A global existence result for the equations describing unsaturated flow in porous media with dynamic capillary pressure

Andro Mikelić

Université de Lyon, Lyon, F-69003, FRANCE;
Université Lyon 1, Institut Camille Jordan, UMR 5208,
Bât. Braconnier, 43, Bd du onze novembre 1918
69622 Villeurbanne Cedex, FRANCE
Email: Andro.Mikelic@univ-lyon1.fr

Abstract

In this paper we investigate the pseudoparabolic equation

$$\partial_t c = \text{ div } \left\{ \frac{k\,(c)}{\mu} \Big(- P_C'\,(c)\,\nabla c + \tau \nabla \partial_t c + g\rho \mathbf{e}_n \Big) \right\},$$

where τ is a positive constant, c is the moisture content, k is the hydraulic conductivity and P_C is the static capillary pressure. This equation describes unsaturated flows in porous media with dynamic capillary pressure - saturation relationship. In general, such models arise in a number of cases when non-equilibrium thermodynamics or extended non-equilibrium thermodynamics are used to compute the flux.

For this equation existence of the travelling wave type solutions was extensively studied. Nevertheless, the existence seems to be known only for the non-degenerate case, when k is strictly positive. We use the approach from statistical hydrodynamics and construct the corresponding entropy functional for the regularized problem. Such approach permits to get existence, for any time interval, of an appropriate weak solution with square integrable first derivatives in x and in t and square integrable time derivative of the gradient. Negative part of such weak solution is small in L^2-norm with respect to x, uniformly in time, as the square root of the relative permeability at the value of the regularization parameter. Next we control the regularized entropy. A fine balance between the regularized entropy and the degeneracy of the capillary pressure permits to get an L^q uniform bound for the time derivative of the gradient. These estimates permit passing to the limit when the regularization parameter tends to zero and obtaining the existence of global nonnegative weak solution.

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1. Introduction

Equations that describe unsaturated flow in porous media, are a special case of the equations describing an immiscible two phase flow, with the non-wetting fluid assumed to be stagnant. The model is based on mass and momentum equations which are coupled to constitutive equations. Following Bear (see [3], pages 487-488), the motion is described by Darcy's law in the following form

$$\mathbf{q} = -\frac{k(c)\rho g}{\mu}\nabla(\frac{p}{\rho g} + x_n), \tag{1}$$

where $\mathbf{q}$ is the volumetric flux, p is the pressure in the wetting phase and the permeability k is a function of the moisture content $c = \varphi_p S$, where φ_p is the porosity and S is the liquid phase saturation. ρ is the wetting phase density and g is the gravity. Assuming no sources or sinks of moisture within the unsaturated flow domain, the incompressible wetting phase and a nondeformable porous medium, the mass conservation equation reduces to

$$\frac{\partial c}{\partial t} + \text{div } \mathbf{q} = 0. \tag{2}$$

To close the system $(1) - (2)$ we need the constitutive equation relating the capillary pressure P_C to the wetting phase saturation S. This important constitutive equations is classically given as an algebraic relationship between P_C and S. Detailed discussion of the relationship could be found e.g. in [3], pages 475-487. Recently, the relationship between P_C and S has been generalized on the basis of thermodynamical arguments by Gray and Hassanizadeh (see [14],[15],[16],[17], [5] and references therein). They derived the following extended relationship:

$$p = -P_C(S) + f(S, \partial_t S), \tag{3}$$

where f is an unspecified function and P_C is decreasing in S. Such relationship includes dynamic effects and reduces to the classical relationship in the equilibrium situation. This kind of relationship could be also found in the classical book by Barenblatt et al. [1]. Furthermore, in the paper [6] dynamic capillary pressure effects occur as upscaling of two-phase flows. The same type of equations can occur in models that use Classical Irreversible Thermodynamics or Extended Irreversible Thermodynamics.

The most simple way of accounting for dynamic memory effects is the following modification of the capillary relation:

$$p = -P_C(c) + \tau \partial_t c \tag{4}$$

where $\tau > 0$ denotes the dynamic capillary coefficient. This coefficient may depend on moisture content as well as on properties of the porous medium. Experimental studies of the dynamic effect are reported in [27], [28], [29] and [30], where also the values of τ were estimated. We suppose it constant.

After inserting (4) into the equations (1)-(2), we get the following nonlinear degenerate pseudoparabolic equation:

$$\partial_t c = \operatorname{div} \left\{ k\left(c\right) \left(-P'_C\left(c\right) \nabla c + \tau \nabla \partial_t c + \mathbf{e}_n \right) \right\}, \tag{5}$$

where all constants except τ are set to be equal to 1.

Mathematical study of pseudoparabolic PDEs goes back to works of Showalter in seventies (see [26] and subsequent works). Nonlinear diffusion equations with a pseudoparabolic regularizing term being the Laplacean of the time derivative are considered in [23] and in [24]. Global existence of a strong solution is proved by writing the problem as a linear elliptic operator, acting on the time derivative, equal to the nonlinear diffusion term. In such situation, the linear elliptic operator, acting on the time derivative, could be inverted and then the standard geometric theory of nonlinear parabolic equations (see e.g. [18]) is applicable.

In our situation the dynamic capillary effects in unsaturated flows are described by a degenerate non-linear second order elliptic operator, acting on the time derivative, at the place of the Laplacean. The invertibility of this nonlinear elliptic operator depends on the solution itself. Importance of the model for multiphase and unsaturated flows through porous media motivated a number of recent papers. Mostly they deal with the travelling wave solutions. In this direction we mention the paper [19], where Hulshof gives a detailed study of possible travelling wave solutions and in particular of the behavior of such travelling waves near fronts where the concentration is zero. Further studies of the travelling waves solutions to the equation (5) are in the papers [8] and [7]. For small- and waiting time behavior of the equations one can consult [20].

Study of the capillarity limit for the linear relaxation model of the dynamic term is in [11]. It is important to mention that it leads to the Buckley-Leverett equation with discontinuous solutions which do not satisfy Oleinik's entropy condition.

Nevertheless, the study of existence of a solution to the equation (5) was not undertaken. For the nonlinear model from [16] there are only papers [4] and [5], where the non-degeneracy was supposed and existence is local in time. Another existence result, also local in time and for a related equation, is in the paper [10], by Düll, where a related pseudoparabolic equation modeling solvent uptake in polymeric solids was studied. Düll proved the short time existence of a solution for the problem in $\mathbb{R}$, supposing non-negative compactly supported initial datum. In the above quoted works by Beliaev and Düll, the problem was written as a system containing a linear elliptic equation and an evolution equation. Nevertheless, it is not easy to see how to get estimates global in time with such approach.

Finally, let us mention existence and uniqueness results in [25], for quasi-linear pseudoparabolic equations with degeneration in the time derivative term and including memory terms.

