilon p(\varepsilon)\sqrt{g(\varepsilon)}dz \\
&\qquad = \int_{\Sigma_\pm} \frac{\partial \mathcal{P}_\pm}{\partial t}\cdot\frac{\partial \mathbf{u}(\varepsilon)}{\partial t}\sqrt{g(\varepsilon)}ds \mp \int_{\Sigma_\pm} V\frac{\partial p(\varepsilon)}{\partial t}\sqrt{g(\varepsilon)}ds.
\end{aligned} \tag{4.3}$$

Then, similarly as in Lemma 12, we obtain

Lemma 13. *There is $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ one has*

$$\|\boldsymbol{\gamma}^\varepsilon(\frac{\partial \mathbf{u}(\varepsilon)}{\partial t})\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))}, \|\frac{\partial p(\varepsilon)}{\partial t}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}))}, \|\varepsilon\nabla^\varepsilon p(\varepsilon)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \le C.$$

From the scaled Korn's inequality from Theorem 10 we directly obtain the following Corollary.

Corollary 14. *Let us suppose Assumption 3 and let $\{\mathbf{u}(\varepsilon), p(\varepsilon)\}$ be the solution for problem (3.8). Then there is $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ one has*

$$\begin{gathered}
\|\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^9))}, \|u_1(\varepsilon)\|_{H^1(0,T;H^1(\Omega))}, \|u_2(\varepsilon)\|_{H^1(0,T;H^1(\Omega))}, \|u_3(\varepsilon)\|_{H^1(0,T;L^2(\Omega))}, \\
\|p(\varepsilon)\|_{H^1(0,T;L^2(\Omega;\mathbb{R}))}, \|\frac{\partial p(\varepsilon)}{\partial z_3}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}))} \le C.
\end{gathered}$$

In addition, there are $u_1, u_2 \in H^1(0,T;H^1(\Omega;\mathbb{R}^3))$, $u_3 \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$, $p^0 \in H^1(0,T;L^2(\Omega;\mathbb{R}))$ and $\boldsymbol{\gamma}^0 \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))$ such that on a subsequence one has

$$\begin{aligned}
&u_j(\varepsilon) \rightharpoonup u_j \text{ weakly in } H^1(0,T;H^1(\Omega)), \qquad j = 1,2, \\
&u_3(\varepsilon) \rightharpoonup u_3 \text{ weakly in } H^1(0,T;L^2(\Omega)), \\
&p(\varepsilon) \rightharpoonup p^0 \text{ weakly in } H^1(0,T;L^2(\Omega;\mathbb{R})), \\
&\frac{\partial p(\varepsilon)}{\partial z_3} \rightharpoonup \frac{\partial p^0}{\partial z_3} \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R})) \quad \text{and weak * in} \quad L^\infty(0,T;L^2(\Omega;\mathbb{R})), \\
&\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon)) \rightharpoonup \boldsymbol{\gamma}^0 \text{ weakly in } H^1(0,T;L^2(\Omega;\mathbb{R}^{3\times 3})).
\end{aligned} \tag{4.4}$$

Since $\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))$ and $\mathbf{u}(\varepsilon)$ are related one expects that the limits $\mathbf{u}= (u_1, u_2, u_3)$ and $\boldsymbol{\gamma}^0$ are related too. The following theorem gives the precise relationship. Its statements are essential in obtaining the limit model in classical elliptic membrane shell derivation as well. However it is different from the corresponding statement, Theorem 13, in [22]. Its proof is a collection of particular statements in the proof of Theorem 4.4-1 in [5]. Therefore we just sketch it here.

Theorem 15. *For any $\mathbf{v} \in \mathcal{V}(\Omega)$ let $\boldsymbol{\gamma}^\varepsilon(\mathbf{v})$ be given by (3.9) and let the tensor $\boldsymbol{\gamma}(\mathbf{v})$ be given by (2.5). Let the family $(\mathbf{w}(\varepsilon))_{\varepsilon>0} \subset \mathcal{V}(\Omega)$ satisfies*

$$\begin{aligned}
&w_j(\varepsilon) \rightharpoonup w_j \text{ weakly in } H^1(\Omega), \quad j = 1,2, \\
&w_3(\varepsilon) \rightharpoonup w_3 \text{ weakly in } L^2(\Omega), \\
&\boldsymbol{\gamma}^\varepsilon(\mathbf{w}(\varepsilon)) \rightharpoonup \tilde{\gamma}^0 \text{ weakly in } L^2(\Omega;\mathbb{R}^{3\times 3})
\end{aligned} \tag{4.5}$$

as $\varepsilon \to 0$. Then $\mathbf{w}= (w_1, w_2, w_3)$ is independent of transverse variable z_3, belongs to $\mathcal{V}_M(\omega) = H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega)$, and satisfies

$$\tilde{\gamma}^0_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{w}), \qquad \alpha, \beta \in \{1,2\}.$$

Proof. From $\boldsymbol{\gamma}^\varepsilon(\mathbf{w}(\varepsilon)) \rightharpoonup \tilde{\boldsymbol{\gamma}}^0$ we obtain that

$$\varepsilon\boldsymbol{\gamma}^\varepsilon(\mathbf{w}(\varepsilon)) = \boldsymbol{\gamma}_z(\mathbf{w}(\varepsilon)) + \varepsilon\boldsymbol{\gamma}_y(\mathbf{w}(\varepsilon)) - \varepsilon\sum_{i=1}^3 w_i(\varepsilon)\boldsymbol{\Gamma}^i(\varepsilon) \to 0$$

strongly in $L^2(\Omega;\mathbb{R}^3)$. From the convergences in (4.5) and asymptotics of $\boldsymbol{\Gamma}^i(\varepsilon)$ given in (4.2) we have that

$$\varepsilon\sum_{i=1}^3 w_i(\varepsilon)\boldsymbol{\Gamma}^i(\varepsilon) \to 0$$

strongly in $L^2(\Omega;\mathbb{R}^{3\times3})$. Also (4.5) implies that $\varepsilon\boldsymbol{\gamma}_y(\mathbf{w}(\varepsilon)) \to 0$ strongly in $H^{-1}(\Omega;\mathbb{R}^{3\times3})$. Finally $\boldsymbol{\gamma}_z(\mathbf{w}(\varepsilon)) \to 0$ strongly in $L^2(\omega;H^{-1}(-1/2,1/2;\mathbb{R}^{3\times3}))$ and weakly in $L^2(\Omega;\mathbb{R}^{3\times3})$. From the definition of $\boldsymbol{\gamma}_z$ in (3.9) we obtain that $\partial_3 w_i(\varepsilon) \to 0$ strongly in $H^{-1}(\Omega)$. Therefore $\mathbf{w}$ is independent of the transverse variable z_3. Then it is straightforward to conclude that $\mathbf{w} \in \mathcal{V}_M(\omega)$.

