On the interface law between a deformable porous medium containing a viscous fluid and an elastic body*

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Abstract

Coupled multiphase flow and geomechanics models are computationally costly and complex to implement. In order to take advantage of petascale and future exascale computing power, parallel domain decomposition offers an opportunity for decoupling realistic subsurface problems. The basic idea is to reduce the complexity of these multiphysics problems by applying the coupling only in those domains where it is needed. Thus, in classical poroelastic modeling, one needs to take into account both the flow and geomechanics effects in the reservoir (pay zone). When the flow-geomechanics-interactive pay zone and the geomechanics-only non-pay zone are in contact, the natural question is what to set at their interface.

In this paper we undertake a rigorous derivation of the interface conditions between a poroelastic medium (*the pay zone*) and an elastic body (*the non-pay zone*). We suppose that the poroelastic medium contains a pore structure of the characteristic size ε and that the fluid/structure interaction regime corresponds to diphasic Biot's law. The question is challenging because the Biot's equations for the poroelastic part contain one unknown more than the Navier equations for the non-pay zone. The solid part of the pay zone (the matrix) is elastic and the pores contain a viscous fluid. The fluid is supposed viscous and slightly compressible. We study the case when the contrast of property is of order ε^2, i.e. the normal stress of the elastic matrix is of the same order as the fluid pressure. We suppose a periodic matrix and obtain the *a priori* estimates. Then we let the characteristic size of the inhomogeneities tend to zero and pass to the limit in the sense of the two-scale convergence. The obtained effective equations represent a two-scale system for 3 displacements and 2 pressures, coupled through the interface conditions with the Navier equations for the elastic displacement in the non-pay zone. We prove uniqueness for the homogenized 2-scale system. Then we introduce several auxiliary problems and obtain a problem without the fast scale. This new system is diphasic quasi-static and corresponds to the diphasic effective behavior already observed in papers by M. Biot on the

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soil consolidation. In the effective equations it is possible to distinguish the effective displacements of the fluid and the solid phase, respectively. The effective stress tensor contains an instantaneous elasticity tensor and the pressure term. We give a detailed study of the effective equations and justify the quasi-static Biot's poroelasticity equations. Furthermore we prove that the appropriate interface conditions at the interface between an elastic and a poroelastic medium are (i) the effective displacement continuity, (ii) the effective normal stress continuity and (iii) the normal Darcy velocity zero from the poroelastic side. In addition we determine the effective boundary conditions for the contact between a poroelastic body and a rigid obstacle, giving us the effective outer boundary conditions.

1 Introduction

Fluid motion and solid deformation are inherently coupled. Unfortunately today current major commercial simulators for multiphase flow in porous media only model porous flow while solid deformation is normally integrated into a study in an ad hoc manner or must be included through complex iterations between one software package that models fluid flow and a separate package that models solid deformations. There are numerous field applications that would benefit from a better understanding and integration of porous flow and solid deformation. Important applications in the geosciences include environmental cleanup, petroleum production, solid waste disposal, and carbon sequestration, while similar issues arise in the biosciences and chemical sciences as well. Examples of field applications include surface subsidence, pore collapse, cavity generation, hydraulic fracturing, thermal fracturing, wellbore collapse, sand production, fault activation, and disposal of drill cuttings. The above phenomena entail both economic as well as environmental concerns. For example, surface subsidence related to both consolidation of surface layers and fluid withdrawals from oil and gas reservoirs have had a significant impact in the greater Houston area over the last century and have resulted in destruction to infrastructure, buildings and private homes. Subsidence caused by oil and gas production also has been an issue of substantial economic importance in the North Sea oil fields. In some cases multi-billion dollar adjustments have been required to production platforms due to the response to unexpected subsidence of the sea floor driven by oil production. Another important related class of problems involves CO2 sequestration, which is proposed as a key strategy for mitigating climate change driven by high levels of anthropogenic CO2 being added to the atmosphere. In a CO2 sequestration project, fluid is injected into a deep subsurface reservoir (rather than being produced or extracted), so that inflation of the reservoir leads to uplift displacement of the overlying surface. As long as a CO2 sequestration site is removed from faults, this uplift is several centimeters, while its wavelength is in tens of kilometers, so that the uplift poses little danger to buildings and infrastructure. Nevertheless the uplift displacements are of great interest for non-intrusive monitoring of CO2 sequestration. Indeed, uplift can be measured with a sub-millimeter precision using Interferometric Synthetic Aperture Radar (InSAR) technology [35]. The feasibility of this approach has been established by measuring the uplift displacements over the first commercial scale CO2 sequestration project conducted by BP in In Salah Algeria ([10], [35]). In contrast, intrusive monitoring via drill holes bored into the reservoir is expensive, with costs of several million dollars per well. Furthermore, such wells are the most likely pathway for future leakage of sequestered CO2 back into the atmosphere. Of course, if a CO2 sequestration site is close to a fault, one should be concerned about triggering instability leading to large surface displacements that may result in significant losses.

With petascale and future exascale computing power, parallel domain decomposition offers an opportunity for treating multiscale and multiphysics phenomena for realistic subsurface field studies. This is essential when modeling basin scale models such as those arising in carbon sequestration. To reduce the computational complexity of multiphysics problems such as poroelasticity, one is motivated by the observation that pore pressure variations and fluid content within the cap rock and higher layers are oftentimes unaffected by the injection or extraction of fluids within the reservoir. This leads to a domain decomposition computational approach of discretizing the poroelastic and elastic models each indepen-

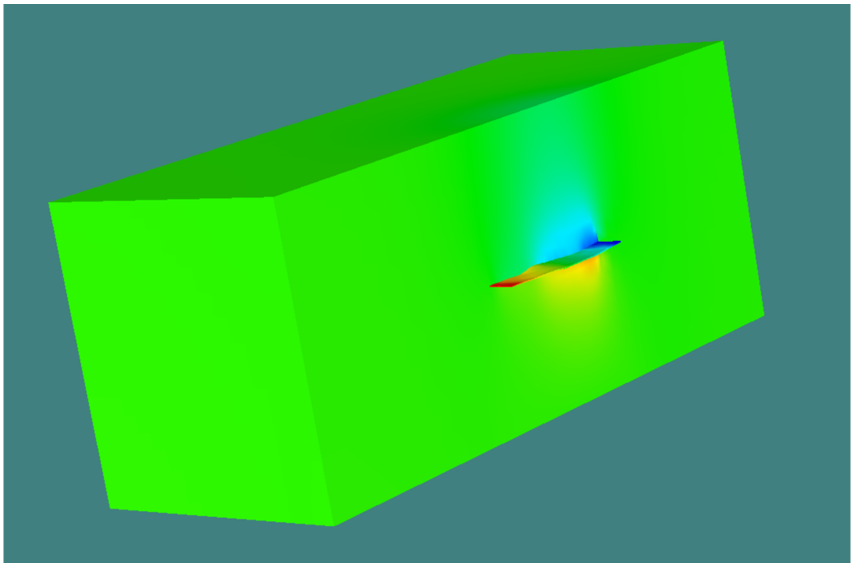

Figure 1: Pay and no pay zone.

dently and defining an interface between the respective regions. In [16], a parallel domain decomposition method was formulated for solving a linear elasticity system. Data across subdomains are transmitted by jumps, as in Discontinuous Galerkin (DG) [17], using mortar finite elements. The global system is reduced to a mortar interface problem and solved in parallel. In [18] building on this work we coupled a time-dependent poroelastic medium (*the pay zone*) with an elastic body (*the non-pay zone*) model in adjacent regions.