We consider the equation (5) in a an open, bounded and connected domain $\Omega \subset \mathbb{R}^n$, with Lipschitz boundary $\partial\Omega$. We decompose the boundary $\partial\Omega$ into Dirichlet part $\partial_D\Omega$ and Neumann part $\partial_N\Omega$, with $\partial\Omega = \overline{\partial_D\Omega} \cup \overline{\partial_N\Omega}$ and $\partial_D\Omega \cap \partial_N\Omega = \emptyset$. We suppose that $\partial_D\Omega$ is measurable with $\mathcal{H}^{n-1}(\partial_D\Omega) > 0$. Let $Q_T = \Omega \times (0,T)$, $T > 0$.

Following the classical textbook [3], we are looking for a solution to the equation (5) satisfying the following initial-boundary conditions:

$$c = c_D \quad \text{on} \quad \Gamma_D = \partial_D\Omega \times (0,T), \tag{6}$$

$$-k(c)\big(-P_C'(c)\nabla c + \tau \nabla \partial_t c + \mathbf{e}_n \big) \cdot \nu = R \ \text{ on } \ \Gamma_N = \partial_N\Omega \times (0,T), \tag{7}$$

$$c = c_i(x) \quad \text{on } \Omega, \tag{8}$$

where $(0,T)$, $T > 0$, is the time interval, ν is the outer normal of Ω, R is the given flux, c_D is the given moisture content at Γ_D and c_i is the initial moisture content.

Our goal is to obtain a global existence of a weak solution, for any time interval, in analogy of the study in [22] of a similar 1D model, describing sequestration of the carbon dioxide in unminable coal seams. Result of this type were obtained in [2] for a degenerate pseudo-parabolic regularization of a nonlinear forward-backward heat equation. As in [22], we observe that our PDE allows a natural generalization of the classic Kullback entropy which is given by $\mathcal{E}''(\varphi) = 1/k(\varphi)$. Following [21], we will use $\mathcal{E}'(\varphi)$ as a test function, with the hope to obtain a convenient a priori estimate. Formal calculation gives the equality

$$\partial_t \int_\Omega \big(\mathcal{E}(c) + \frac{\tau}{2}|\nabla c|^2\big) dx - \int_\Omega P_C'(c)|\nabla c|^2 \, dx = \text{ low order terms.} \tag{9}$$

This estimate has remarkable property that gradients are not multiplied by k. Nevertheless, it is typical for unsaturated flows that $k(0) = 0$ and that $-P_C'$ tends to infinity when c tends to zero. These degenerate coefficients and presence of the initial and the boundary conditions lead to unbounded non-integrable $\mathcal{E}'$. The equality (9) cannot be directly used and we can not follow the approach from [13] to get the entropy estimates. As in [22], we have to regularize and then to obtain the entropy estimate and an additional estimate for the time derivative for the regularized problem. Consequently, our calculations are more complicated than in the literature.

Study of the model requires to precise assumptions on the coefficients and on the data:

(H1) After [3], we suppose that there are constants $\beta > 0$, $C_k > 0$, and a non-negative function $f \in C_0^\infty(\mathbb{R})$ such that k is given by

$$k(z) = \frac{C_k \, z^\beta}{1 + C_k \, z^\beta \, f(z)}, \quad z \in [0, 1]. \tag{10}$$

Typical exemple is given by the Brooks–Corey relation

$$k(z) = C_k z^{2/\Lambda+3}, \tag{11}$$

where $\Lambda > 0$ is the Brooks and Corey exponent.

(H2) Concerning the capillary pressure, we deal only with its derivative. After [3], we suppose that there exist $\lambda > 0$, $C_p > 0$, $M_p > 0$, and an arbitrary non-negative function $g \in C_0^\infty(\mathbb{R})$ such that $-P_C'$ is written as

$$-P_C'(z) = \frac{C_p \, z^{-\lambda}}{1 + M_p \, z^\lambda \, g(z)} \quad z \in [0, 1]. \tag{12}$$

Typical exemple is given again by the Brooks–Corey relation

$$-P_C'(z) = C_p z^{-1/\Lambda-1}, \tag{13}$$

where $\Lambda > 0$ is, as above, the Brooks and Corey exponent.

Since we will construct a non-negative solution, there is no need to extend $-P_C'$ to $(-\infty, 0)$. Extension for $z > 1$ is obvious.

(H3) The product of the functions k and P_C' is bounded on $[0,1]$. Consequently, $\beta \geq \lambda$. We extend k and P_C' to $(1, +\infty)$ by their values at $z = 1$.

(H4) We assume that $c_D \in C^1([0,T]; H^1(\Omega))$, $0 < c_{Dmin} \leq c_D(x,t) \leq 1$ a.e. on Q_T. The initial condition c_i belongs to $H^1(\Omega)$ and $0 \leq c_i \leq 1$ a.e. on Ω. Keeping c_D strictly positive is essential for obtaining a priori estimates. Similarly, we will impose in (**H6**) on initial data to be non-zero almost everywhere.

(H5) We suppose that the flux on $\partial_N \Omega$ verifies

$$\begin{aligned} R(x,t,z) = R_0(x,t)\,\zeta(z);\ R_0 \in C^1(\bar{\Gamma}_N \times [0,T])),\ \text{with } R_0 \geq 0; \\ \zeta \in C_0^\infty(\mathbb{R}), \quad \zeta(z) \geq 0, \text{ for } \ z > 0,\ \zeta(0) = 0, \\ \text{and } \ z\zeta(z) \geq 0 \text{ for } \ z < 0. \end{aligned} \tag{14}$$

It is important to have R compatible with degeneration of the liquid phase.

We introduce now the *definition of a weak solution*. We have

Definition 1. *Let*

$$V = (z \in H^1(\Omega) \big| \ z|_{\partial_D \Omega} = 0). \tag{15}$$

Then the variational formulation corresponding to the problem (5) -(8) is

$$\begin{aligned} &\textit{Find} \quad c \in H^1(Q_T) \quad \textit{such that} \quad 0 \leq c(x,t) \quad \textit{a.e. on } Q_T; \\ &c - c_D \in L^2(0,T;V), \ \ k(c)\nabla\partial_t c \in L^2(Q_T), \qquad \textit{and satisfying} \\ &-\int_0^T \int_\Omega c\,\frac{\partial v}{\partial t}\,dx\,dt - \int_\Omega c_i(x)\,v(x,0)\,dx + \int_0^T \int_{\partial_N \Omega} R\,v\,d\Gamma\,dt \\ &+ \int_0^T \int_\Omega \tau\,k(c)\nabla(\partial_t c)\nabla v\,dx\,dt - \int_0^T \int_\Omega k(c) P_C'(c)\nabla c \nabla v\,dx\,dt \\ &\qquad + \int_0^T \int_\Omega k(c)\,\partial_{x_n} v\,dx\,dt = 0. \end{aligned} \tag{16}$$

for all $v \in H^1(0,T;V)$ *such that* $v|_{t=T} = 0$, In order to prove existence of at least one weak solution for problem (16), we need a regularized problem. It corresponds to the regularized coefficients k and P'_C given by

$$k_\varepsilon(z) = k(\varepsilon + z^+) \quad \text{and} \quad P'_{C\varepsilon}(z) = P'_C(\varepsilon + z^+), \quad z \in \mathbb{R}, \tag{17}$$

with $\varepsilon > 0$ and $z^+ = \sup\{z, 0\}$.