Now, the convergences $\gamma^\varepsilon_{\alpha\beta}(\mathbf{w}(\varepsilon)) \rightharpoonup \tilde{\gamma}^0_{\alpha\beta}$, $\alpha,\beta \in \{1,2\}$ from (4.5), using the definition of $\boldsymbol{\gamma}^\varepsilon$ from (3.9), imply

$$\tilde{\gamma}^0_{\alpha\beta} = \lim_{\varepsilon\to0}\left(\frac{1}{2}(\partial_\alpha w_\beta(\varepsilon) + \partial_\beta w_\alpha(\varepsilon)) - \sum_{i=1}^3 w_i(\varepsilon)\Gamma^i_{\alpha\beta}(\varepsilon)\right).$$

Using the asymptotics of $\boldsymbol{\Gamma}^i(\varepsilon)$ from (4.2) together with the remaining convergences in (4.5) yield

$$\tilde{\gamma}^0_{\alpha\beta} = \frac{1}{2}(\partial_\alpha w_\beta + \partial_\beta w_\alpha) - \sum_{i=1}^3 w_i\Gamma^i_{\alpha\beta}(0) = \gamma_{\alpha\beta}(\mathbf{w}).$$

□

Remark 16. In order to apply Theorem 15 we need pointwise convergences for every $t \in [0,T]$. The estimates from Corollary 14 (i.e., Lemma 12 and Lemma 13) imply that we are in the same position as in Remark 14 from [22] for $u_1(\varepsilon), u_2(\varepsilon), p(\varepsilon)$ and $\boldsymbol{\gamma}^\varepsilon(\boldsymbol{u}(\varepsilon))$, i.e.

$$\begin{aligned}
&u_\alpha(\varepsilon)(t) \rightharpoonup u_\alpha(t) \text{ weakly in } H^1(\Omega) \qquad \text{for every} \quad t \in [0,T], \quad \alpha \in \{1,2\},\\
&p(\varepsilon)(t) \rightharpoonup p^0(t) \text{ weakly in } L^2(\Omega),\\
&\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))(t) \rightharpoonup \boldsymbol{\gamma}^0(t) \text{ weakly in } L^2(\Omega;\mathbb{R}^{3\times3}),
\end{aligned}$$

for every $t \in [0,T]$. In the case of $u_3(\varepsilon)$ we argue similarly and obtain

$$u_3(\varepsilon)(t) \rightharpoonup u_3(t) \text{ weakly in } L^2(\Omega;\mathbb{R}^3) \qquad \text{for every} \quad t \in [0,T]. \tag{4.6}$$

Thus we may apply Theorem 15, with $\mathbf{w}(\varepsilon) = \mathbf{u}(\varepsilon)(t)$, for each $t \in [0,T]$ and conclude that the limit points of $\{\mathbf{u}(\varepsilon)(t)\}$ belong to $\mathcal{V}_M(\omega)$.

Moreover we conclude that

$$\gamma^0_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{u}), \qquad \alpha,\beta \in \{1,2\}. \tag{4.7}$$

5 Derivation of the elliptic membrane model

We now derive the limit model. We do it by taking the limit in (3.8) for two choices of test functions. In the third step we prove the strong convergence of the strain and pressure. Finally in the fourth step we prove the strong convergence of displacements.

Step 1 (Identification of γ^0_{i3}). We take the limit as $\varepsilon \to 0$ in the first equation in (3.8) divided by ε and obtain

$$\int_\Omega \mathcal{C}\left(\mathbf{Q}(0)\boldsymbol{\gamma}^0\mathbf{Q}(0)^T\right) : \left(\mathbf{Q}(0)\boldsymbol{\gamma}_z(\mathbf{v})\mathbf{Q}(0)^T\right)\sqrt{g(0)}dz - \alpha\int_\Omega p^0 \operatorname{tr}\left(\mathbf{Q}(0)\boldsymbol{\gamma}_z(\mathbf{v})\mathbf{Q}(0)^T\right)\sqrt{g(0)}dz = 0,$$
$$\mathbf{v} \in \mathcal{V}(\Omega), \text{ a.e. } t \in [0,T].$$

This equation is the same as in the first step of derivation the limit model in the flexural case, see page 382 in [22]. Thus using the definition of $\boldsymbol{\gamma}_z$ and the space $\mathcal{V}(\Omega)$ and arguing as in [22] we identify the third row/column of $\boldsymbol{\gamma}^0$. Here we also use (4.7).

Lemma 17.

$$\gamma^0_{13} = \gamma^0_{31} = \gamma^0_{23} = \gamma^0_{32} = 0,$$
$$\gamma^0_{33} = \frac{\alpha}{\tilde{\lambda}+2} p^0 - \frac{\tilde{\lambda}}{\tilde{\lambda}+2} \mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}).$$

From this lemma and Theorem 15 we have that $\boldsymbol{\gamma}^0$ is of the following form

$$\boldsymbol{\gamma}^0 = \begin{bmatrix} \boldsymbol{\gamma}(\mathbf{u}) & & 0 \\ & & 0 \\ 0 & 0 & \frac{\alpha}{\tilde{\lambda}+2} p^0 - \frac{\tilde{\lambda}}{\tilde{\lambda}+2} \mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}) \end{bmatrix}. \tag{5.1}$$

Step 2 (Identification of the limit model). Now we take the limit in (3.8), after division of both equations by ε, for test functions independent of the transversal variable z_3, i.e., $\mathbf{v} \in H^1_0(\omega; \mathbb{R}^3)$, such that

$$\boldsymbol{\gamma}_z(\mathbf{v}) = 0.$$

Thus $\boldsymbol{\gamma}^\varepsilon(\mathbf{v}) = \boldsymbol{\gamma}_y(\mathbf{v}) - \sum_{i=1}^3 v_i \boldsymbol{\Gamma}^i(0)$. The equations are

$$\begin{aligned}
&\int_\Omega \mathcal{C}\left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))\mathbf{Q}(\varepsilon)^T\right) : \left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\mathbf{v})\mathbf{Q}(\varepsilon)^T\right)\sqrt{g(\varepsilon)}dz \\
&\qquad - \alpha \int_\Omega p(\varepsilon)\,\mathrm{tr}\left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\mathbf{v})\mathbf{Q}(\varepsilon)^T\right)\sqrt{g(\varepsilon)}dz = \int_{\Sigma_\pm} \mathcal{P}_\pm \cdot \mathbf{v}\sqrt{g(\varepsilon)}ds, \\
&\qquad \mathbf{v} \in H^1_0(\omega;\mathbb{R}^3), \text{ a.e. } t \in [0,T], \\
&\int_\Omega \beta \frac{\partial p(\varepsilon)}{\partial t} q\sqrt{g(\varepsilon)}dz + \int_\Omega \alpha \frac{\partial}{\partial t}\mathrm{tr}\left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))\mathbf{Q}(\varepsilon)^T\right) q\sqrt{g(\varepsilon)}dz \\
&+ \varepsilon^2 \int_\Omega \mathbf{Q}(\varepsilon)\nabla^\varepsilon p(\varepsilon) \cdot \mathbf{Q}(\varepsilon)\nabla^\varepsilon q\sqrt{g(\varepsilon)}dz = \mp \int_{\Sigma_\pm} Vq\sqrt{g(\varepsilon)}ds, \qquad q \in H^1(\Omega).
\end{aligned}$$

In the limit when $\varepsilon \to 0$ we obtain

$$\begin{aligned}
&\int_\Omega \mathcal{C}\left(\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\right) : \left(\mathbf{Q}(\boldsymbol{\gamma}_y(\mathbf{v}) - \sum_{i=1}^3 v_i \boldsymbol{\Gamma}^i(0))\mathbf{Q}^T\right)\sqrt{a}dz \\
&- \alpha \int_\Omega p^0\,\mathrm{tr}\left(\mathbf{Q}(\boldsymbol{\gamma}_y(\mathbf{v}) - \sum_{i=1}^3 v_i \boldsymbol{\Gamma}^i(0))\mathbf{Q}^T\right)\sqrt{a}dz = \int_{\Sigma_\pm} \mathcal{P}_\pm \cdot \mathbf{v}\sqrt{a}ds, \\
&\quad \mathbf{v} \in H^1_0(\omega;\mathbb{R}^3), \text{ a.e. } t \in [0,T], \\
&\int_\Omega \beta \frac{\partial p^0}{\partial t} q\sqrt{a}dz + \int_\Omega \alpha \frac{\partial}{\partial t}\mathrm{tr}\left(\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\right) q\sqrt{a}dz + \int_\Omega \frac{\partial p^0}{\partial z_3}\mathbf{Q}\mathbf{e}_3 \cdot \frac{\partial q}{\partial z_3}\mathbf{Q}\mathbf{e}_3\sqrt{a}dz \\
&\quad = \mp \int_{\Sigma_\pm} Vq\sqrt{a}ds, \qquad q \in H^1(\Omega).
\end{aligned} \tag{5.2}$$