Typical example of a pay-zone for a petroleum reservoir (inserted in between non-pay zones) is shown in Figure 1.

Each model was discretized independently on non-matching grids and the systems were coupled using DG jumps and mortars. At each time step, an interface problem is solved, with subdomain solves performed in parallel. We also proposed an algorithm where the computation of the displacement is time-lagged. We showed that in each case, the matrix of the interface problem is positive definite. Error estimates were established. This algorithm can also viewed as a multiscale method since the mortar space can be chosen to be of higher order; see [18] for details.

In this paper we undertake a rigorous derivation of the interface conditions between the pay zone and the non-pay zone. We suppose that the poroelastic medium contains a pore structure of the characteristic size ε and that the fluid/structure interaction regime corresponds to diphasic Biot's law. The question is challenging because the Biot's equations for the poroelastic part contain one unknown more than the Navier equations for the non-pay zone. The solid part of the pay zone (the matrix) is elastic and the pores contain a viscous fluid. We show through homogenization the mathematical correctness in the limit of applying a multidomain or multiblock methodology for Biot systems for treating coupled geomechanical and fluid flow problems. This approach generalizes to multiple subdomains.

We suppose that the poroelastic medium contains a pore structure of the characteristic size ε and that the fluid/structure interaction regime corresponds to diphasic Biot's law. The question is challenging because the Biot's equations for the poroelastic part contain one unknown more than the Navier equations for the non-pay zone. The solid part of the pay zone (the matrix) is elastic and the pores contain a viscous fluid. The fluid is supposed viscous and slightly compressible. We study the case when the contrast of property is of order ε^2, i.e. the normal stress of the elastic matrix is of the same order as the fluid pressure. We suppose a periodic matrix and obtain the *a priori* estimates. Then we let the characteristic size of the inhomogeneities tend to zero and pass to the limit in the sense of the two-scale convergence. The obtained effective equations represent a two-scale system for 3 displacements and 2 pressures, coupled through the interface conditions with the Navier equations for the elastic displacement in the non-pay zone.

The theory of flow-deformation coupling in porous media originated from the research of Biot on the three-dimensional consolidation of saturated soft soil under loads. A collection of Biot's seminal papers can be found in [36]. In this paper our aim is to derive from the first principles

1. the quasi-static Biot's system from consolidation theory,
2. the interface conditions between a poroelastic medium and an elastic medium,
3. the boundary conditions at a closed outer boundary for the quasi-static Biot system.

We start by recalling the fluid-structure interaction modeling.

We consider an incompressible fluid of density ρ_f and dynamic viscosity η. It is initially contained in a domain $\Omega_f(0) \in C^2$. The fluid flow, in Eulerian description, is described by the incompressible Navier-Stokes system

$$\rho_f\Big(\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v}\Big) = \text{ Div } \sigma^F + \rho_F \mathbf{f} \quad \text{in} \quad \Omega_F(t), \tag{1.1}$$

$$\text{div } \mathbf{v} = 0 \quad \text{in} \quad \Omega_F(t), \tag{1.2}$$

$$\mathbf{v}(x,0) = 0 \quad \text{in} \quad \Omega_F(0), \tag{1.3}$$

where $\sigma^F = 2\eta D(\mathbf{v}) - pI$ is the fluid stress tensor and $D(\mathbf{v})_{ij} = (\partial_{x_i} v_j + \partial_{x_j} v_i)/2$ is the rate of strain tensor. $\mathbf{f}$ are exterior bulk forces, p the pressure and $\mathbf{v}$ the Eulerian velocity field (the spatial velocity).

Simultaneously with the fluid, we consider an elastic structure. The reference configuration $\Omega_s(0)$ is the elastic domain at $t = 0$. The deformation of the elastic structure is described in terms of its displacement $\mathbf{u}(X,t) = \Phi(X,t,0) - X$, where $\Phi(X,t,s)$ denotes the Lagrangian flow (the configuration), i.e. the position at time t of the particle located at X at time s. The deformation gradient is $F_{ij} = \partial_{X_j}\Phi_i$. Here the stress measures the force per unit *nondeformed* area and we use the first Piola-Kirchhoff stress tensor P. It is linked to the Cauchy stress σ^s by

$$P(X,t) = J\sigma^s(x,t)F^{-\tau}, \quad J = detF. \tag{1.4}$$

Now the balance of momentum reads

$$\rho_{ref}\frac{\partial \mathbf{u}(X,t)}{\partial t} = \text{ Div } P + \rho_{ref}\mathbf{u} \quad \text{in} \quad \Omega_s(0) \tag{1.5}$$

Since our structure is elastic, we can write the first Piola-Kirchhoff stress tensor P as $P(X,t) = \hat{P}(X,F(x,t))$, where F is the deformation gradient. The stress-strain law for hyperelastic materials reads $\hat{P}_{ij} = \rho_{ref}(\partial W/\partial F_{ij})$, where $W(X,F)$ is a stored energy function. For a frame indifferent, homogeneous and isotropic hyperelastic material we have

$$P = \alpha_0 F + \alpha_1 FF^\tau F + \alpha_2 FF^\tau FF^\tau F, \tag{1.6}$$

where α_i are functions of the invariants of $F^\tau F$. For details we refer to [24] .

In [24] the linear elasticity is introduced as small displacements from a given deformation. We have $P_{ij}(X,t) \approx \sum_{k,l} \frac{\partial \hat{P}_{ij}}{\partial F_{k,l}}(I)\frac{\partial u_k}{\partial X_l}(X,t)$. The quantity $C_{ijkl}(X) = \frac{\partial \hat{P}_{ij}}{\partial F_{k,l}}(I)$ is the fourth order elasticity tensor. In the special case of a homogeneous isotropic linear elastic material, there are constants λ and μ_s called Lamé moduli such that $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu_s(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. We suppose that we deal with a linearly elastic solid structure.

For the fluid and elastic domains we prescribe classical boundary conditions at a part of the boundary. Nevertheless, we should discuss the interface conditions at the contact boundary between the fluid and the solid structure.

The fluid equations are coupled with the equations for the solid structure through the lateral boundary conditions requiring continuity of velocity and continuity (balance) of forces. Depending on the size of the displacement, the coupling is evaluated at the non-deformed interface $\Sigma(0)$ in case when the deformations are expected to be small (*i.e., linear coupling*), or, at the deformed interface, $\Sigma(t)$, when the deformations are expected to be large (*i.e., nonlinear coupling*). In either case, the coupling is performed in the Lagrangian framework, namely, with respect to the reference configuration $\Sigma(0)$. More specifically, if we assume nonlinear coupling, then we require that the fluid velocity evaluated at the deformed interface $\Sigma(t) = \{(X + \mathbf{u}(X,t), t) \mid X \in \Sigma(0)\}$ equals the Lagrangian velocity of the structure. This reads

$$\mathbf{v}(X + \mathbf{u}(X,t), t) = \frac{\partial \mathbf{v}}{\partial t}(X,t) \quad \text{on } \Sigma(0), \tag{1.7}$$

Next we consider balance of forces by requiring that $\sigma^F n = \sigma^s n$ on $\Sigma(t)$. The fluid contact force is typically given in Eulerian coordinates. To perform the coupling in the Lagrangian framework we need the equality $\sigma^s(x,t) = J^{-1} P(X,t) F^\tau$. Hence we have

$$\left(\sigma^F (X + \mathbf{u}(X,t), t) - J^{-1} P(X,t) F^\tau (X,t)\right) \mathbf{n}(X + \mathbf{u}(X,t), t) = 0 \quad \text{on } \Sigma(0) \tag{1.8}$$

To the system (1.1)-(1.3, (1.4), (1.5) , (1.6) , (1.7) and (1.8) , we add the initial data for the displacement

$$\mathbf{u} = 0 \quad \text{on } \Sigma(0) \times \{0\}. \tag{1.9}$$

and, boundary conditions of Dirichlet and Neumann type at outer boundary.