Now we introduce the *definition of a weak solution for the regularized problem* by

Definition 2. *The variational formulation corresponding to the regularized problem (5) -(8) is*

Find $c_\varepsilon \in H^1(Q_T)$ *such that* $c_\varepsilon - c_D \in L^2(0,T;V)$, $\nabla \partial_t c_\varepsilon \in L^2(Q_T)$ *and satisfying*

$$\begin{aligned} &-\int_0^T \int_\Omega c_\varepsilon \frac{\partial v}{\partial t}\, dx\, dt - \int_\Omega c_i(x)\, v(x,0)\, dx + \int_0^T \int_{\partial_N \Omega} R\, v\, d\Gamma\, dt \\ &+ \int_0^T \int_\Omega \tau\, k_\varepsilon(c_\varepsilon) \nabla(\partial_t c_\varepsilon) \nabla v\, dx\, dt - \int_0^T \int_\Omega k_\varepsilon(c_\varepsilon) P'_{C\varepsilon}(c_\varepsilon) \nabla c_\varepsilon \nabla v\, dx\, dt \\ &+ \int_0^T \int_\Omega k(c_\varepsilon)\, \partial_{x_n} v\, dx\, dt = 0. \end{aligned} \tag{18}$$

for all $v \in H^1(0,T;V)$ *such that* $v|_{t=T} = 0$,

The results we prove in the paper are the following:

Theorem 3. *Under the hypotheses* **(H1)- (H5)** *there is a weak solution* c_ε *for the regularized problem (5) -(8), satisfying (18).*

Theorem 4. *Let us suppose the hypothesis*

(H6) *The initial moisture content satisfies the finite entropy condition*

$$\beta > 2 \quad \text{and} \quad \int_\Omega c_i^{2-\beta}(x) dx < +\infty. \tag{19}$$

Under the hypotheses **(H1)- (H6)**, *there is a constant* $C > 0$, *independent of* ε, *such that every weak solution* c_ε *for the regularized problem (5) -(8),* *satisfies :*

$$\|c_\varepsilon^-\|_{L^\infty(0,T;L^2(\Omega))} \le C\varepsilon^{\beta/2}; \quad meas\ \{c_\varepsilon \le 0\} \le C\varepsilon^{\beta-2} \tag{20}$$

$$\left\|(|c_\varepsilon| + \varepsilon)^{2-\beta}\right\|_{L^\infty(0,T;L^1(\Omega))} \le C, \quad \beta > 2 \tag{21}$$

$$\|\nabla c_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le C \tag{22}$$

$$\|\partial_t c_\varepsilon\|_{L^2(Q_T)} \le C \tag{23}$$

$$\left\|\sqrt{k_\varepsilon(c_\varepsilon)}\nabla\partial_t c_\varepsilon\right\|_{L^2(Q_T)} \le C \tag{24}$$

$$\left\|\nabla \int_0^{c_\varepsilon} \sqrt{-P'_{C\varepsilon}(\xi)}d\xi\right\|_{L^2(Q_T)} \le C, \tag{25}$$

where $c_\varepsilon^- = \inf\{c_\varepsilon, 0\}$.

Theorem 5. *Let us suppose the hypotheses* **(H1)-(H6)**. *If, in addition, the exponents β and λ verify*

$$\beta \ge \lambda > 4, \ \ for\ n = 1, 2. \tag{26}$$

$$\beta \ge \lambda > 2, \quad \frac{10}{3} + \frac{\beta}{3} < \lambda, \ \ for\ n = 3, \tag{27}$$

then there is a weak solution for the problem (5) -(8), satisfying (16).

Remark 6. *We note that the condition (27) implies $\lambda > 5$ for $n = 3$.*

It would be interesting to have L^∞-bounds for weak solutions. Physical moisture content should be always smaller or equal to φ_p, consequently the dimensionless c should be element of $[0, 1]$.

Such estimates for pseudoparabolic equations exist in the literature. A classical reference is the paper [9] by DiBenedetto and Pierre, with general comparison and maximum principle for a large class of pseudoparabolic equations. Nevertheless, due to the position of the nonlinearity, our equation can not be written as a time derivative of of an elliptic operator applied to the solution equals an elliptic operator of the solution, and getting L^∞-bounds, using methods from [9], is not clear.

Value $c = 1$ corresponds to the completely saturated flows, where we have only the liquid phase. Modeling of the transition to the completely saturated regime is out of scope of this paper. Note that in our situation introducing Kirchhhoff's potential does not solve the difficulty. Our model is consistent with the single phase flow only for $g = 0$. In this case we can suppose $P_C(1) = 0$ without loosing generality and we have the following result

Theorem 7. *Let us suppose the hypotheses of Theorem 5. In addition let* $g = 0$, $-P_C(c_D) + \partial_t c_D \leq 0$ *and* $P_C(1) = 0$*. Then there is a weak solution for the problem (5) -(8), satisfying (16) and such that* $c(x,t) \leq 1$ *a.e. on* Q_T*.*

2. Proof of Theorem 3

$\boxed{\text{1 STEP}}$

In this section, we will establish the existence of a weak solution for the regularized problem (5) -(8), satisfying (18).

It is easy to see that

$$0 < m_k \leqslant k_\varepsilon(S) \leqslant \|k\|_\infty, \tag{28}$$

$$0 < m_{kP} \leqslant k_\varepsilon \, P'_{C\varepsilon}(S) \leqslant \|k \, P'_C\|_\infty. \tag{29}$$

Constants m_k and m_{kP} depend on ε.