First note that

$$\mathbf{Q}^T\mathbf{Q} = \begin{bmatrix} \mathbf{A}^c & 0 \\ 0 & 1 \end{bmatrix} \tag{5.3}$$

Since

$$\boldsymbol{\gamma}_y(\mathbf{v}) - v_i\boldsymbol{\Gamma}^i(0) = \begin{bmatrix} & \boldsymbol{\gamma}(\mathbf{v}) & \frac{1}{2}\partial_1 v_3 + \sum_{\sigma=1}^2 v_\sigma b^\sigma_1 \\ & & \frac{1}{2}\partial_2 v_3 + \sum_{\sigma=1}^2 v_\sigma b^\sigma_2 \\ \frac{1}{2}\partial_1 v_3 + \sum_{\sigma=1}^2 v_\sigma b^\sigma_1 & \frac{1}{2}\partial_2 v_3 + \sum_{\sigma=1}^2 v_\sigma b^\sigma_2 & 0 \end{bmatrix}$$

using (5.3) we get

$$\operatorname{tr}\left(\mathbf{Q}(\boldsymbol{\gamma}_y(\mathbf{v})-\sum_{i=1}^{3} v_i\boldsymbol{\Gamma}^i(0))\mathbf{Q}^T\right)=\operatorname{tr}\left(\mathbf{Q}^T\mathbf{Q}(\boldsymbol{\gamma}_y(\mathbf{v})-\sum_{i=1}^{3} v_i\boldsymbol{\Gamma}^i(0))\right)=\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{v}). \tag{5.4}$$

Next, using Lemma 17 yields

$$\begin{aligned}\operatorname{tr}\left(\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\right)&=\operatorname{tr}\left(\mathbf{Q}^T\mathbf{Q}\boldsymbol{\gamma}^0\right)=\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{u})+\gamma^0_{33}\\&=\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{u})+\frac{\alpha}{\tilde\lambda+2}p^0-\frac{\tilde\lambda}{\tilde\lambda+2}\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{u})=\frac{2}{\tilde\lambda+2}\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{u})+\frac{\alpha}{\tilde\lambda+2}p^0.\end{aligned} \tag{5.5}$$

Further, using (5.3) and Lemma 17 we compute

$$\mathbf{Q}^T\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\mathbf{Q}=\begin{bmatrix}\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})\mathbf{A}^c & 0\\ 0 & \gamma^0_{33}\end{bmatrix}=\begin{bmatrix}\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})\mathbf{A}^c & 0\\ 0 & \frac{\alpha}{\tilde\lambda+2}p^0-\frac{\tilde\lambda}{\tilde\lambda+2}\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{u})\end{bmatrix}. \tag{5.6}$$

Now the main elastic term in the first equation in (5.2) is computed:

$$\begin{aligned}&\int_\Omega \mathcal{C}\left(\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\right):\left(\mathbf{Q}(\boldsymbol{\gamma}_y(\mathbf{v})-\sum_{i=1}^{3} v_i\boldsymbol{\Gamma}^i(0))\mathbf{Q}^T\right)\sqrt{a}dz\\
&=\int_\Omega \tilde\lambda\operatorname{tr}\left(\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\right)\operatorname{tr}\left(\mathbf{Q}(\boldsymbol{\gamma}_y(\mathbf{v})-\sum_{i=1}^{3} v_i\boldsymbol{\Gamma}^i(0))\mathbf{Q}^T\right)+2\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T:\mathbf{Q}(\boldsymbol{\gamma}_y(\mathbf{v})-\sum_{i=1}^{3} v_i\boldsymbol{\Gamma}^i(0))\mathbf{Q}^T\sqrt{a}dz\\
&=\int_\Omega \tilde\lambda\left(\frac{2}{\tilde\lambda+2}\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{u})+\frac{\alpha}{\tilde\lambda+2}p^0\right)\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{v})\sqrt{a}dz\\
&\quad+\int_\Omega 2\mathbf{Q}^T\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\mathbf{Q}:(\boldsymbol{\gamma}_y(\mathbf{v})-\sum_{i=1}^{3} v_i\boldsymbol{\Gamma}^i(0))\sqrt{a}dz\\
&=\int_\Omega \frac{2\tilde\lambda}{\tilde\lambda+2}(\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{u}))(\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{v}))+\frac{\tilde\lambda\alpha}{\tilde\lambda+2}p^0\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{v})\sqrt{a}dz+\int_\Omega 2\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{v})\sqrt{a}dz.\end{aligned}$$

Let us define the tensor (it usually appears in plate and shell theories!)

$$\tilde{\mathcal{C}}\mathbf{E}=\frac{2\tilde\lambda}{\tilde\lambda+2}\operatorname{tr}(\mathbf{E})\mathbf{I}+2\mathbf{E},\qquad \mathbf{E}\in\mathbb{R}^{2\times 2}_{sym}.$$

The first equation in (5.2) now becomes: for all $\mathbf{v}\in H^1_0(\omega;\mathbb{R}^3)$ one has

$$\begin{aligned}&\int_\omega \tilde{\mathcal{C}}(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})):\boldsymbol{\gamma}(\mathbf{v})\mathbf{A}^c\sqrt{a}dz_1dz_2+\int_\Omega \frac{\tilde\lambda\alpha}{\tilde\lambda+2}p^0\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{v})\sqrt{a}dz-\alpha\int_\Omega p^0\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{v})\sqrt{a}dz\\
&\quad=\int_\omega(\mathcal{P}_++\mathcal{P}_-)\cdot\mathbf{v}\sqrt{a}dz_1dz_2.\end{aligned}$$

By density of $H^1_0(\omega;\mathbb{R}^3)$ in $\mathcal{V}_M(\omega)$ the above equation implies:

$$\begin{aligned}&\int_\omega \tilde{\mathcal{C}}(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})):\boldsymbol{\gamma}(\mathbf{v})\mathbf{A}^c\sqrt{a}dz_1dz_2-\frac{2\alpha}{\tilde\lambda+2}\int_\omega\int_{-1/2}^{1/2}p^0dz_3\mathbf{A}^c:\boldsymbol{\gamma}(\mathbf{v})\sqrt{a}dz_1dz_2\\
&\quad=\int_\omega(\mathcal{P}_++\mathcal{P}_-)\cdot\mathbf{v}\sqrt{a}dz_1dz_2.,\qquad \mathbf{v}\in\mathcal{V}_M(\omega).\end{aligned} \tag{5.7}$$

Equation (5.7) is the classical equation of the membrane shell model with the addition of the pressure p^0 term.

Using (5.5), the second equation in (5.2) can be now written by

$$
\begin{aligned}
&\int_\Omega \left(\beta + \frac{\alpha^2}{\tilde{\lambda}+2}\right) \frac{\partial p^0}{\partial t} q\sqrt{a}dz + \int_\Omega \alpha \frac{\partial}{\partial t}\left(\frac{2}{\tilde{\lambda}+2}\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u})\right) q\sqrt{a}dz + \int_\Omega \frac{\partial p^0}{\partial z_3}\frac{\partial q}{\partial z_3}\sqrt{a}dz \\
&\quad = \mp \int_{\Sigma_\pm} Vq\sqrt{a}ds, \qquad q \in H^1(\Omega).
\end{aligned} \tag{5.8}
$$

The elliptic membrane poroelastic shell model is given by (5.7), (5.8).

Step 3 (The strong convergences of strain and pressure).