The nonlinear and nonlocal interface condition (1.7) is very difficult to handle and we will linearize it supposing infinitesimal displacements around initial configuration. Similarly, a linearization will be applied to the condition (1.8).

This paper is concentrated on the case which corresponds to the presence of both the fluid and elastic matrix in the effective diphasic macroscopic behavior.

Original equations of Biot describe this particular situation and his heuristic modeling assures a kind of Darcy law for the difference between effective velocities of the solid and fluid part. Asymptotic modeling of this case attracted great attention in the literature and we mention only research undertaken by Auriault [4], Burridge and Keller [9], Levy [22], Nguetseng [26], Sanchez-Palencia [28] . The approach is to set *the dimensionless viscosity* to be $\mu\varepsilon^2$ and then to study the 2-scale asymptotic expansion. For the fully dynamic high frequency regime and for a slightly compressible linear case, the rigorous results are in the well-known book by Sanchez-Palencia [28] . This problem was one of the first applications of the two-scale convergence method in the papers by Nguetseng [25] and [26]. The rigorous homogenization using the two-scale convergence method (fast and slow scales separation) and comparison to the Biot's models ([6], [7] and [8]) is analyzed in the papers by Mikelić et al [12], [14] and [15].)

Contrary to the advanced theory of poroelasticity, modeling of the interface and/or boundary conditions is generally done empirically. To the best of our knowledge, the major contribution is due to Showalter and coworkers in [29], [30], [31], [32], [33] and [34]. The spirit of their approach is to accept the macroscale poroelasticity equations as fundamental and then to propose boundary conditions which lead to mathematically well-posed initial/boundary value problems. Once having well posed coupled problems, it is possible to develop sophisticated numerical algorithms. We refer to the publications [16], [17] and [18] by Wheeler and coworkers. For the fully nonlinear interface coupling we refer to [19].

However, the physical interpretation to be ascribed to these ad hoc interface and boundary conditions seems obscure. There is a need of obtaining interface and boundary conditions from first principles, which we undertake in this paper.

In Section 2 we formulate the geometry, state the linearized slightly compressible pore scale model and state the upscaling result. Relationship of the derived homogenized model to the physical microscopic problem is given in Theorem 1, which is a dimensional version of Theorem 3 in Section 6. It also contains the approximation through correctors in the corresponding norms. The main body of the paper begins in Section 3 in which we carefully derive a non-dimensional form, in order to justify the quasi-static approximation. In Section 4 we derive a priori estimates uniform with respect to ε. Since the acceleration terms are small, these estimates require a careful attention. We prove Proposition 3 giving us an adapted tool for handling incomplete estimates. It is based on the fine properties of the geometry and on the second Korn inequality. Precise a priori estimates allow two-scale compactness, established in Proposition 5. Here we take into the account the boundary conditions for Darcy type terms. The fundamental result is obtained in Theorem 2, which provides the interface conditions and the effective equations in the two-scale form. In Section 5 the fast and slow scales are separated and an effective quasi-static Biot system is obtained, together with the effective interface and boundary conditions. Finally the strong convergence of the correctors is proved in Section 6.

2 The pore scale model

2.1 Geometry

Structure of the porous media met in the nature is frequently very complicated. In order to model their behavior it is necessary to make some hypothesis on it.

Here we make the following geometry assumptions:

- In general it is supposed that there are two connected phases, a solid and a fluid one. The solid phase is deformable. In addition, the porous medium is assumed to be heterogeneous at the microscopic (pore) level but statistically homogeneous at macroscopic level.
- The characteristic length of the non-homogeneities is ℓ. Since the theory for the physical velocities, pressures and other quantities is very complicated, one prefers working with averaged quantities over characteristic volumes being of order ℓ^3.

A representative example of the above geometry is the *periodic* porous medium with connected fluid and solid phases. It is obtained by a periodic arrangement of the pores. The formal description goes along the following lines:

First we define the geometrical structure inside the unit cell $\mathscr{Y} =]0,1[^3$. Let $\mathscr{Y}_s$ (the solid part) be a closed subset of $\bar{\mathscr{Y}}$ and $\mathscr{Y}_f = \mathscr{Y} \backslash \mathscr{Y}_s$ (the fluid part). Now we make periodic repetition of $\mathscr{Y}_s$ over $\mathbb{R}^n$ and set $\mathscr{Y}_s^k = \mathscr{Y}_s + k$, $k \in \mathbb{Z}^n$. Obviously the obtained closed set $E_s = \bigcup_{k \in \mathbb{Z}^n} \mathscr{Y}_s^k$ is a closed subset of $\mathbb{R}^n$ and $E_f = \mathbb{R}^n \backslash E_s$ in an open set in $\mathbb{R}^n$. Following Allaire [2] we make the following assumptions on $\mathscr{Y}_f$ and E_f:

(H1) $\mathscr{Y}_f$ is an open connected set of strictly positive measure, with a Lipschitz boundary and $\mathscr{Y}_s$ has strictly positive measure in $\bar{\mathscr{Y}}$ as well.

(H2) E_f and the interior of E_s are open sets with the boundary of class $C^{0,1}$, which are locally located on one side of their boundary. Moreover both E_f and E_s are connected.

For simplicity we consider the "porous" domain (**the pay zone**) $\Omega_L = (0,L)^3$. We suppose the pay-zone is covered with a regular mesh of size ℓ, each cell being a cube $\mathscr{Y}_i^\ell$, with $1 \le i \le N(\ell) = L^3 \ell^{-3}[1+0(1)]$. Each cube $\mathscr{Y}_i^\ell$ is homeomorphic to $\mathscr{Y}$, by the linear homeomorphism Π_i^ℓ (that is set by translation and homothety of ratio $1/\ell$).

We define

$$\mathscr{Y}_{S_i}^\ell = (\Pi_i^\ell)^{-1}(\mathscr{Y}_s) \qquad \text{and} \qquad \mathscr{Y}_{f_i}^\ell = (\Pi_i^\ell)^{-1}(\mathscr{Y}_f)$$

For sufficiently small $\ell > 0$ we consider the sets

$$T_\ell = \{k \in \mathbb{Z}^3 | \mathscr{Y}_{S_k}^\ell \subseteq \Omega_L\}$$

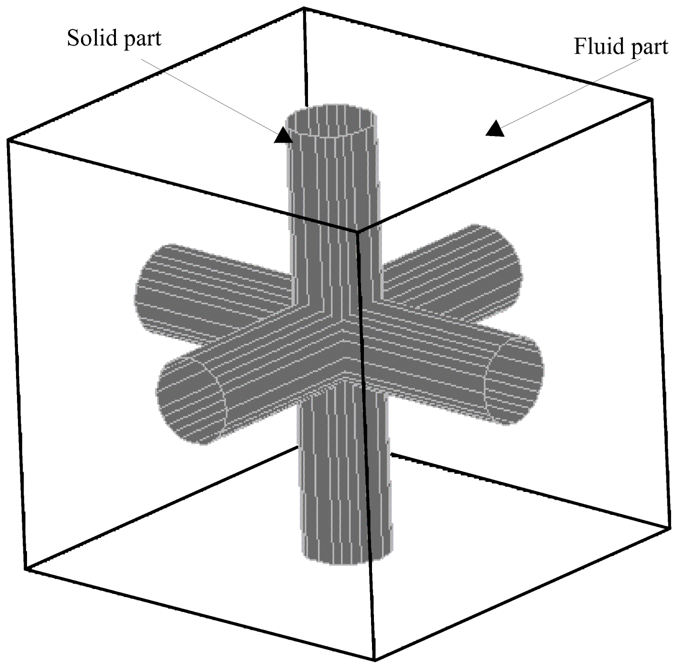

Figure 2: *A pore with its solid part.*

and define

$$\Omega_s = \bigcup_{k \in T_\ell} \mathscr{Y}^\ell_{S_k}, \quad \Gamma = \partial\Omega_s \setminus \partial\Omega_L, \quad \Omega_f = \Omega_L \setminus \Omega_s.$$

The domains Ω_s and Ω_f represent, respectively, the solid and fluid parts of a porous medium Ω_L. For simplicity we suppose $L/\ell \in \mathbb{N}$.