$\boxed{\text{2. STEP: Galerkin approximation}}$

We solve the problem (18) by introducing the corresponding approximate problem. Let $(e_j)_{j\in\mathbb{N}}$ is a basis of V and let $V_N = \text{span}\,\{e_1, \cdots, e_N\}$. The Galerkin approximation for the problem (18) reads as follows:

$$\text{Find} \quad c_{\varepsilon N} = c_D + \sum_{j=1}^{N} \alpha_j(t) e_j(x),\ \alpha_j \in C^1[0, T\,] \text{ such that}$$

$$\begin{aligned}\int_\Omega \partial_t c_{\varepsilon N}\, e_l\, dx + \int_{\partial_N\Omega} R\, e_l\, d\Gamma - \int_\Omega k_\varepsilon\, P'_{C\varepsilon}(c_{\varepsilon N}) \nabla c_{\varepsilon N} \nabla e_l\, dx \\ + \int_\Omega \tau\, k_\varepsilon(c_{\varepsilon N}) \nabla(\partial_t c_{\varepsilon N}) \nabla e_l\, dx + \int_\Omega k(c_{\varepsilon N})\, \partial_{x_n} e_l\, dx = 0,\end{aligned} \tag{30}$$

for all $l \in \{1, \ldots, N\}$ and satisfying the initial condition

$$c_{\varepsilon N}(0) = c_D(x, 0) + \Pi_N(c_i - c_D|_{t=0}) \tag{31}$$

Π_N is the projector on the finite-dimensional space V_N.
We can rewrite (30) as

$$\begin{cases} A(\alpha)\dfrac{d\alpha}{dt} = B(\alpha)\alpha + \mathbf{F}(\alpha, t), \\ \alpha_k(0) = (e_k, c_i - c_D|_{t=0}), \qquad k = 1, \ldots, N. \end{cases} \tag{32}$$

where

$$A_{l,j} = \int_\Omega e_j\, e_l\, dx + \int_\Omega \tau\, k_\varepsilon(c_{\varepsilon N})\nabla e_j \nabla e_l\, dx \tag{33}$$

$$B_{l,j} = \int_\Omega k_\varepsilon\, P'_{C\varepsilon}(c_{\varepsilon N})\nabla e_j \nabla e_l\, dx \tag{34}$$

$$\begin{aligned} F_l = \int_\Omega k_\varepsilon\, P'_{C\varepsilon}(c_{\varepsilon N})\nabla c_D \nabla e_l\, dx - \int_{\Gamma_N} R\, e_l\, d\Gamma - \int_\Omega \partial_t c_D\, e_l\, dx \\ - \int_\Omega \tau\, k_\varepsilon(c_{\varepsilon N})\nabla(\partial_t c_D)\nabla e_l\, dx - \int_\Omega k(c_{\varepsilon N})\, \partial_{x_n} e_l\, dx\, dt. \end{aligned} \tag{35}$$

Obviously, the matrix $A = A(\alpha)$ is symmetric positive definite matrix, depending smoothly on α. As $F = F(\alpha, t)$ and B are continuously differentiable functions of α and continuous functions of t, the Cauchy-Lipschitz theorem implies that

Lemma 8. *There exists $T_N > 0$ such that the problem* (30)-(31) *has a unique solution belonging to $C^1([0, T_N\ ]; V_N)$.*

3 STEP: A priori estimates for the regularized problem

Proposition 9. *Under the assumptions* **(H1)-(H5)***, the solution of the approximate problem (30)-(31) exists for all times $T < +\infty$ and belongs to $C^1([0, T\ ]; V_N)$.*

Proof. We test (30) by $\partial_t(c_{\varepsilon N} - c_D)$ and obtain

$$\begin{aligned} \int_\Omega (\partial_t c_{\varepsilon N})^2 dx - \int_\Omega \partial_t c_{\varepsilon N}\, \partial_t c_D\, dx + \int_{\partial_N \Omega} R\, \partial_t c_{\varepsilon N}\, d\Gamma - \int_{\partial_N \Omega} R\, \partial_t c_D\, d\Gamma \\ + \int_\Omega \tau\, k_\varepsilon(c_{\varepsilon N})\nabla(\partial_t c_{\varepsilon N})\nabla(\partial_t c_{\varepsilon N}) dx - \int_\Omega \tau\, k_\varepsilon(c_{\varepsilon N})\nabla(\partial_t c_{\varepsilon N})\nabla(\partial_t c_D) dx + \\ \int_\Omega k(c_{\varepsilon N})\, \partial_{x_n} \partial_t(c_{\varepsilon N} - c_D)\, dx - \int_\Omega k_\varepsilon\, P'_{C\varepsilon}(c_{\varepsilon N})\nabla c_{\varepsilon N} \nabla(\partial_t c_{\varepsilon N})\, dx \\ + \int_\Omega k_\varepsilon\, P'_{C\varepsilon}(c_{\varepsilon N})\nabla c_{\varepsilon N} \nabla(\partial_t c_D)\, dx = 0 \end{aligned} \tag{36}$$

Next

$$\nabla c_{\varepsilon N}(t) - \nabla c_{\varepsilon N}(0) = \int_0^t \partial_\xi \nabla c_{\varepsilon N}\, d\xi,$$

and consequently

$$\|\nabla c_{\varepsilon N}(t)\|^2_{L^2(\Omega)} \leq 2(\|\nabla c_i\|^2_{L^2(\Omega)} + \left\|\int_0^t \partial_\xi \nabla c_{\varepsilon N}\, d\xi\right\|^2_{L^2(\Omega)}) \leq$$
$$C + Ct\int_0^t \left\|\partial_\xi \nabla c_{\varepsilon N}\right\|^2_{L^2(\Omega)} d\xi \tag{37}$$

Now we use the assumptions **(H1)-(H5)**, the estimates (28)-(29), (37) and the Trace theorem (see [12]) to conclude that (36) implies

$$\int_\Omega (\partial_t c_{\varepsilon N})^2 dx + \ m_k \int_\Omega \tau\, k_\varepsilon(c_{\varepsilon N})|\nabla(\partial_t c_{\varepsilon N})|^2 dx$$
$$\leq C + C\frac{\|k\ P'_C\|^2_\infty}{m_k}\ t\int_0^t \int_\Omega \left|\nabla(\partial_\xi c_{\varepsilon N})\right|^2 dx\, d\xi, \tag{38}$$

Using Gronwall's inequality we obtain from (38) that

$$\max_{0\leq t\leq T_N} \int_\Omega |\nabla(\partial_t c_{\varepsilon N})|^2 dx \leq C. \tag{39}$$

Hence the maximal solution for the approximate problem (30)-(31) exists for all times $T < +\infty$ and belongs to $C^1([0, T\ ]; V_N)$. □

We consider by now the problem on $[0, T]$, $T > 0$. A direct consequence of the calculations from the proof of Proposition 9 are the following estimates:

Corollary 10. *Under the hypotheses* **(H1)-(H5)**, *there is a constant C independent of N such that*

$$\|\nabla c_{\varepsilon N}\|_{L^2(0,T;\, L^2(\Omega)^n)} \leq C \tag{40}$$
$$\|\partial_t c_{\varepsilon N}\|_{L^2(0,T;\, L^2(\Omega))} \leq C \tag{41}$$
$$\|\sqrt{k_\varepsilon(c_{\varepsilon N})}\nabla\,(\partial_t c_{\varepsilon N})\|_{L^2(0,T;\, L^2(\Omega)^n)} \leqslant C \tag{42}$$

Proposition 11. *Under the hypotheses* **(H1)-(H5)**, *the solution $c_{\varepsilon N}$ of the approximate problem (30)-(31) converge to a function $c_\varepsilon \in H^1(Q_T)$, $\partial_t \nabla c_\varepsilon \in L^2(Q_T)$, satisfying equation (18) in the limit when $N \to +\infty$.*