As mentioned in Remark 6 the limit problem can be decoupled and the displacement can be calculated independently of the pressure, by solving the elliptic membrane shell model with modified coefficients. However the same decoupling cannot be done for the three-dimensional problem and thus the result of the strong convergence for the classical elliptic membrane shell derivation, see [5], cannot be applied directly to the poroelastic case. Hence we adapt the ideas from [5]. Let

$$
\begin{aligned}
\Lambda(\varepsilon)(t) = \frac{1}{2}\int_\Omega &\mathcal{C}\left(\mathbf{Q}(\varepsilon)\left(\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))(t) - \boldsymbol{\gamma}^0(t)\right)\mathbf{Q}(\varepsilon)^T\right) \\
&: \left(\mathbf{Q}(\varepsilon)\left(\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))(t) - \boldsymbol{\gamma}^0(t)\right)\mathbf{Q}(\varepsilon)^T\right)\sqrt{g(\varepsilon)} + \frac{1}{2}\beta\int_\Omega (p(\varepsilon)(t) - p^0(t))^2\sqrt{g(\varepsilon)}dz \\
&+ \varepsilon^2\int_0^t\int_\Omega (\nabla^\varepsilon p(\varepsilon) - \nabla^\varepsilon p^0)\mathbf{Q}(\varepsilon)^T \cdot (\nabla^\varepsilon p(\varepsilon) - \nabla^\varepsilon p^0)\mathbf{Q}(\varepsilon)^T\sqrt{g(\varepsilon)}dz.
\end{aligned}
$$

We will show that $\Lambda(\varepsilon)(t) \to \Lambda(t)$ as ε tends to zero for all $t \in [0, T]$. Since $\Lambda(\varepsilon) \geq 0$ then $\Lambda \geq 0$ as well. After some calculation we will show that actually $\Lambda = 0$. This will give the strong convergences in (4.4).

Since we have only weak convergences in (4.4) to find the limit of $\Lambda(\varepsilon)$ we first remove quadratic terms in it using (3.8). We insert $\mathbf{v} = \dfrac{\partial \mathbf{u}(\varepsilon)}{\partial t}$ and $q = p(\varepsilon)$ in (3.8) divided by ε and sum up the equations. We integrate this equation over time and use the initial conditions $p(\varepsilon)|_{t=0} = 0$ and $\mathbf{u}(\varepsilon)|_{t=0} = 0$. The left hand side of the obtained equation, which is the same as in [22, page 387], we insert in $\Lambda(\varepsilon)$ to obtain

$$
\begin{aligned}
\Lambda(\varepsilon)(t) = &\int_0^t\int_{\Sigma_\pm} \mathcal{P}_\pm \cdot \frac{\partial \mathbf{u}(\varepsilon)}{\partial t}\sqrt{g(\varepsilon)}dsd\tau \mp \int_0^t\int_{\Sigma_\pm} Vp(\varepsilon)\sqrt{g(\varepsilon)}dsd\tau \\
&- \int_\Omega \mathcal{C}\left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon))(t)\mathbf{Q}(\varepsilon)^T\right) : \left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^0(t)\mathbf{Q}(\varepsilon)^T\right)\sqrt{g(\varepsilon)}dz \\
&- \beta\int_\Omega p(\varepsilon)p^0\sqrt{g(\varepsilon)}dz \\
&- 2\varepsilon^2\int_0^t\int_\Omega \mathbf{Q}(\varepsilon)\nabla^\varepsilon p(\varepsilon)(t) \cdot \mathbf{Q}(\varepsilon)\nabla^\varepsilon p^0(t)\sqrt{g(\varepsilon)}dzd\tau \\
&+ \frac{1}{2}\int_\Omega \mathcal{C}\left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^0\mathbf{Q}(\varepsilon)^T\right) : \left(\mathbf{Q}(\varepsilon)\boldsymbol{\gamma}^0\mathbf{Q}(\varepsilon)^T\right)\sqrt{g(\varepsilon)}dz + \frac{1}{2}\beta\int_\Omega (p^0)^2\sqrt{g(\varepsilon)}dz \\
&+ \varepsilon^2\int_0^t\int_\Omega \mathbf{Q}(\varepsilon)\nabla^\varepsilon p^0 \cdot \mathbf{Q}(\varepsilon)\nabla^\varepsilon p^0\sqrt{g(\varepsilon)}dzd\tau.
\end{aligned}
$$

We take ε to zero and using weak convergences from Corollary 14 obtain that $\Lambda(\varepsilon)(t) \to \Lambda(t) \geq 0$, where

$$
\begin{aligned}
\Lambda(t) = &\int_0^t\int_\omega (\mathcal{P}_+ + \mathcal{P}_-) \cdot \frac{\partial \mathbf{u}}{\partial t}\sqrt{a}dsd\tau \mp \int_0^t\int_{\Sigma_\pm} Vp^0\sqrt{a}dsd\tau \\
&- \frac{1}{2}\int_\Omega \mathcal{C}\left(\mathbf{Q}\boldsymbol{\gamma}^0(t)\mathbf{Q}^T\right) : \left(\mathbf{Q}\boldsymbol{\gamma}^0(t)\mathbf{Q}^T\right)\sqrt{a}dz - \frac{1}{2}\beta\int_\Omega p^0(t)^2\sqrt{a}dz - \int_0^t\int_\Omega \left(\frac{\partial p^0}{\partial z_3}\right)^2\sqrt{a}dzd\tau.
\end{aligned} \tag{5.9}
$$

Next we use the limit equations (5.7) and (5.8) to simplify the expression for Λ. We insert $\frac{\partial \mathbf{u}}{\partial t}$ as a test function in (5.7), p^0 in (5.8) and sum up the equations. The anti-symmetric terms cancel out as before. Then we integrate the equation over time and use the initial conditions to obtain

$$\frac{1}{2}\int_\omega \tilde{\mathcal{C}}(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u}(t))) : (\boldsymbol{\gamma}(\mathbf{u}(t))\mathbf{A}^c)\sqrt{a}dz_1dz_2 + \frac{1}{2}\int_\Omega\left(\beta + \frac{\alpha^2}{\tilde{\lambda}+2}\right)(p^0(t))^2\sqrt{a}dz$$
$$+\int_0^t\int_\Omega\left(\frac{\partial p^0}{\partial z_3}\right)^2\sqrt{a}dzd\tau = \int_0^t\int_\omega(\mathcal{P}_+ + \mathcal{P}_-)\cdot\frac{\partial \mathbf{u}}{\partial t}\sqrt{a}dsd\tau \mp \int_0^t\int_{\Sigma_\pm} Vp^0\sqrt{a}dsd\tau.$$

Inserting the above equation into (5.9) yields

$$\begin{aligned}\Lambda(t) = {}& \frac{1}{2}\int_\omega \tilde{\mathcal{C}}(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u}(t))) : (\boldsymbol{\gamma}(\mathbf{u}(t))\mathbf{A}^c)\sqrt{a}dz_1dz_2 + \frac{1}{2}\int_\Omega \frac{\alpha^2}{\tilde{\lambda}+2}(p^0(t))^2\sqrt{a}dz \\ & - \frac{1}{2}\int_\Omega \mathcal{C}\left(\mathbf{Q}\boldsymbol{\gamma}^0(t)\mathbf{Q}^T\right) : \left(\mathbf{Q}\boldsymbol{\gamma}^0(t)\mathbf{Q}^T\right)\sqrt{a}dz.\end{aligned} \tag{5.10}$$