By assumption the porous medium Ω_L is in contact with an elastic medium (**the non-pay zone**), $\Omega_{el} = (0,L)^2 \times (-L,0)$. They are separated by a **contact interface** $\Sigma = (0,L)^2 \times \{0\}$.

The complete domain under consideration is $\Omega_{total} = \Omega_L \cup \Sigma \cup \Omega_{el}$, shown in Figure 3 .

2.2 Pore level first principles fluid-structure model

We now introduce a dimensional form of the equations. The proposed first principle, pore level, model is based on a set of characteristic values for the physical parameters. This leads to a reduced model, allowing only the most important physics of the problem. The most important simplifications which could be done were:

a) dropping the inertial term in the flow equation,

b) linearization due to the linear elastic nature of the solid skeleton and

c) linearization of the fluid-solid interface conditions.

Furthermore, there is a relationship between the non-dimensional numbers and the typical size of the homogeneities. This plays a decisive role in the structure of the effective equations obtained in the limit when the scale parameter ε tends to zero.

Our geometry corresponds to the initial Lagrangean configuration.

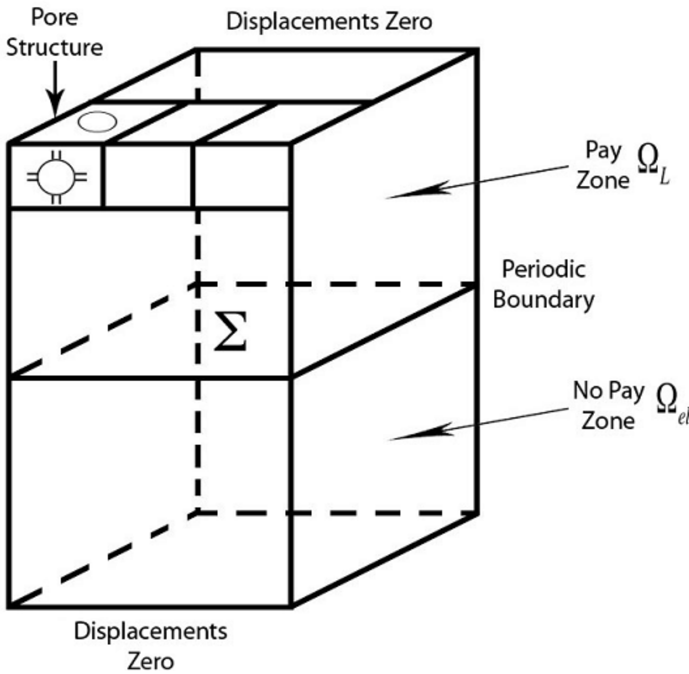

Figure 3: Ω_{total}, *Pay zone (Ω_L), Non-pay zone (Ω_{el}), Interface* Σ *and Boundary Connditions.*

More precisely, we suppose that the solid part of the porous medium Ω_L is a *linear elastic solid continuum* and start by recalling the basic equations:

Let $e(\mathbf{w})$ be the strain tensor defined by

$$(e(\mathbf{w}))_{i,j} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right), \quad i,j = 1,2,3$$

and $\sigma(\mathbf{w})$ be the stress tensor

$$\sigma(\mathbf{w}) = Ae(\mathbf{w}). \tag{2.1}$$

In the case when the solid structure is homogeneous and isotropic, the elasticity coefficients A are given by the Lamé coefficients λ and μ and the stress tensor has the form:

$$\sigma(\mathbf{w}) = \lambda \nabla \cdot (\mathbf{w})I + 2\mu e(\mathbf{w}) = \frac{\nu\Lambda}{(1+\nu)(1-2\nu)}\nabla \cdot (\mathbf{w})I + \frac{\Lambda}{1+\nu}e(\mathbf{w}), \tag{2.2}$$

where Λ is Young's modulus and ν Poisson' ratio.

The pore space is filled with a slightly compressible viscous Newtonian fluid with viscosity η and density ρ_f. If the fluid bulk modulus is E_f than the mass conservation equation has the form

$$\frac{1}{\rho_f E_f}\frac{\partial p}{\partial t} + \text{ div } \frac{\partial \mathbf{u}_f}{\partial t} = 0, \tag{2.3}$$

where p is the fluid pressure and $\mathbf{u}_f$ is the fluid displacement.

We suppose that the momentum equation reduces to the non-stationary Stokes system in $\{\partial_t \mathbf{u}_f, p\}$.

Due to the linearization of the interface fluid/structure, we impose the displacements continuity and the contact forces continuity at the fixed interface Γ. $\mathbf{n}$ is the exterior unit normal.

Our system reads as follows:

$$\rho_f \frac{\partial^2 \mathbf{u}_f}{\partial t^2} + \nabla p = \eta \Delta \frac{\partial \mathbf{u}_f}{\partial t} + \rho_f \mathbf{F} \ \text{in} \quad \Omega_f \tag{2.4}$$

$$\frac{p}{\rho_f E_f} + \nabla \cdot \mathbf{u}_f = 0 \ \text{in} \quad \Omega_f \tag{2.5}$$

$$\rho_s \frac{\partial^2 \mathbf{u}_s}{\partial t^2} = \ \text{div}(Ae(\mathbf{u}_s)) + \rho_s \mathbf{F} \ \text{in} \quad \Omega_s \cup \Omega_{el} \cup (\Sigma \cap \overline{\Omega}_s) \tag{2.6}$$

$$\mathbf{u}_s = \mathbf{u}_f \ \text{on} \quad \Gamma \cup (\Sigma \cap \overline{\Omega}_f) \quad \text{(displacement continuity at the interface)} \tag{2.7}$$

$$\sigma^f = -pI + 2\eta e(\frac{\partial \mathbf{u}_f}{\partial t}) \quad \text{(fluid stress)} \tag{2.8}$$

$$\sigma^s = Ae(\mathbf{u}_s) \quad \text{(stress in solid)} \tag{2.9}$$

$$(-pI + 2\eta e(\frac{\partial \mathbf{u}_f}{\partial t}))\mathbf{n} = Ae(\mathbf{u}_s)\mathbf{n} \ \text{on} \, \Gamma \cup (\Sigma \cap \overline{\Omega}_f). \tag{2.10}$$

It is natural to set

$$\mathbf{u} = \begin{cases} \mathbf{u}_s, & \text{in } \Omega_s \cup \Sigma \cup \Omega_{el} \\ \mathbf{u}_f, & \text{in } \Omega_f. \end{cases} \tag{2.11}$$