Proof. By the weak compactness and by the Aubin-Lions compactness theorem we can extract a subsequence $\{c_{\varepsilon N}\}$, denoted again by the same subscripts, and $c_\varepsilon \in H^1(Q_T)$, $\partial_t \nabla c_\varepsilon \in L^2(Q_T)$ such that we have

$$\begin{aligned}
& c_{\varepsilon N} \to c_\varepsilon \ \text{ weakly in } L^2(Q_T) \\
& \nabla c_{\varepsilon N} \to \nabla c_\varepsilon \ \text{ weakly in } L^2(Q_T) \\
& \partial_t c_{\varepsilon N} \to \partial_t c_\varepsilon \ \text{ weakly in } L^2(Q_T) \\
& \nabla\left(\partial_t c_{\varepsilon N}\right) \to \nabla\left(\partial_t c_\varepsilon\right) \ \text{ weakly in } L^2(Q_T) \\
& c_{\varepsilon N} \to c_\varepsilon \ \text{ strongly in } \ L^2(Q_T) \ \text{ and a.e. on } \ Q_T,
\end{aligned}$$

when $N \to \infty$. It is straightforward to prove that the limit function c_ε satisfies equation (18) and the initial and boundary conditions. $\square$

3. Proof of Theorem 4

$\boxed{\text{1. STEP}}$

We test the variational equation (18) by

$$\psi = \int_{c_D}^{c_\varepsilon} \frac{d\xi}{k_\varepsilon(\xi)} \ \in V.$$

ψ is linked to the regularized entropy, corresponding to $1/k_\varepsilon(c_\varepsilon)$. We get

$$\begin{aligned}
\int_0^t \int_\Omega \partial_t c_\varepsilon \int_{c_D}^{c_\varepsilon} \frac{d\xi}{k_\varepsilon(\xi)}\, dx\, dt + \int_0^t \int_{\partial_N\Omega} R \int_{c_D}^{c_\varepsilon} \frac{d\xi}{k_\varepsilon(\xi)}\, d\Gamma\, dt \\
+ \int_0^t \int_\Omega \tau\, k_\varepsilon(c_\varepsilon) \nabla(\partial_t c_\varepsilon)\left(\frac{\nabla c_\varepsilon}{k_\varepsilon(c_\varepsilon)} - \frac{\nabla c_D}{k_\varepsilon(c_D)}\right) dx\, dt \\
- \int_0^t \int_\Omega k_\varepsilon(c_\varepsilon) P'_{C\varepsilon}(c_\varepsilon) \nabla c_\varepsilon \left(\frac{\nabla c_\varepsilon}{k_\varepsilon(c_\varepsilon)} - \frac{\nabla c_D}{k_\varepsilon(c_D)}\right) dx\, dt \\
+ \int_0^t \int_\Omega k(c_\varepsilon) \left(\frac{\partial_{x_n} c_\varepsilon}{k_\varepsilon(c_\varepsilon)} - \frac{\partial_{x_n} c_D}{k_\varepsilon(c_D)}\right) dx\, dt = 0
\end{aligned} \tag{43}$$

Now, we focus on the first term in (43). It transforms as follows:

$$\partial_t c_\varepsilon \int_{c_D}^{c_\varepsilon} \frac{d\xi}{k_\varepsilon(\xi)} = \partial_t \left(c_\varepsilon \int_{c_D}^{c_\varepsilon} \frac{d\xi}{k_\varepsilon(\xi)} - \int_0^{c_\varepsilon} \frac{\xi\, d\xi}{k_\varepsilon(\xi)}\right) + c_\varepsilon \frac{\partial_t c_D}{k_\varepsilon(c_D)}. \tag{44}$$

It is natural to set now for the regularized entropy density

$$\mathcal{E}_\varepsilon(v) = \int_0^v \int_{c_D}^u \frac{d\xi}{k_\varepsilon(\xi)}\, du = v \int_{c_D}^v \frac{d\xi}{k_\varepsilon(\xi)} - \int_0^v \frac{\xi\, d\xi}{k_\varepsilon(\xi)} + \text{ an affine function of } v.$$

Nevertheless we should be careful with its behavior when $\varepsilon \to +0$. By (10), we have $1/k_\varepsilon(u) = (\varepsilon + u)^{-\beta}/C_k + O(1)$, $1 \geq u \geq 0$. Consequently, for $1 \geq u \geq 0$, the principal part of the entropy density is

$$c_\varepsilon \int_{c_D}^{c_\varepsilon} \frac{d\xi}{k_\varepsilon(\xi)} - \int_0^{c_\varepsilon} \frac{\xi\, d\xi}{k_\varepsilon(\xi)} = \frac{(c_\varepsilon + \varepsilon)^{2-\beta} + (\beta - 2)c_\varepsilon(c_D + \varepsilon)^{1-\beta} - \varepsilon^{2-\beta}}{C_k\,(1-\beta)(2-\beta)} + O(1). \tag{45}$$

Now we set for the entropy density $\mathcal{E}_\varepsilon(v) = \Psi_\varepsilon(v) + G_\varepsilon(v)$, where

$$\Psi_\varepsilon(u) = \begin{cases} \dfrac{u^2\varepsilon^{-\beta}}{2C_k} + \dfrac{u\{(\varepsilon + c_D)^{1-\beta} - \varepsilon^{1-\beta}\}}{C_k(\beta - 1)} + \dfrac{\varepsilon^{2-\beta}}{C_k\,(1-\beta)(2-\beta)}, & \text{for } \ u < 0; \\[2ex] \dfrac{(u+\varepsilon)^{2-\beta} + (\beta - 2)u(c_D + \varepsilon)^{1-\beta}}{C_k\,(1-\beta)(2-\beta)}, & \text{for } \ 0 \leq u \leq 1; \\[2ex] \dfrac{(u-1)^2(1+\varepsilon)^{-\beta}}{2C_k} + \dfrac{u}{C_k(\beta-1)}\{(\varepsilon + c_D)^{1-\beta} - (1+\varepsilon)^{1-\beta}\} & \\ + \dfrac{(1+\varepsilon)^{2-\beta} + (\beta - 2)(1+\varepsilon)^{1-\beta}}{C_k\,(1-\beta)(2-\beta)}, & \text{for } \ u > 1. \end{cases} \tag{46}$$

and G_ε is a smooth bounded real function on $\mathbb{R}$, bounded uniformly with respect to ε. We note that

$$\Psi_\varepsilon(u) \geq \Psi_\varepsilon^0(u) = \begin{cases} \dfrac{u^2\varepsilon^{-\beta}}{2C_k} + \dfrac{\varepsilon^{2-\beta}}{C_k\,(1-\beta)(2-\beta)}, & \text{for } \ u < 0; \\[2ex] \dfrac{(u+\varepsilon)^{2-\beta}}{C_k\,(1-\beta)(2-\beta)}, & \text{for } \ 0 \leq u \leq 1; \\[2ex] \dfrac{(u-1)^2(1+\varepsilon)^{-\beta}}{2C_k} + \dfrac{(1+\varepsilon)^{2-\beta}}{C_k\,(1-\beta)(2-\beta)}, & \text{for } u > 1, \end{cases} \tag{47}$$

and that $\Psi_\varepsilon^0(u) \geq C(|u| + \varepsilon)^{2-\beta}$.