Next we compute the elastic energy using (5.1), (5.5), (5.6):

$$\begin{aligned}&\int_\Omega \mathcal{C}\left(\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\right) : \left(\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\right)\sqrt{a}dz \\
&= \int_\Omega \tilde{\lambda}(\operatorname{tr}\left(\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\right))^2 + 2\mathbf{Q}^T\mathbf{Q}\boldsymbol{\gamma}^0\mathbf{Q}^T\mathbf{Q} : \boldsymbol{\gamma}^0\sqrt{a}dz \\
&= \int_\Omega \tilde{\lambda}\left(\frac{2}{\tilde{\lambda}+2}\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}) + \frac{\alpha}{\tilde{\lambda}+2}p^0\right)^2 + 2\begin{bmatrix}\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})\mathbf{A}^c & 0 \\ 0 & \frac{\alpha}{\tilde{\lambda}+2}p^0 - \frac{\tilde{\lambda}}{\tilde{\lambda}+2}\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u})\end{bmatrix} : \boldsymbol{\gamma}^0\sqrt{a}dz \\
&= \int_\Omega \tilde{\lambda}\left(\frac{2}{\tilde{\lambda}+2}\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}) + \frac{\alpha}{\tilde{\lambda}+2}p^0\right)^2 \\
&\quad + 2\left(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}) + \left(\frac{\alpha}{\tilde{\lambda}+2}p^0 - \frac{\tilde{\lambda}}{\tilde{\lambda}+2}\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u})\right)^2\right)\sqrt{a}dz \\
&= \int_\Omega \Big(\frac{4\tilde{\lambda}}{(\tilde{\lambda}+2)^2}(\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}))^2 + \frac{4\tilde{\lambda}\alpha}{(\tilde{\lambda}+2)^2}\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u})p^0 + \frac{\tilde{\lambda}\alpha^2}{(\tilde{\lambda}+2)^2}(p^0)^2 + 2\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}) \\
&\quad + \frac{2\alpha^2}{(\tilde{\lambda}+2)^2}(p^0)^2 - \frac{4\tilde{\lambda}\alpha}{(\tilde{\lambda}+2)^2}\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u})p^0 + \frac{2\tilde{\lambda}^2}{(\tilde{\lambda}+2)^2}(\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}))^2\Big)\sqrt{a}dz \\
&= \int_\Omega \frac{2\tilde{\lambda}}{\tilde{\lambda}+2}(\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u}))^2 + \frac{\alpha^2}{\tilde{\lambda}+2}(p^0)^2 + 2\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})\mathbf{A}^c : \boldsymbol{\gamma}(\mathbf{u})\sqrt{a}dz \\
&= \int_\Omega \tilde{\mathcal{C}}(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})) : \boldsymbol{\gamma}(\mathbf{u})\mathbf{A}^c + \frac{\alpha^2}{\tilde{\lambda}+2}(p^0)^2\sqrt{a}dz.\end{aligned}$$

Inserting this into (5.10) we obtain that $\Lambda(t) = 0$ and thus $\Lambda(\varepsilon)(t) \to 0$ for every $t \in [0,T]$. Since $\Lambda(\varepsilon) : [0,T] \to \mathbb{R}$ is an equicontinuous family, strong convergences of the strain tensor and the pressure follow

$$\begin{aligned}&\boldsymbol{\gamma}^\varepsilon(\mathbf{u}(\varepsilon)) \to \boldsymbol{\gamma}^0 \qquad \text{strongly in } C([0,T];L^2(\Omega;\mathbb{R}^{3\times3})), \\ &p(\varepsilon) \to p^0 \qquad \text{strongly in } C([0,T];L^2(\Omega)), \\ &\frac{\partial p(\varepsilon)}{\partial z_3} \to \frac{\partial p^0}{\partial z_3} \qquad \text{strongly in } L^2(0,T;L^2(\Omega)).\end{aligned} \tag{5.11}$$

Step 4 (The strong convergences of the displacements). The setting is more complicated than in the proof of Theorem 4.4-1 from [5] because the problem is time dependent. Moreover, the setting is more complicated than in the flexural case from [22]. The first convergence in (5.11) implies

$$\gamma_{\alpha\beta}(\mathbf{u}(\varepsilon)) \to \gamma_{\alpha\beta}(\mathbf{u}) \text{ strongly in } C([0,T];L^2(\Omega)), \quad \alpha,\beta \in \{1,2\}. \tag{5.12}$$

Let us now denote by $\bar{\cdot}$ the operator of averaging over z_3, i.e.,

$$\overline{v}(\cdot) = \int_{-1/2}^{1/2} v(\cdot, z_3) dz_3.$$

Then from (5.12) we obtain

$$\gamma_{\alpha\beta}(\overline{\mathbf{u}(\varepsilon)}) \to \gamma_{\alpha\beta}(\overline{\mathbf{u}}) \text{ strongly in } C([0,T];L^2(\omega)), \quad \alpha,\beta \in \{1,2\}.$$

Note that $\overline{\mathbf{u}} = \mathbf{u}$ since $\mathbf{u}$ is independent of z_3. The inequality of Korn's type on an elliptic surface, see Lemma 4 ([5, Theorem 2.7-3]), for $\overline{\mathbf{u}(\varepsilon)} - \overline{\mathbf{u}}$ implies

$$\|\overline{u_1(\varepsilon)} - u_1\|^2_{H^1(\omega)} + \|\overline{u_2(\varepsilon)} - u_2\|^2_{H^1(\omega)} + \|\overline{u_3(\varepsilon)} - u_3\|^2_{L^2(\omega)} \leq C_M \|\boldsymbol{\gamma}(\overline{\mathbf{u}(\varepsilon)}) - \boldsymbol{\gamma}(\overline{\mathbf{u}})\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}.$$

Application of (5.12) yields

$$\begin{aligned} &\overline{u_\alpha(\varepsilon)} \to u_\alpha \text{ strongly in } C([0,T];H^1(\omega)), \qquad \alpha \in \{1,2\}, \\ &\overline{u_3(\varepsilon)} \to u_3 \text{ strongly in } C([0,T];L^2(\omega)). \end{aligned} \tag{5.13}$$

Next we prove that

$$u_3(\varepsilon) \to u_3 \text{ strongly in } C([0,T];L^2(\Omega)). \tag{5.14}$$

From the Poincare type estimate

$$|u_3(\varepsilon)(\cdot, z_3) - \overline{u_3(\varepsilon)}(\cdot)| \leq C\sqrt{\int_{-1/2}^{1/2} (\partial_3 u_3(\varepsilon)(\cdot, y_3))^2 dy_3}$$

we obtain

$$\|u_3(\varepsilon) - \overline{u_3(\varepsilon)}\|_{C([0,T];L^2(\Omega))} \leq C\|\partial_3 u_3(\varepsilon)\|_{C([0,T];L^2(\Omega))}.$$

From $\gamma^\varepsilon_{33}(\mathbf{u}(\varepsilon)) = \frac{1}{\varepsilon}\partial_3 u_3(\varepsilon) \to \gamma^0_{33}$ strongly in $C([0,T];L^2(\Omega))$ we conclude $\partial_3 u_3(\varepsilon) \to 0$ strongly in $C([0,T];L^2(\Omega))$. Together with the last convergence in (5.13) this implies (5.14).

In the remaining part of the proof we prove

$$u_\alpha(\varepsilon) \to u_\alpha \text{ strongly in } C([0,T];H^1(\Omega)), \qquad \alpha \in \{1,2\}. \tag{5.15}$$

It follows from the Korn inequality and the Lions lemma (see [14]). Let us denote $\mathbf{u}'(\varepsilon) = (u_1(\varepsilon), u_2(\varepsilon), 0)$, $\mathbf{u}' = (u_1, u_2, 0)$. Then, using the Korn inequality, the convergence

$$\mathbf{e}(\mathbf{u}'(\varepsilon)) \to \mathbf{e}(\mathbf{u}') \text{ strongly in } C([0,T];L^2(\Omega;\mathbb{R}^{3\times 3})) \tag{5.16}$$

is equivalent to (5.15) ($\mathbf{e}(\mathbf{u})$ denotes the symmetrized gradient). Convergence in (5.16) will be obtained component by component. Since for $\alpha,\beta \in \{1,2\}$

$$e_{\alpha\beta}(\mathbf{u}'(\varepsilon)) = \gamma^\varepsilon_{\alpha\beta}(\mathbf{u}(\varepsilon)) + \sum_{i=1}^{3} u_i(\varepsilon)\Gamma^i_{\alpha\beta}(\varepsilon),$$

using the first convergence in (5.11) for $\gamma^\varepsilon_{\alpha\beta}(\mathbf{u}(\varepsilon))$, Remark 16 for $u_1(\varepsilon)$ and $u_2(\varepsilon)$, (5.14) for $u_3(\varepsilon)$ and (4.2) we obtain

$$e_{\alpha\beta}(\mathbf{u}'(\varepsilon)) \to \gamma_{\alpha\beta}(\mathbf{u}) + \sum_{i=1}^{3} u_i\Gamma^i_{\alpha\beta}(0) = e_{\alpha\beta}(\mathbf{u}') \text{ strongly in } C([0,T];L^2(\Omega)), \qquad \alpha,\beta = 1,2. \tag{5.17}$$

Since $e_{33}(\mathbf{u}'(\varepsilon)) = e_{33}(\mathbf{u}') = 0$ the convergence of this component is trivial.