We suppose that $L >> \ell$ and we include the nonhomogeneous boundary conditions in the forcing term $\mathbf{F}$. At $t = 0$ we suppose for simplicity that displacements and velocities are zero:

$$\mathbf{u}|_{\{t=0\}} = \partial_t \mathbf{u}|_{\{t=0\}} = 0 \text{ on } (0,L)^2 \times (-L,L). \tag{2.12}$$

Next we suppose that

$$\{\mathbf{u}, p\} \quad \text{is periodic in } (x_1, x_2) \ \text{with period} \quad L; \tag{2.13}$$

$$\mathbf{u} = 0 \quad \text{on} \quad \{x_3 = L\} \cup \{x_3 = -L\}. \tag{2.14}$$

2.3 The rigorously obtained dimensional upscaled model and the interface conditions

Let the effective tensorial coefficients A^H, $\mathscr{B}^H$ and K be defined by (5.6), (5.7) and (5.9), respectively. Following the results from Section 5, they are positive definite and symmetric. Let the constant M be defined by $M = \kappa_{co}|\mathscr{Y}_f| + M_0$, with M_0 given by (5.11). They have the following meaning in Biot's theory:

SYMBOL	*QUANTITY*	*UNITY*
$\mathscr{K} = K\ell^2$	permeability	Darcy
$\lvert\mathscr{Y}_f\rvert I - \mathscr{B}^H$	pressure-storage coupling coefficient	dimensionless
ρ_s	solid grain density	kg/m^3
η	fluid viscosity	$kg/$m sec
$\lvert\mathscr{Y}_f\rvert$	porosity	$0 < \lvert\mathscr{Y}_f\rvert < 1$
M	combined porosity and compressibility of the fluid and solid	dimensionless
$\mathscr{G} = A^H \Lambda$	Gassman's tensor	Pa

Table 1: *Effective coefficients*

Then in the limit when $\varepsilon = \ell / L_{obs} \to 0$, the system (2.4)-(2.14) can be approximated by the following

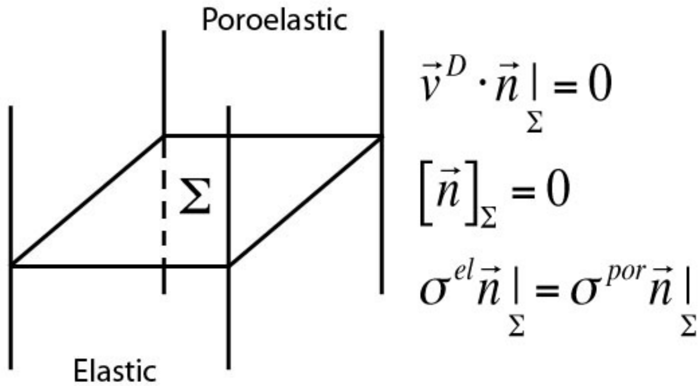

Figure 4: *Interface Conditions.*

elliptic-parabolic system of PDEs, valid on $(0,L)^2 \times (-L,L)$:

$$\sigma^{por} = \mathscr{G} e_x(\mathbf{u})\} - (|\mathscr{Y}_f| I - \mathscr{B}^H) p^0 \quad \text{in } (0,L)^3 \times (0,T), \text{(the total poroelasticity tensor)}, \tag{2.15}$$

$$-\operatorname{div}_x\{\sigma^{por}\} = \bar{\rho}\mathbf{F}(x,t) \quad \text{in } (0,L)^3 \times (0,T), \tag{2.16}$$

$$\sigma^{el} = A e_x(\mathbf{u}) \quad \text{in } (0,L)^3 \times (0,T), \text{(the elasticity tensor)}, \tag{2.17}$$

$$-\operatorname{div}_x\{\sigma^{el}\} = \rho_s \mathbf{F}(x,t) \quad \text{in } (0,L)^2 \times (-L,0) \times (0,T), \tag{2.18}$$

$$[\mathbf{u}]_\Sigma = 0 \quad \text{and} \quad \sigma^{por}\mathbf{e}^3|_\Sigma = \sigma^{el}\mathbf{e}^3|_\Sigma \quad \text{for all } t \in (0,T), \text{ (the displacements-stress interface conditions)}, \tag{2.19}$$

$$\mathbf{v}^D = \frac{\mathscr{K}}{\eta}(\rho_f \mathbf{F} - \nabla_x p^0) \quad \text{in } (0,L)^3 \times (0,T), \text{ (Darcy's law for effective relative velocity)}, \tag{2.20}$$

$$\partial_t\Big(\frac{M}{\Lambda} p^0 + \operatorname{div}_x((|\mathscr{Y}_f| I - \mathscr{B}^H)\mathbf{u}\Big) + \operatorname{div}_x\{\mathbf{v}^D\} = 0 \quad \text{in } (0,L)^3 \times (0,T), \tag{2.21}$$

$$\mathbf{v}^D \cdot \mathbf{e}^3 = 0 \quad \text{on } (\Sigma \cup \{x_3 = L\}) \times (0,T), \text{ (the non-penetration condition at the interface)}, \tag{2.22}$$

$$p^0|_{t=0} = 0 \quad \text{on } (0,L)^3; \quad \mathbf{u} = 0 \quad \text{on } \{x_3 = \pm L\} \times (0,T), \tag{2.23}$$

$$\{\mathbf{u}, p^0\} \quad \text{is periodic in } (x_1,x_2) \quad \text{with period } L. \tag{2.24}$$

Remark 1 The expression $\frac{M}{\Lambda} p^0 + \operatorname{div}_x((|\mathscr{Y}_f| I - \mathscr{B}^H)\mathbf{u}$ represents the change in the effective porosity.

The interface conditions are summarized in Figure 4. The properties are summarized below:

Proposition 1 Let $\mathbf{F} \in H_0^2(\mathbb{R}_+; L^2(\Omega)^3)$. Let $L_{obs} \approx L$ be the observation length and let $V_L = \{\varphi \in H^1(\Omega_L) \mid \varphi$ is periodic in (x_1,x_2) with period $L\}$. The homogenized equations read (2.15)-(2.24) or in variational form

$$\text{Find } \{\mathbf{u}, p^0\} \in H^2(0,T;V) \times H^2(0,T;V_1) \text{ such that}$$

$$\int_{\Omega_L} \Big(\mathscr{G} e_x(\mathbf{u}) - (|\mathscr{Y}_f| I - \mathscr{B}^H) p^0 I\Big) : e_x(\varphi)\, dx + \int_{\Omega_{el}} A e_x(\mathbf{u}) : e_x(\varphi)\, dx =$$
$$\int_{\Omega_L} \bar{\rho}\mathbf{F}\varphi\, dx + \int_{\Omega_{el}} \rho_s \mathbf{F}\varphi\, dx, \quad \varphi \in V_L; \tag{2.25}$$

$$\frac{\partial}{\partial t}\int_{\Omega_L} \frac{M}{\Lambda} p^0 \xi\, dx + \int_{\Omega_L} (|\mathscr{Y}_f| I - \mathscr{B}^H)\xi : e_x(\frac{\partial \mathbf{u}}{\partial t})\, dx - \int_{\Omega_L} \frac{\mathscr{K}}{\eta}(\rho_f \mathbf{F} - \nabla_x p^0)\nabla\xi\, dx = 0, \ \forall \xi \in V_L \tag{2.26}$$

$$p^0|_{t=0} = 0 \quad \text{on } \Omega_L. \tag{2.27}$$

System (2.25)-(2.27) has a unique solution.