Next we insert (46) and (44) into (43) and obtain

$$\int_\Omega \mathcal{E}_\varepsilon(c_\varepsilon(x,t))\,dx + \frac{\tau}{2}\int_\Omega |\nabla c_\varepsilon(x,t)|^2 dx - \int_0^t\int_\Omega P'_{C\varepsilon}(c_\varepsilon)|\nabla c_\varepsilon|^2\,dx\,dt =$$

$$\int_\Omega \mathcal{E}_\varepsilon(c_i(x))\,dx + \frac{\tau}{2}\int_\Omega |\nabla c_i(x)|^2 dx - \int_0^t\int_\Omega c_\varepsilon \frac{\partial_t c_D}{k_\varepsilon(c_D)}\,dx\,dt$$

$$-\int_0^t\int_\Omega \frac{k_\varepsilon(c_\varepsilon)}{k_\varepsilon(c_D)}\,P'_{C\varepsilon}(c_\varepsilon)\,\nabla c_\varepsilon\,\nabla c_D\,dx\,dt - \int_0^t\int_{\partial_N\Omega} R\int_{c_D}^{c_\varepsilon}\frac{d\xi}{k_\varepsilon(\xi)}\,d\Gamma\,dt$$

$$-\int_0^t\int_\Omega \frac{k(c_\varepsilon)}{k_\varepsilon(c_\varepsilon)}\,\partial_{x_n}c_\varepsilon\,dx\,dt + \int_0^t\int_\Omega \frac{k(c_\varepsilon)}{k_\varepsilon(c_D)}\,\partial_{x_n}c_D\,dx\,dt$$

$$+\int_0^t\int_\Omega \tau\,\frac{k_\varepsilon(c_\varepsilon)}{k_\varepsilon(c_D)}\nabla(\partial_t c_\varepsilon)\nabla c_D\,dx\,dt \tag{48}$$

We note that

$$\int_\Omega \mathcal{E}_\varepsilon(c_i(x))\,dx \le C(1+\int_\Omega c_i^{2-\beta}(x)dx + \int_\Omega c_i(x)c_D^{1-\beta}(x,0)dx) < +\infty, \tag{49}$$

by the hypothesis **(H6)**). Next using the hypothesis (**H5**) we have

$$\int_0^t\int_{\partial_N\Omega} R\int_0^{c_\varepsilon}\frac{d\xi}{k_\varepsilon(\xi)}\,d\Gamma\,dt \ge 0.$$

Now using the Cauchy-Schwartz inequality, (49) and hypotheses **(H1)-(H6)** we find out that (48) implies

$$\int_\Omega \Psi^0_\varepsilon(c_\varepsilon)\,dx + \frac{\tau}{2}\int_\Omega |\nabla c_\varepsilon|^2 dx + \int_0^t\int_\Omega \left|\nabla\int_0^{c_\varepsilon}\sqrt{-P'_{C\varepsilon}(\xi)}d\xi\right|^2 dx\,dt$$

$$\le C(\delta) + \delta\int_0^t\int_\Omega \tau\,k_\varepsilon(c_\varepsilon)|\nabla(\partial_t c_\varepsilon)|^2\,dx\,dt, \tag{50}$$

where $\delta > 0$ is small.

$\boxed{\text{2 STEP}}$

Our next step is to test the variational equation (18) by $\partial_t c_\varepsilon - \partial_t c_D$. We get

$$\int_0^t\int_\Omega \partial_t c_\varepsilon\,(\partial_t c_\varepsilon - \partial_t c_D)\,dx\,dt + \int_0^t\int_{\partial_N\Omega} R\,(\partial_t c_\varepsilon - \partial_t c_D)\,d\Gamma\,dt +$$

$$\int_0^t\int_\Omega \tau\,k_\varepsilon(c_\varepsilon)\nabla(\partial_t c_\varepsilon)\nabla\partial_t(c_\varepsilon - c_D)dx\,dt + \int_0^t\int_\Omega k(c_\varepsilon)\,\partial_{x_n}\partial_t(c_\varepsilon - c_D)dx\,dt$$

$$-\int_0^t\int_\Omega k_\varepsilon(c_\varepsilon)P'_{C\varepsilon}(c_\varepsilon)\nabla c_\varepsilon\,\nabla(\partial_t c_\varepsilon - \partial_t c_D)dx\,dt = 0 \tag{51}$$

In (51) only non-trivial estimate is for the boundary flux. It reads

$$
\begin{aligned}
|\int_0^t \int_{\partial_N \Omega} R\, \partial_t (c_\varepsilon - c_D)\, d\Gamma\, dt| &= |\int_0^t \int_{\partial_N \Omega} R_0(x,t) \partial_t (c_\varepsilon - c_D) \zeta(c_\varepsilon)\, d\Gamma\, dt| \\
&\le |\int_0^t \int_{\partial_N \Omega} R_0(x,\xi) \partial_t c_D(x,\xi) \zeta(c_\varepsilon)\, d\Gamma\ d\xi| + \\
|\int_{\partial_N \Omega} R_0(x,0) \int_0^{c_i} \zeta(\xi)\ d\xi\, d\Gamma\ | &+ |\int_{\partial_N \Omega} R_0(x,t) \int_0^{c_\varepsilon(t)} \zeta(\xi)\ d\xi\, d\Gamma\ | + \\
&|\int_0^t \int_{\partial_N \Omega} \partial_t R_0(x,\xi) \int_0^{c_\varepsilon(t)} \zeta(\xi)\ d\xi\, d\Gamma dt| \le C. \qquad (52)
\end{aligned}
$$

Using the hypotheses **(H1)-(H6)**, the Trace theorem (see e.g. [12]), (52) and the Cauchy-Schwartz inequality, we obtain the following inequality

$$
\begin{aligned}
\int_0^t \int_\Omega (\partial_t c_\varepsilon)^2\, dx\, dt + \int_0^t \int_\Omega \tau\, k_\varepsilon(c_\varepsilon) |\nabla(\partial_t c_\varepsilon)|^2\, dx\, dt \le C(1+ \\
\|k\ P_c'\|_\infty \int_0^t \int_\Omega \left| \nabla \int_0^{c_\varepsilon} \sqrt{-P_{C\,\varepsilon}'(\xi)} d\xi \right|^2 dx\, dt + \sup_{t\in(0,T)} \int_\Omega |\nabla c_\varepsilon|^2\ dx). \qquad (53)
\end{aligned}
$$

By (50) we have

$$
\begin{aligned}
&\int_0^t \int_\Omega \left| \nabla \int_0^{c_\varepsilon} \sqrt{-P_{C\,\varepsilon}'(\xi)} d\xi \right|^2 dx\, dt + \sup_{t\in(0,T)} \int_\Omega |\nabla c_\varepsilon|^2\ dx \\
&\le C + \delta \int_0^t \int_\Omega \tau\, k_\varepsilon(c_\varepsilon) |\nabla(\partial_t c_\varepsilon)|^2 dx\, dt \quad \text{with small} \quad \delta. \qquad (54)
\end{aligned}
$$

After inserting (54) into (53), we conclude that there exists constants independent of ε such that (23)-(25) hold true.