For the convergence of $e_{\alpha 3}(\mathbf{u}'(\varepsilon)) = \frac{1}{2}\partial_3 u_\alpha(\varepsilon)$ we apply the Lions lemma. Thus we first prove that

$$\partial_3 u_\alpha(\varepsilon), \partial_{13} u_\alpha(\varepsilon), \partial_{23} u_\alpha(\varepsilon), \partial_{33} u_\alpha(\varepsilon) \to 0 \text{ strongly in } C([0,T];H^{-1}(\Omega)), \quad \alpha = 1,2. \tag{5.18}$$

We start with the expression

$$\frac{1}{\varepsilon}\partial_3 u_\alpha(\varepsilon) = 2\gamma^\varepsilon_{\alpha 3}(\mathbf{u}(\varepsilon)) - \partial_\alpha u_3(\varepsilon) + 2\sum_{i=1}^{3} u_i(\varepsilon)\Gamma^i_{\alpha 3}(\varepsilon). \tag{5.19}$$

The first and the third term on the right hand side converge strongly in $C([0,T];L^2(\Omega))$, while the middle one converges only in $C([0,T];H^{-1}(\Omega))$ by (5.14). Thus we obtain that $\partial_3 u_\alpha(\varepsilon) \to 0$ in $C([0,T];H^{-1}(\Omega))$. Differentiating (5.19) with respect to z_3 yields

$$\frac{1}{\varepsilon}\partial_{33} u_\alpha(\varepsilon) = 2\partial_3\gamma^\varepsilon_{\alpha 3}(\mathbf{u}(\varepsilon)) - \partial_\alpha\partial_3 u_3(\varepsilon) + 2\sum_{i=1}^{3} \partial_3(u_i(\varepsilon)\Gamma^i_{\alpha 3}(\varepsilon)).$$

Since $\partial_3 u_3(\varepsilon) \to 0$ in $C([0,T];L^2(\Omega))$ the convergence of $\partial_{33} u_\alpha(\varepsilon)$ in $C([0,T];H^{-1}(\Omega))$ is obtained. Differentiating (5.17) with respect to z_3 for $\alpha = \beta$ yields

$$\partial_{\alpha 3} u_\alpha(\varepsilon) = \partial_3 e_{\alpha\alpha}(\mathbf{u}'(\varepsilon)) \to \partial_3 e_{\alpha\alpha}(\mathbf{u}') = 0 \text{ strongly in } C([0,T];H^{-1}(\Omega)), \qquad \alpha = 1,2.$$

Since

$$\partial_{13} u_2(\varepsilon) = \varepsilon\partial_1\gamma^\varepsilon_{23}(\mathbf{u}(\varepsilon)) - \varepsilon\partial_2\gamma^\varepsilon_{13}(\mathbf{u}(\varepsilon)) + \partial_3 e_{12}(\mathbf{u}'(\varepsilon)) + \varepsilon\left(\partial_1\sum_{\tau=1}^{2} u_\tau(\varepsilon)\Gamma^\tau_{23}(\varepsilon) - \partial_2\sum_{\tau=1}^{2} u_\tau(\varepsilon)\Gamma^\tau_{13}(\varepsilon)\right)$$

and since all terms on the right hand side strongly converge in $C([0,T];H^{-1}(\Omega))$ we obtain the strong convergence of $\partial_{13} u_2(\varepsilon)$ in the same space. It then implies that the term

$$\partial_{23} u_1(\varepsilon) = 2\partial_3\gamma^\varepsilon_{12}(\mathbf{u}(\varepsilon)) - \partial_{13} u_2(\varepsilon) + 2\partial_3\sum_{i=1}^{3} u_i(\varepsilon)\Gamma^i_{12}(\varepsilon)$$

converges strongly in $C([0,T];H^{-1}(\Omega))$. Thus we have proved (5.18). A consequence of Lions lemma is that the spaces

$$v \in L^2(\Omega) \leftrightarrow (v, \partial_1 v, \partial_2 v, \partial_3 v) \in H^{-1}(\Omega)^4$$

are isomorphic. Therefore the spaces

$$v \in C([0,T];L^2(\Omega)) \leftrightarrow (v, \partial_1 v, \partial_2 v, \partial_3 v) \in C([0,T];H^{-1}(\Omega))^4$$

are also isomorphic. Therefore (5.18) implies that $e_{\alpha 3}(\mathbf{u}'(\varepsilon)) = \frac{1}{2}\partial_3 u_\alpha(\varepsilon) \to 0$ strongly in $C([0,T];L^2(\Omega))$. Therefore we have proved (5.16) and then (5.15) follows by the Korn inequality. Thus we have proved the strong convergence of displacements

$$\mathbf{u}(\varepsilon) \to \mathbf{u} \qquad \text{strongly in } C([0,T];H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)).$$

A Spherical surface

Let $\omega = [0,d_1] \times [0,d_2]$, where $d_1 \in (0,2\pi], d_2 \in (0,\pi]$, with one of the strict inequalities $d_1 < 2\pi$ or $d_2 < \pi$ holding, and let (φ,θ) denotes the generic point in ω. Let $R > 0$. We define a spherical shell by the parametrization

$$\mathbf{X} : \omega \to \mathbb{R}^3, \qquad \mathbf{X}(\varphi,\theta) = (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)^T.$$

Then the extended covariant basis of the shell $S = \mathbf{X}(\omega)$ is given by

$$\begin{aligned}
\mathbf{a}_1(\varphi,\theta) &= \partial_\varphi\mathbf{X}(\varphi,\theta) = R(-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)^T,\\
\mathbf{a}_2(\varphi,\theta) &= \partial_\theta\mathbf{X}(\varphi,\theta) = R(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)^T,\\
\mathbf{a}_3(\varphi,\theta) &= \frac{\mathbf{a}_1(\varphi,\theta) \times \mathbf{a}_2(\varphi,\theta)}{|\mathbf{a}_1(\varphi,\theta) \times \mathbf{a}_2(\varphi,\theta)|} = (-\sin\theta\cos\varphi, -\sin\theta\sin\varphi, -\cos\theta)^T.
\end{aligned}$$

The contravariant basis is biorthogonal and is given by

$$\begin{aligned}
\mathbf{a}^1(\varphi,\theta) &= \frac{1}{R}(-\sin\varphi/\sin\theta, \cos\varphi/\sin\theta, 0)^T,\\
\mathbf{a}^2(\varphi,\theta) &= \frac{1}{R}(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)^T,\\
\mathbf{a}^3(\varphi,\theta) &= (-\sin\theta\cos\varphi, -\sin\theta\sin\varphi, -\cos\theta)^T.
\end{aligned}$$