Proof. It is sufficient to establish uniqueness. For $\mathbf{F}=0$ we set $\varphi=\mathbf{u}$ and $\xi=p^0$ in (2.25)-(2.27) and add the resulting equalities. We thus obtain

$$\int_{\Omega_L} \mathscr{G} e_x(\mathbf{u}) : e_x(\mathbf{u})\, dx + \int_{\Omega_{el}} A e_x(\mathbf{u}) : e_x(\mathbf{u})\, dx + \frac{M}{2\Lambda}\frac{d}{dt}\int_{\Omega_L} (p^0(t))^2\, dx + \int_{\Omega_L} \frac{\mathscr{K}}{\eta} \nabla p^0(t)\cdot\nabla p^0(t)\, dx = 0, \tag{2.28}$$

implying $\mathbf{u}=0$ and $p^0=0$. □

Theorem 1 Let the functions $\{\mathbf{w}^{ij},\mathbf{w}^0,\mathbf{q}^i\}$, $i,j=1,2,3$, be given by 5.1, 5.2 and 5.4, respectively. Let $\mathbf{F}\in H^2_0(\mathbb{R}_+;L^2(\Omega)^3)$. Then, in the limit $\varepsilon=\dfrac{\ell}{L_{obs}}\to 0$, we have

$$\int_0^t\int_{\Omega_f}\left|\ell\partial_t\nabla\mathbf{u}_f(x,\tau)-\sum_{j=1}^3\nabla_y\mathbf{q}^j(\frac{x}{\ell})\frac{\ell^2}{\eta}(F_j(x,\tau)\rho_f-\frac{\partial p^0(x,\tau)}{\partial x_j})\right|^2\frac{\eta\, dx d\tau}{\Lambda\ell^4 L_{obs}}\to 0; \tag{2.29}$$

$$\int_0^t\int_{\Omega_f}\left|\partial_t\mathbf{u}_f(x,\tau)-\partial_t\mathbf{u}(x,\tau)-\sum_{j=1}^3\mathbf{q}^j(\frac{x}{\ell})\frac{\ell^2}{\eta}(F_j(x,\tau)\psi_f-\frac{\partial p^0(x,\tau)}{\partial x_j})\right|^2\frac{\eta\, dx d\tau}{\Lambda\ell^4 L_{obs}}\to 0; \tag{2.30}$$

$$\int_0^t\int_{\Omega_f}\left|\operatorname{div}\mathbf{u}_f(x,\tau)+\frac{p^0(x,\tau)}{\rho_f E_f}\right|^2\frac{\Lambda\, dx d\tau}{\eta L^3_{obs}}\to 0. \tag{2.31}$$

Furthermore, for every $t>0$ the limit $\varepsilon=\dfrac{\ell}{L_{obs}}\to 0$ yields

$$\frac{1}{L^3_{obs}}\int_{\Omega_s}|\mathbf{u}_s(x,t)-\mathbf{u}(x,t)|^2\, dx\le C\frac{\ell^4}{L^2_{obs}}; \tag{2.32}$$

$$\int_{\Omega_s} Ae\Big(\mathbf{u}_s(x,t)-\mathbf{u}(x,t)-\ell\sum_{i,j=1}^3 e_{ij}(\mathbf{u}(x,t))\mathbf{w}^{ij}(\frac{x}{\ell})-\ell p^0(x,t)\mathbf{w}^0(\frac{x}{\ell})\Big) : e\Big(\mathbf{u}_s(x,t)-\mathbf{u}(x,t)-\ell p^0(x,t)\mathbf{w}^0(\frac{x}{\ell}) -\ell\sum_{i,j=1}^3 e_{ij}(\mathbf{u}(x,t))\mathbf{w}^{ij}(\frac{x}{\ell})\Big)\, dx+\int_{\Omega_{el}} Ae\left(\mathbf{u}_s(x,t)-\mathbf{u}(x,t)\right) : e\left(\mathbf{u}_s(x,t)-\mathbf{u}(x,t)\right)\frac{dx}{\Lambda\ell^2 L^2_{obs}}\to 0. \tag{2.33}$$

We next establish Theorem 3, which is a dimensionless version of Theorem 1.

3 The dimensionless model

SYMBOL	*QUANTITY*	*CHARACTERISTIC VALUE*
Λ	Young's modulus	7e9 Pa
ρ_f	fluid density	1e3 kg/m^3
ρ_s	solid grain density	2.65e3 kg/m^3
η	fluid viscosity	1e−3 $kg/$m sec
ℓ	typical pore size	1e−5 m
d	characteristic displacement	$d<\ell$
L_{obs}	observation length	5000 m
ε	small parameter	$\varepsilon=\ell/L_{obs}=0.2\text{e}{-8}$
P	characteristic fluid pressure	$P=\Lambda\varepsilon$
E_f	pore fluid bulk modulus	1e6 Pa

Table 2: *Data description*

3.1 Derivation of the dimensionless equations

Let T_c be the characteristic time, L_{obs} the observation length, Λ be the characteristic size of the elastic moduli and P the characteristic fluid pressure. For the elastic displacement we suppose that its characteristic size is $d = \ell$ and expect that it is not oscillatory at the leading order. In the porous medium we expect the pressure to be the dominant part of the fluid stress and to balance the elastic contact force at the interfaces. Consequently, the dimensionless small parameter is $\varepsilon = \ell/L_{obs} = d/L_{obs}$ and (2.10) implies that

$$P = \Lambda\varepsilon. \tag{3.1}$$

With this choice and for simplicity, changing the variables and the unknowns by

$$\begin{aligned} &t \to T_c t,\ x \to L_{obs}x,\ \Omega_s \to \Omega_s^\varepsilon,\ \Omega_f \to \Omega_f^\varepsilon, \Omega_{el} \to L_{obs}\Omega_{el}\ , \Sigma \to L\Sigma,\ A \to \Lambda A,\ \Gamma \to L\Gamma^\varepsilon, \\ &\Omega_L \to L_{obs}\Omega_1,\ \mathbf{u} \to \ell\mathbf{u}^\varepsilon, \mathbf{F} \to F_0\mathbf{F}\ \text{ and } p \to Pp^\varepsilon \end{aligned} \tag{3.2}$$

the system (2.4)-(2.6) becomes

$$\kappa_s \frac{\partial^2 \mathbf{u}_s^\varepsilon}{\partial t^2} = \operatorname{div}(Ae(\mathbf{u}_s^\varepsilon)) + \frac{\rho_s L_{obs}^2 F_0}{\Lambda\ell}\mathbf{F} \ \text{ in } \Omega_s^\varepsilon \cup \Omega_{el} \tag{3.3}$$

$$\kappa_f \frac{\partial^2 \mathbf{u}_f^\varepsilon}{\partial t^2} + \nabla p^\varepsilon = \frac{\eta}{\Lambda T_c}\Delta\frac{\partial \mathbf{u}_f^\varepsilon}{\partial t} + \frac{\rho_f L_{obs}^2 F_0}{\Lambda\ell}\mathbf{F} \ \text{ in } \Omega_f^\varepsilon, \tag{3.4}$$

$$\kappa_{co} p^\varepsilon + \operatorname{div} \mathbf{u}_f^\varepsilon = 0 \ \text{ in } \Omega_f^\varepsilon, \tag{3.5}$$

with

$$\kappa_{co} = \frac{\Lambda}{\rho_f E_f}, \quad \kappa_f = \frac{\rho_f}{\Lambda}\Big(\frac{L_{obs}}{T_c}\Big)^2, \quad \kappa_s = \frac{\rho_s}{\Lambda}\Big(\frac{L_{obs}}{T_c}\Big)^2. \tag{3.6}$$

Next we choose the characteristic time by setting

$$T_c = \eta/(\Lambda\varepsilon^2) \qquad \text{(Terzaghi's time).} \tag{3.7}$$

We note that with such a choice of T_c the velocity of the solid structure deformation ℓ/T_c equals the Darcy velocity $\dfrac{\ell^2 P}{\eta L_{obs}} = \dfrac{\ell\Lambda\varepsilon^2}{\eta}$. Hence the velocity in both fluid and solid are of the same order and we expect to keep track of both of them after the upscaling process. We note that the choice of T_c is equivalent to the choice of the dimensionless size parameter $\varepsilon = \ell/L_{obs} = \sqrt{\dfrac{\eta}{\Lambda T_c}}$.