Inserting (25) into (50) gives the estimate (22). Finally, (47) gives the entropy estimate (21) and the approximative positivity estimate (20). This proves Theorem 4.

4. Proof of Theorem 5

For passing to the limit $\varepsilon \to 0$, we miss only an estimate on $\nabla(\partial_t c_\varepsilon)$ in $L^{r_0}(Q_T)$, for some $r_0 > 1$, independent of ε.

Proposition 12. *Let $n \leq 3$ and let us suppose the hypotheses* **(H1)-(H6)**. *Then under the assumptions (26)–(27) any weak solution c_ε for the regularized problem (5) -(8), satisfies*

$$\|\nabla(\partial_t c_\varepsilon)\|_{L^{r_0}(Q_T)} \leq C, \quad \text{with} \quad r_0 \in (1,2), \tag{55}$$

where the constants C and r_0 do not depend on ε.

Proof. First, by (12) we have $-P'_C(y) = C_p\, y^{-\lambda} + O(1)$, $y > 0$. Then the condition $\sup_y -k(y)\, P'_C(y) < +\infty$ implies that

$$\beta \geq \lambda. \tag{56}$$

Next we note that

$$P'_C(\varepsilon + c_\varepsilon^+)|\nabla c_\varepsilon|^2 = P'_C(\varepsilon + c_\varepsilon^+)|\nabla c_\varepsilon^+|^2 + P'_C(\varepsilon)|\nabla c_\varepsilon^-|^2,$$

where $c_\varepsilon^+ = \sup\{c_\varepsilon, 0\}$. Hence estimate (24) reads

$$-\int_{Q_T} P'_C(\varepsilon + c_\varepsilon^+)|\nabla c_\varepsilon^+|^2 \; dxdt - \int_{Q_T} P'_C(\varepsilon)|\nabla c_\varepsilon^-|^2 \; dxdt \leq C.$$

The leading order part of $\displaystyle\int_0^{c_\varepsilon^+} \sqrt{-P'_{C\varepsilon}(\xi)}\; d\xi$ is

$$\int_0^{c_\varepsilon^+} \sqrt{C_p}(\xi+\varepsilon)^{-\lambda/2}\, d\xi = \frac{\sqrt{C_p}}{1-\lambda/2}((c_\varepsilon^+ + \varepsilon)^{1-\lambda/2} - \varepsilon^{1-\lambda/2}).$$

Therefore instead of considering

$$\int_0^{c_\varepsilon^+} \sqrt{-P'_{C\varepsilon}(\xi)} d\xi \quad \text{we take the function} \quad \frac{\sqrt{C_p}}{1-\lambda/2}\, \varepsilon^{1-\lambda/2} + \int_0^{c_\varepsilon^+} \sqrt{-P'_{C\varepsilon}(\xi)} d\xi.$$

and the estimate (25) implies

$$\nabla(c_\varepsilon^+ + \varepsilon)^{1-\lambda/2} \in L^2(Q_T). \tag{57}$$

Next we use that $c_\varepsilon = c_D$ on Γ_D and (57) and Poincaré's inequality for $H^1(\Omega)$ (see e.g. [12]) imply

$$(c_\varepsilon^+ + \varepsilon)^{1-\lambda/2} \in L^2(0,T; H^1(\Omega)). \tag{58}$$

Now we apply Sobolev embedding theorems (see again [12]) and obtain

$$(c_\varepsilon^+ + \varepsilon)^{1-\lambda/2} \in L^2(0,T;C(\bar\Omega)) \quad \text{for} \;\; n=1, \tag{59}$$

$$(c_\varepsilon^+ + \varepsilon)^{1-\lambda/2} \in L^2(0,T;L^r(\Omega)), \;\; \forall\, r \in (1,+\infty), \quad \text{for} \;\; n=2, \tag{60}$$

$$(c_\varepsilon^+ + \varepsilon)^{1-\lambda/2} \in L^2(0,T;L^{\frac{2\,n}{n-2}}(\Omega)), \quad \text{for} \;\; n>2. \tag{61}$$

By the entropy estimate (21), $(c_\varepsilon^+ + \varepsilon)^{2-\beta} \in L^\infty(0,T;L^1(\Omega))$, and we would like to conclude that (21) and (58) imply $k_\varepsilon^{-\gamma} \in L^1(Q_T)$ for some $\gamma > 1$. Idea is to achieve this integrability by interpolating between (58) and (21). We will considered separately the cases $n = 1, 2,$ and 3, respectively.

We start with the case $n = 3$ and we undertake to estimate the integral $\displaystyle\int_\Omega (c_\varepsilon^+ + \varepsilon)^{-\gamma\,\beta} dx$ with $\gamma > 1$. For $6 > \theta > 0$, we use Hölder's inequality with $p = \dfrac{6}{\theta}$ and $p_1 = \dfrac{6}{6-\theta}$, to obtain the estimate

$$\int_\Omega (c_\varepsilon^+ + \varepsilon)^{-\gamma\,\beta} dx = \int_\Omega (c_\varepsilon^+ + \varepsilon)^{(1-\lambda/2)\theta} (c_\varepsilon^+ + \varepsilon)^{(2-\beta)\theta_1} dx$$
$$\leq (\int_\Omega (c_\varepsilon^+ + \varepsilon)^{(1-\lambda/2)6} dx)^{\theta/6} (\int_\Omega (c_\varepsilon^+ + \varepsilon)^{(2-\beta)6\,\theta_1/(6-\theta)} dx)^{(6-\theta)/6}$$

Due to (21), we have $\theta_1 = 1 - \theta/6$. Next, from (61) with $n = 3$, we have

$$(\int_\Omega (c_\varepsilon^+ + \varepsilon)^{(1-\lambda/2)6} dx)^{1/3} \in L^1(0,T),$$

and we find that $\theta = 2$. Finally, we see that (21) and (58) imply

$$\int_0^T \int_\Omega (c_\varepsilon^+ + \varepsilon)^{-\gamma\beta}\, dxdt \leq \int_0^T (\int_\Omega (c_\varepsilon^+ + \varepsilon)^{(1-\lambda/2)6} dx)^{1/3} dt \,\cdot$$
$$\max_{0\leq t\leq T} \int_\Omega (c_\varepsilon^+ + \varepsilon)^{2-\beta} dx < +\infty$$