The covariant $\mathbf{A}_c = (a_{\alpha\beta})$ and contravariant $\mathbf{A}^c = (a^{\alpha\beta})$ metric tensors are respectively given by

$$\mathbf{A}_c = \begin{bmatrix} R^2\sin^2\theta & 0 \\ 0 & R^2 \end{bmatrix}, \qquad \mathbf{A}^c = \begin{bmatrix} \frac{1}{R^2}\frac{1}{\sin^2\theta} & 0 \\ 0 & \frac{1}{R^2} \end{bmatrix}$$

and the area element is now $\sqrt{a}dS = \sqrt{\det \mathbf{A}_c}dS = R^2\sin\theta dS$. The covariant and mixed components of the curvature tensor are now given by

$$\begin{aligned}
b_{11} &= \mathbf{a}^3\cdot\partial_\varphi\mathbf{a}_1 = R\sin^2\theta, \qquad b_{12} = \mathbf{a}^3\cdot\partial_\theta\mathbf{a}_1 = 0,\\
b_{21} &= \mathbf{a}^3\cdot\partial_\varphi\mathbf{a}_2 = 0, \qquad b_{22} = \mathbf{a}^3\cdot\partial_\theta\mathbf{a}_2 = R,\\
b_1^1 &= \sum_{\sigma=1}^{2} a^{1\sigma}b_{\sigma 1} = a^{11}b_{11} = \frac{1}{R}, \qquad b_2^1 = \sum_{\sigma=1}^{2} a^{1\sigma}b_{\sigma 2} = 0,\\
b_1^2 &= \sum_{\sigma=1}^{2} a^{2\sigma}b_{\sigma 1} = 0, \qquad b_2^2 = \sum_{\sigma=1}^{2} a^{2\sigma}b_{\sigma 2} = a^{22}b_{22} = \frac{1}{R^2}R = \frac{1}{R}.
\end{aligned}$$

For Christoffel symbols $\Gamma^\sigma_{\alpha\beta} = \mathbf{a}^\sigma\cdot\partial_\beta\mathbf{a}_\alpha$ one has

$$\boldsymbol{\Gamma}^1 = (\Gamma^1_{\alpha\beta}) = \begin{bmatrix} 0 & \operatorname{ctg}\theta \\ \operatorname{ctg}\theta & 0 \end{bmatrix}, \qquad \boldsymbol{\Gamma}^2 = (\Gamma^2_{\alpha\beta}) = \begin{bmatrix} -\sin\theta\cos\theta & 0 \\ 0 & 0 \end{bmatrix}.$$

Now the displacement vector $\tilde{\mathbf{v}}$ in the canonical coordinates is rewritten in the local basis $\tilde{\mathbf{v}} = \mathbf{Q}\mathbf{v} = v_1\mathbf{a}^1 + v_2\mathbf{a}^2 + v_3\mathbf{a}^3$. Note that contravariant basis is different than the usual basis associated with the spherical coordinates. One has

$$v_1 = R\sin\theta v_\varphi, \quad v_2 = Rv_\theta, \quad v_3 = -v_r.$$

Similarly, $\tilde{\mathcal{P}}_\pm = \mathbf{Q}^{-T}\mathcal{P}_\pm = (\mathcal{P}_\pm)_1\mathbf{a}_1 + (\mathcal{P}_\pm)_2\mathbf{a}_2 + (\mathcal{P}_\pm)_3\mathbf{a}_3$. Thus

$$(\mathcal{P}_\pm)_1 = \frac{1}{R\sin\theta}(\mathcal{P}_\pm)_\varphi, \quad (\mathcal{P}_\pm)_2 = \frac{1}{R}(\mathcal{P}_\pm)_\theta, \quad (\mathcal{P}_\pm)_3 = -(\mathcal{P}_\pm)_r$$

and

$$\mathcal{P}_\pm\cdot\mathbf{v} = (\mathcal{P}_\pm)_1 v_1 + (\mathcal{P}_\pm)_2 v_2 + (\mathcal{P}_\pm)_3 v_3 = (\mathcal{P}_\pm)_\varphi v_\varphi + (\mathcal{P}_\pm)_\theta v_\theta + (\mathcal{P}_\pm)_r v_r.$$

Inserting the geometry coefficients into the strain $\boldsymbol{\gamma}$ we obtain

$$\begin{aligned}
\boldsymbol{\gamma}(\mathbf{v}) &= \begin{bmatrix} \partial_1 v_1 - \sum_{\kappa=1}^2 \Gamma^\kappa_{11}v_\kappa - b_{11}v_3 & \frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) - \sum_{\kappa=1}^2 \Gamma^\kappa_{12}v_\kappa - b_{12}v_3 \\ \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2) - \sum_{\kappa=1}^2 \Gamma^\kappa_{21}v_\kappa - b_{21}v_3 & \partial_2 v_2 - \sum_{\kappa=1}^2 \Gamma^\kappa_{22}v_\kappa - b_{22}v_3 \end{bmatrix}\\
&= R\begin{bmatrix} \sin\theta\partial_\varphi v_\varphi + \sin\theta\cos\theta v_\theta + \sin^2\theta v_r & \frac{1}{2}(\partial_\varphi v_\theta + \sin\theta\partial_\theta v_\varphi) - \cos\theta v_\varphi \\ \frac{1}{2}(\sin\theta\partial_\theta v_\varphi + \partial_\varphi v_\theta) - \cos\theta v_\varphi & \partial_\theta v_\theta + v_r \end{bmatrix}.
\end{aligned}$$

Next

$$\begin{aligned}
\tilde{\mathcal{C}}(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u})) : \boldsymbol{\gamma}(\mathbf{v})\mathbf{A}^c &= \frac{2\tilde{\lambda}}{\tilde{\lambda}+2}\operatorname{tr}(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u}))\operatorname{tr}(\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{v})) + 2\mathbf{A}^c\boldsymbol{\gamma}(\mathbf{u}) : \boldsymbol{\gamma}(\mathbf{v})\mathbf{A}^c\\
&= \frac{2\tilde{\lambda}}{\tilde{\lambda}+2}\frac{1}{R^2}\left(\frac{1}{\sin^2\theta}(\sin\theta\partial_\varphi u_\varphi + \sin\theta\cos\theta u_\theta + \sin^2\theta u_r) + (\partial_\theta u_\theta + u_r)\right)
\end{aligned}$$

$$\left(\frac{1}{\sin^2\theta}(\sin\theta\partial_\varphi v_\varphi+\sin\theta\cos\theta v_\theta+\sin^2\theta v_r)+(\partial_\theta v_\theta+v_r)\right)$$
$$+2\frac{1}{R^2}\left(\frac{1}{\sin^4\theta}(\sin\theta\partial_\varphi u_\varphi+\sin\theta\cos\theta u_\theta+\sin^2\theta u_r)(\sin\theta\partial_\varphi v_\varphi+\sin\theta\cos\theta v_\theta+\sin^2\theta v_r)\right.$$
$$+\frac{2}{\sin^2\theta}(\frac{1}{2}(\partial_\varphi u_\theta+\sin\theta\partial_\theta u_\varphi)-\cos\theta u_\varphi)(\frac{1}{2}(\partial_\varphi v_\theta+\sin\theta\partial_\theta v_\varphi)-\cos\theta v_\varphi)$$
$$\left.+(\partial_\theta u_\theta+u_r)(\partial_\theta v_\theta+v_r)\right)$$
$$=\frac{2\tilde\lambda}{\tilde\lambda+2}\frac{1}{R^2}\left(\frac{1}{\sin\theta}(\partial_\varphi u_\varphi+\cos\theta u_\theta+\sin\theta u_r)+\partial_\theta u_\theta+u_r\right)$$
$$\left(\frac{1}{\sin\theta}(\partial_\varphi v_\varphi+\cos\theta v_\theta+\sin\theta v_r)+\partial_\theta v_\theta+v_r\right)$$
$$+2\frac{1}{R^2}\left(\frac{1}{\sin^2\theta}(\partial_\varphi u_\varphi+\cos\theta u_\theta+\sin\theta u_r)(\partial_\varphi v_\varphi+\cos\theta v_\theta+\sin\theta v_r)\right.$$
$$\left.+\frac{1}{2\sin^2\theta}(\partial_\varphi u_\theta+\sin\theta\partial_\theta u_\varphi-2\cos\theta u_\varphi)(\partial_\varphi v_\theta+\sin\theta\partial_\theta v_\varphi-2\cos\theta v_\varphi)+(\partial_\theta u_\theta+u_r)(\partial_\theta v_\theta+v_r)\right).$$