In applications to large reservoirs we anticipate that L_{obs} of significant size and T_c not too short. With our data from Table 3 we have $T_c = 0.41$ days. We note that T_c increases as L_{obs}^2.

Therefore we then observe that the coefficients describing acceleration in dimensionless form satisfy

$$\begin{cases} \kappa_f = \dfrac{\rho_f}{\Lambda}\Big(\dfrac{L_{obs}}{T_c}\Big)^2 = \rho_f\Lambda\Big(\dfrac{\ell\varepsilon}{\eta}\Big)^2 = \dfrac{\rho_f \ell^2}{\eta T_c} = 0.28\ \mathrm{e}-8 = \kappa_f^0\varepsilon,\ \kappa_f^0 = O(1); \\ \kappa_s = \dfrac{\rho_s}{\Lambda}\Big(\dfrac{L_{obs}}{T_c}\Big)^2 = \rho_s\Lambda\Big(\dfrac{\ell\varepsilon}{\eta}\Big)^2 = \dfrac{\rho_s \ell^2}{\eta T_c} = 0.742\ \mathrm{e}-8 = \kappa_s^0\varepsilon,\ \kappa_s^0 = O(1); \\ \kappa_{co} = \dfrac{\Lambda}{\rho_f E_f} = 7 = O(1). \end{cases} \tag{3.8}$$

Concerning the forcing terms, we have

$$\psi_f = \rho_f \frac{F_0 L_{obs}^2}{\ell\Lambda} = O(1) = \psi_s = \rho_s \frac{F_0 L_{obs}^2}{\ell\Lambda}. \tag{3.9}$$

Summarizing the above results we obtained the following dimensionless system of PDEs in $\Omega = \Omega_1 \cup \Sigma \cup \Omega_{el}$:

$$\varepsilon \kappa_f^0 \frac{\partial^2 \mathbf{u}^\varepsilon}{\partial t^2} + \nabla p^\varepsilon = \varepsilon^2 \Delta \frac{\partial \mathbf{u}^\varepsilon}{\partial t} + \psi_f \mathbf{F} \ \text{ in } \quad \Omega_f^\varepsilon \tag{3.10}$$

$$\kappa_{co} p^\varepsilon + \nabla \cdot \mathbf{u}^\varepsilon = 0 \ \text{ in } \quad \Omega_f^\varepsilon \tag{3.11}$$

$$\varepsilon \kappa_s^0 \frac{\partial^2 \mathbf{u}^\varepsilon}{\partial t^2} = \ \operatorname{div}(Ae(\mathbf{u}^\varepsilon)) + \psi_s \mathbf{F} \ \text{ in } \quad \Omega_s^\varepsilon \cup \Omega_{el} \cup (\Sigma \cap \overline{\Omega_s^\varepsilon}) \tag{3.12}$$

$$[\mathbf{u}^\varepsilon] = 0 \ \text{ on } \quad \Gamma^\varepsilon \cup (\Sigma \cap \overline{\Omega_f^\varepsilon}) \tag{3.13}$$

$$\sigma^{f,\varepsilon} = -p^\varepsilon I + 2\varepsilon^2 e(\frac{\partial \mathbf{u}^\varepsilon}{\partial t}) \ \text{ in } \quad \Omega_f^\varepsilon \tag{3.14}$$

$$\sigma^{s,\varepsilon} = Ae(\mathbf{u}^\varepsilon) \ \text{ in } \quad \Omega_s^\varepsilon \cup \Omega_{el} \cup (\Sigma \cap \overline{\Omega_s^\varepsilon}) \tag{3.15}$$

$$(-p^\varepsilon I + 2\varepsilon^2 e(\frac{\partial \mathbf{u}^\varepsilon}{\partial t}))\mathbf{n} = Ae(\mathbf{u}^\varepsilon)\mathbf{n} \ \text{ on } \quad \Gamma^\varepsilon \cup (\Sigma \cap \overline{\Omega_f^\varepsilon}), \tag{3.16}$$

$$\mathbf{u}^\varepsilon|_{\{t=0\}} = \partial_t \mathbf{u}^\varepsilon|_{\{t=0\}} = 0 \text{ on } \quad \Omega_L \cup \Sigma \cup \Omega_{el}, \tag{3.17}$$

$$\mathbf{u}^\varepsilon = 0 \quad \text{on} \quad \{x_3 = L/L_{obs}\} \cup \{x_3 = -L/L_{obs}\}, \tag{3.18}$$

$$\{\mathbf{u}^\varepsilon, p^\varepsilon\} \quad \text{ is periodic in } (x_1, x_2) \text{ with period } \quad \frac{L}{L_{obs}}. \tag{3.19}$$

Let $\Omega = \Omega_1 \cup \Sigma \cup \Omega_{el}$. The functional space corresponding to (3.10)-(3.19) is

$$V = \Big\{ \varphi \in H^1(\Omega)^3 \,|\quad \varphi \text{ is periodic in } (x_1, x_2) \text{ with period } \quad \frac{L}{L_{obs}} \\ \text{and } \ \varphi = 0 \text{ on } \{x_3 = L/L_{obs}\} \cup \{x_3 = -L/L_{obs}\} \Big\} \tag{3.20}$$

The variational formulation which corresponds to (3.10)-(3.19) is given by:

Find $\mathbf{u}^\varepsilon \in H^1(0,T;V)$ with $\dfrac{d^2\mathbf{u}^\varepsilon}{dt^2} \in L^2(0,T;L^2(\Omega)^3)$ such that

$$\frac{d^2}{dt^2} \int_\Omega \varepsilon \kappa^\varepsilon \mathbf{u}^\varepsilon(t) \varphi \, dx + \frac{d}{dt} \int_{\Omega_f^\varepsilon} 2\varepsilon^2 e(\mathbf{u}^\varepsilon(t)) : e(\varphi)\, dx \\ + \int_{\Omega_s^\varepsilon \cup \Omega_{el} \cup (\Sigma \cap \overline{\Omega_s^\varepsilon})} Ae(\mathbf{u}^\varepsilon(t)) : e(\varphi)\, dx + \int_{\Omega_f^\varepsilon} \frac{\operatorname{div}\mathbf{u}^\varepsilon}{\kappa_{co}} \operatorname{div}\varphi \, dx \ = \ \int_\Omega \psi^\varepsilon \mathbf{F} \varphi \, dx, \\ \forall \varphi \in H^1_{per}(\Omega)^3, \qquad (a.e.) \text{ in }]0,T[, \tag{3.21}$$

where

$$\kappa^\varepsilon = \kappa_f^0 \chi_{\Omega_f^\varepsilon} + \kappa_s^0 \chi_{\Omega_s^\varepsilon}, \quad \psi^\varepsilon = \psi_f \chi_{\Omega_f^\varepsilon} + \psi_s \chi_{\Omega_s^\varepsilon} \tag{3.22}$$

and with initial conditions (3.17).