The above calculation gives us γ:

$$-\gamma\,\beta = (1 - \frac{\lambda}{2})\,2 + (2-\beta)\,\frac{2}{3} = \frac{10}{3} - \lambda - \frac{2\,\beta}{3}$$

Therefore $\gamma > 1$ if and only if $\beta \geq \lambda > 2$ and

$$\frac{-1}{\beta}(\frac{10}{3} - \lambda - \frac{2\,\beta}{3}) > 1 \quad \Longleftrightarrow \quad \frac{10}{3} + \frac{\beta}{3} < \lambda \tag{62}$$

If $n=2$, we apply Hölder's inequality with arbitrary finite $r>2$ and with $p=r/\theta$, $\quad 1/p_1 = 1-\theta/r$. Again we get $\theta=2$ and $\theta_1 = 1-2/r$. Hence now

$$-\gamma\,\beta = (1-\frac{\lambda}{2})\,2 + (2-\beta)(1-\frac{2}{r}) = 4-\frac{4}{r}-\lambda-\beta(1-\frac{2}{r}). \tag{63}$$

In order to have $\gamma>1$, we conclude that for $n=2$, λ and β should satisfy $\beta\geq\lambda>4$.

For $n=1$, from the estimates (59) and (21) we get

$$\begin{aligned}\int_\Omega (c_\varepsilon^+ + \varepsilon)^{-\gamma\,\beta}dx &= \int_\Omega (c_\varepsilon^+ + \varepsilon)^{(1-\lambda/2)\,2}(c_\varepsilon^+ + \varepsilon)^{2-\beta}dx \\ &\leq (\max_{x\in\bar\Omega}(c_\varepsilon^+ + \varepsilon)^{1-\lambda/2})^2 \int_\Omega (c_\varepsilon^+ + \varepsilon)^{2-\beta}dx \in L^1(0,T)\end{aligned}$$

In order to have $\gamma>1$, λ and β should satisfy

$$-\gamma\,\beta = 4-\lambda-\beta < -\beta \quad \text{ i.e. } \quad \beta\geq\lambda>4. \tag{64}$$

Now we are in situation to establish an estimate in $L^q(Q_T)$, $q>1$, for $\nabla(\partial_t c_\varepsilon)$. It follows by applying Hölder's inequality:

$$\begin{aligned}\int_0^T\int_\Omega |\nabla(\partial_t c_\varepsilon)|^q dx\,dt &= \int_0^T\int_\Omega \left|k_\varepsilon^{1/2}\,\nabla(\partial_t c_\varepsilon)\right|^q\,k_\varepsilon^{-q/2}dx\,dt \\ \leq (\int_0^T\int_\Omega k_\varepsilon\,|\nabla(\partial_t c_\varepsilon)|^2 dx\,dt)^{2/q}&\cdot(\int_0^T\int_\Omega k_\varepsilon^{-q/(2-q)}dx\,dt)^{1-2/q}.\end{aligned} \tag{65}$$

For $n=3$, (62) gives $q(3) = \dfrac{2(3\lambda+2\beta-10)}{3\lambda+5\beta-10}$. For $n=2$, $\beta\geq\lambda>4$ implies that there exists $r\geq 2$ such that $\lambda>4+2(\beta-2)/r$. Then using (63), we obtain $q(2) = 2(4-\lambda-\beta+2(\beta-2)/r)/(4-\lambda-2\beta+2(\beta-2)/r)\in(1,2)$. Finally, for $n=1$, we have $q(1)=2(4-\lambda-\beta)/(4-\lambda-2\beta)\in(1,2)$. Now $k_\varepsilon^{-q/(2-q)}\in L^1(Q_T)$, for $q=q(n)$ defined above and (55) holds true. $\square$

Now the estimates (20)-(25) and (55) imply that there exists a subsequence of c_ε, denoted by the same subscripts, and $c\in H^1(Q_T)$, $\partial_t\nabla c\in L^{r_0}(Q_T)$, for some $r_0\in(1,2)$ such that

$$\left\{\begin{array}{ll} c_\varepsilon\to c, & \text{strongly in } \ L^2(Q_T) \\ c_\varepsilon\rightharpoonup c, & \text{weakly in } \ H^1(Q_T) \\ \nabla\partial_t c_\varepsilon\rightharpoonup\nabla\partial_t c, & \text{weakly in } \ L^{r_0}(Q_T), \\ c_\varepsilon^- = \inf\{c_\varepsilon,0\}\to c^-=0, & \text{strongly in } \ L^2(Q_T). \end{array}\right. \tag{66}$$

Next the estimate (24) implies that, after passing to a subsequence,

$$\sqrt{k_\varepsilon(c_\varepsilon)}\nabla\partial_t c_\varepsilon \rightharpoonup q \quad \text{weakly in} \quad L^2(Q_T), \quad \text{as } \varepsilon \to 0.$$

Furthermore, $\nabla\partial_t c_\varepsilon \rightharpoonup \nabla\partial_t c$ weakly in $L^{r_0}(Q_T)$ and $\sqrt{k_\varepsilon(c_\varepsilon)} \to \sqrt{k(c)}$ strongly in $L^r(Q_T)$, as $\varepsilon \to 0$, for all $r < \infty$. Hence $q = \sqrt{k(c)}\nabla\partial_t c \in L^r(Q_T)$, for all $r < r_0$, and $\sqrt{k(c)}\nabla\partial_t c \in L^2(Q_T)$.

It is straightforward to check that c is a weak solution for the problem (5) -(8), satisfying (16). This completes the proof of Theorem 5.

5. Proof of Theorem 7

Let $p = -P_C(c) + \tau\partial_t c$. Then p satisfies the variational equation

$$\int_0^t \int_\Omega p\, v\, dx\, d\xi + \tau \int_0^t \int_{\partial_N\Omega} R\, v\, d\Gamma\, d\xi + \\ \tau \int_0^t \int_\Omega k(c)\nabla p \nabla v\, dx\, d\xi = -\int_0^t \int_\Omega P_C\, v\, dx\, d\xi\ . \tag{67}$$

for all $v \in L^2(0,T;V)$.

Next we test (67) by p_+ and get $p \le 0$ a.e. on Q_T.

Therefore we have $\partial_t c \le P_C(c)$ and $c \in C[0,T]$ a.e. on Ω. Let us suppose that at the instant $t = t_1$ c reaches value 1 on a subset of Ω of positive measure. c is a continuous function of t with values in $L^2(\Omega)$. If it crosses the value 1 after reaching it for $t = t_1$, then

$$c(x,t) - c(x,t_1) \le \frac{1}{\tau}\int_{t_1}^t P_C(c(x,\xi))\,\mathrm{d}\,\xi = 0$$

for x from a subset of Ω of positive measure. This contradicts the hypothesis that $c(t) > 1$ and therefore $c(x,t) \le 1$ a.e. on Q_T which proves the Theorem.

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