The insertion of the above expression for $\tilde{\mathcal{C}}$ into (3.15) gives the equations of the spherical membrane shell. They read: find $\{(u_\varphi,u_\theta,u_r)\}\in C([0,T];\mathcal{V}_M(\omega))$, satisfying the system

$$\int_0^{d_1}\int_0^{d_2}\left(\left(\frac{2\tilde\lambda}{\tilde\lambda+2}+\frac{4\alpha^2}{(\tilde\lambda+2)\big(\beta(\tilde\lambda+2)+\alpha^2\big)}\right)\frac{1}{R^2}\left(\frac{1}{\sin\theta}(\partial_\varphi u_\varphi+\cos\theta u_\theta+\sin\theta u_r)+\partial_\theta u_\theta+u_r\right)\right.$$
$$\left(\frac{1}{\sin\theta}(\partial_\varphi v_\varphi+\cos\theta v_\theta+\sin\theta v_r)+\partial_\theta v_\theta+v_r\right)$$
$$+2\frac{1}{R^2}\left(\frac{1}{\sin^2\theta}(\partial_\varphi u_\varphi+\cos\theta u_\theta+\sin\theta u_r)(\partial_\varphi v_\varphi+\cos\theta v_\theta+\sin\theta v_r)\right.$$
$$+\frac{1}{2\sin^2\theta}(\partial_\varphi u_\theta+\sin\theta\partial_\theta u_\varphi-2\cos\theta u_\varphi)(\partial_\varphi v_\theta+\sin\theta\partial_\theta v_\varphi-2\cos\theta v_\varphi)$$
$$\left.\left.+(\partial_\theta u_\theta+u_r)(\partial_\theta v_\theta+v_r)\right)\right)R^2\sin\theta d\varphi d\theta$$
$$=\int_0^{d_1}\int_0^{d_2}\left(((\mathcal{P}_+)_\varphi+(\mathcal{P}_-)_\varphi)v_\varphi+((\mathcal{P}_+)_\theta+(\mathcal{P}_-)_\theta)v_\theta+((\mathcal{P}_+)_r+(\mathcal{P}_-)_r)v_r\right)R^2\sin\theta d\varphi d\theta,$$
$$(v_\varphi,v_\theta,v_r)\in\mathcal{V}_M(\omega).$$

Then $\int_{-1/2}^{1/2}p^0dr$ is calculated from (3.14) and the fluctuation of the pressure across the thickness can be calculated from (3.17).

Remark 18. In the case of the whole sphere we are not in the elliptic membrane case since the boundary conditions are different. However if we assume that the body is loaded radially (i.e. $\mathcal{P}_\pm, V$ are functions of time only) we can search for the radial solution of the three-dimensional problem (3.8) (i.e. $u_\varphi=u_\theta=0$ and u_r is independent of φ and θ). The asymptotic analysis is based on the a priori estimates used in Theorem 10. In the case of the whole sphere and radial solutions, a simple calculation gives

$$\|\boldsymbol{\gamma}^\varepsilon(\mathbf{v})\|^2_{L^2(\Omega;\mathbb{R}^{3\times3})}=\int_0^{2\pi}\int_0^{\pi}\int_{-1/2}^{1/2}\left((R\sin^2\theta-\varepsilon z_3)^4v_3^2+(R-\varepsilon z_3)^2v_3^2+\frac{1}{\varepsilon^2}\partial_3v_3(z_3)^2\right)dz_3d\theta d\varphi$$
$$\geq\int_0^{2\pi}\int_0^{\pi}\int_{-1/2}^{1/2}\left(R^2/4v_3^2+\frac{1}{\varepsilon^2}\partial_3v_3(z_3)^2\right)dz_3d\theta d\varphi\geq C^2\|v_3\|^2_{H^1(\Omega)},$$

for $R\geq\varepsilon$ and $1\gg\varepsilon$. Then the subsequent analysis follows as above and the limit model in this case can be obtained by specialization of the above equations for spherical geometry.

Thus for loading depending only on time for the solution of the shell problem we obtain the following relation for u_r and p^0:

$$\frac{3\tilde{\lambda}+2}{\tilde{\lambda}+2}4u_r - 2R\frac{2\alpha}{\tilde{\lambda}+2}\int_{-1/2}^{1/2} p^0 dr = ((\mathcal{P}_+)_r + (\mathcal{P}_-)_r)R^2. \tag{A.20}$$

$$\begin{aligned}
&\frac{d}{dt}\int_{-1/2}^{1/2}\left(\beta + \frac{\alpha^2}{\tilde{\lambda}+2}\right)p^0 q R^2 dr + 2R\frac{2\alpha}{\tilde{\lambda}+2}\int_{-1/2}^{1/2}\partial_t u_r q dr + R^2\int_{-1/2}^{1/2}\frac{\partial p^0}{\partial r}\frac{\partial q}{\partial r}dr = \\
&V(q(-1/2)-q(1/2))R^2 \qquad \text{in} \quad \mathcal{D}'(0,T), \qquad q\in H^1(-1/2,1/2), \\
&p^0 = 0 \qquad \text{at} \quad t=0.
\end{aligned} \tag{A.21}$$

From (A.21) for constant test functions we obtain

$$\left(\beta + \frac{\alpha^2}{\tilde{\lambda}+2}\right)\partial_t \int_{-1/2}^{1/2} p^0 R^2 dr + 2R\frac{2\alpha}{\tilde{\lambda}+2}\partial_t u_r = 0.$$

Inserting u_r from(A.20), after some calculations, we obtain

$$\partial_t \int_{-1/2}^{1/2} p^0 = -\frac{R\alpha}{\beta(3\tilde{\lambda}+2) + \alpha^2\frac{3\tilde{\lambda}+6}{\tilde{\lambda}+2}}\partial_t((\mathcal{P}_+)_r + (\mathcal{P}_-)_r). \tag{A.22}$$

After partial integration in (A.21), we obtain

$$\begin{aligned}
&\left(\beta + \frac{\alpha^2}{\tilde{\lambda}+2}\right)\partial_t p^0 R^2 + 2R\frac{2\alpha}{\tilde{\lambda}+2}\partial_t u_r - R^2\partial_{rr}p^0 = 0, \\
&\partial_r p^0|_{r=\pm 1/2} = V, \\
&p^0 = 0 \qquad \text{at} \quad t=0.
\end{aligned}$$

Here we calculate u_r from (A.20) with $\partial_t \int_{-1/2}^{1/2} p^0 dr$ replaced using (A.22) and obtain the boundary value problem for p^0

$$\begin{aligned}
&\left(\beta + \frac{\alpha^2}{\tilde{\lambda}+2}\right)\partial_t p^0 + R\frac{\alpha(\beta(\tilde{\lambda}+2)+\alpha^2)}{\beta(3\tilde{\lambda}+2)(\tilde{\lambda}+2)+\alpha^2(3\tilde{\lambda}+6)}\partial_t((\mathcal{P}_+)_r + (\mathcal{P}_-)_r) - \partial_{rr}p^0 = 0, \\
&\partial_r p^0|_{r=\pm 1/2} = V, \\
&p^0 = 0 \qquad \text{at} \quad t=0.
\end{aligned}$$

This equation has the same structure as the equation in [31] for spherical poroelastic membrane, however the coefficients are obviously not the same since in [31] there are no inverse Biot's coefficient β and the effective stress coefficient α. In the constitutive law (2.6) Taber takes $\alpha = 1$ and in (2.8) β seems to be 1 as well. Hence, our consideration establish rigorously the results from [31].

B Acknowledgements

The research of A.M. was supported in part by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

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