4 Uniform a priori estimates and two-scale convergence result

4.1 A Priori Estimates for (3.21), (3.17)

For proving the a priori estimates, we need the following auxiliary results:

Lemma 1 (see e.g. [28]) Let $\varphi \in V(\Omega_f^\varepsilon) = \{\varphi \in H^1(\Omega_f^\varepsilon)^3 \,| \varphi = 0 \ \text{ on } \ \Gamma^\varepsilon |\ \varphi \text{ is periodic in } (x_1, x_2) \text{ with period } L/L_{obs} \}$. Then we have

$$\|\varphi\|_{L^2(\Omega_f^\varepsilon)^3} \le C\varepsilon \|\nabla_x \varphi\|_{L^2(\Omega_f^\varepsilon)^9}. \tag{4.1}$$

Next let $\varphi \in H^1(\Omega_1)^3$ be periodic in (x_1, x_2) with period L/L_{obs} and such that $\varphi = 0$ on $\{x_3 = a\} \cap \overline{\Omega^\varepsilon}_s$ for some $a \in [0, L/L_{obs}]$. Then we have

$$\|\varphi\|_{L^2(\{x_3=a\})^3} \le C\varepsilon^{1/2} \|e(\varphi)\|_{L^2(\Omega_1)^9}. \tag{4.2}$$

Lemma 2 (see [27]) Under the hypothesis on the geometry from subsection 2.1, there is a linear extension operator $P_\varepsilon : H^1(\Omega_s^\varepsilon)^3 \to H^1(\Omega_1)^3$ such that

$$P_\varepsilon \eta = \eta, \quad \forall \eta \in \mathscr{R} = \{\eta \in \mathbb{R}^3 \mid \eta = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \} \text{ (rigid body motions)} \tag{4.3}$$

$$||P_\varepsilon \mathbf{v}||_{H^1(\Omega_1)^3} \le c_1 ||\mathbf{v}||_{H^1(\Omega_s^\varepsilon)^3}, \tag{4.4}$$

$$||P_\varepsilon \mathbf{v}||_{L^2(\Omega_1)^3} + ||e(P_\varepsilon \mathbf{v})||_{L^2(\Omega_1)^9} \le c_2 \bigg(||\mathbf{v}||_{L^2(\Omega_s^\varepsilon)^3} + ||e(\mathbf{v})||_{L^2(\Omega_s^\varepsilon)^9} \bigg) \tag{4.5}$$

$$||\nabla P_\varepsilon \mathbf{v}||_{L^2(\Omega_1)^9} \le c_3 ||\nabla \mathbf{v}||_{L^2(\Omega_s^\varepsilon)^9}, \tag{4.6}$$

$$||e(P_\varepsilon \mathbf{v})||_{L^2(\Omega_1)^9} \le c_4 ||e(\mathbf{v})||_{L^2(\Omega_s^\varepsilon)^9} \tag{4.7}$$

for every $\mathbf{v} \in H^1(\Omega_s^\varepsilon)^3$, with the constants $c_1, \ldots, c_4$ independent on ε and $\mathbf{v}$.

Proposition 2 Let $\xi \in C([0,T], H^1(\Omega_1))^3$ such that

$\xi|_{t=0} = 0$, ξ is periodic in (x_1, x_2) with period L/L_{obs} and $\xi = 0$, for a.e. $(t,x) \in (0,T) \times \{x_3 = L/L_{obs}\}$.

Then the following estimate holds for all $t \in [0,T]$, with a constant C independent of ε,

$$||\xi(t)||_{L^2(\Omega_1)^3} \le C \left\{ ||e(\xi(t))||_{L^2(\Omega_s^\varepsilon)^9} + \varepsilon \int_0^t ||e(\partial_\tau \xi(\tau)||_{L^2(\Omega_f^\varepsilon)^9} d\tau \right\}. \tag{4.8}$$

Remark 2 We prove (4.8) using a particular H^1-extension. Nevertheless, we note that this extension will not satisfy the boundary conditions at the outer boundary and at the interface. This makes the proof non-trivial.

Proof. For convenience of the reader we provide a proof similar to one from [20]. For every $t \in [0,T]$, let $\hat{\xi}(t) = P_\varepsilon \xi$ be the H^1-extension of $\xi\big|_{\Omega_s^\varepsilon}$ to Ω_1, as in Lemma 2. Let $\omega(t) = \hat{\xi}(t) - \xi(t)$ on Ω_f^ε and 0 elsewhere. Then for every $t \in (0,T)$, we have $\omega(t) \in H^1(\Omega_f^\varepsilon)^3$ and $\omega(t)$ is periodic in (x_1, x_2). Due to the periodicity, only possible rigid body motion is translation by a constant vector. Thus Poincare's inequality (4.1) together with the Second Korn Inequality imply that

$$||\omega(t)||_{L^2(\Omega_f^\varepsilon)^3} \le C\varepsilon ||\nabla \omega(t)||_{L^2(\Omega_f^\varepsilon)^9} \le C\varepsilon ||e(\omega(t))||_{L^2(\Omega_f^\varepsilon)^9}, \tag{4.9}$$

for all $t \in (0,T)$. Using (4.2), (4.9), and the properties of the extension $\hat{\xi}$ we obtain

$$\begin{aligned} &||\xi(t)||_{L^2(\Omega_f^\varepsilon)^3} \le ||\hat{\xi}(t)||_{L^2(\Omega_f^\varepsilon)^3} + C\varepsilon \left\{ ||e(\hat{\xi}(t))||_{L^2(\Omega_f^\varepsilon)^9} + ||e(\xi(t))||_{L^2(\Omega_f^\varepsilon)^9} \right\} \\ &\le C| \int_{\{x_3 = L/L_{obs}\}} \hat{\xi}(t)\, dS| + C||e(\xi(t))||_{L^2(\Omega_s^\varepsilon)^9} + C\varepsilon ||e(\xi(t))||_{L^2(\Omega_f^\varepsilon)^9} \\ &\le C||e(\xi(t))||_{L^2(\Omega_s^\varepsilon)^9} + C\varepsilon ||e(\xi(t))||_{L^2(\Omega_f^\varepsilon)^9} \end{aligned} \tag{4.10}$$

Next, we remark that using $\xi(0) = 0$, we have

$$||e(\xi(t))||_{L^2(\Omega_f^\varepsilon)^9} \le || \int_0^t e(\partial_\tau \xi(\tau)) d\tau ||_{L^2(\Omega_f^\varepsilon)^9} \le \int_0^t ||e(\partial_\tau \xi(\tau))||_{L^2(\Omega_f^\varepsilon)^9} d\tau. \tag{4.11}$$

Estimates (4.10), and (4.11) now imply (4.8), and the proposition is proved. □

Proposition 3 Let $\tilde{\Omega_s^\varepsilon} = \Omega_s^\varepsilon \cup \Omega_{el} \cup (\Sigma \cap \overline{\Omega_s^\varepsilon})$ and let us suppose $\mathbf{F} \in H_0^2(\mathbb{R}_+; L^2(\Omega)^3)$. Then we have

$$\|\frac{\partial^3 \mathbf{u}^\varepsilon}{\partial t^3}\|_{L^\infty(0,T;L^2(\Omega)^3)} + \|e\left(\frac{\partial^2 \mathbf{u}^\varepsilon}{\partial t^2}\right)\|_{L^\infty(0,T;L^2(\tilde{\Omega_s^\varepsilon})^9)} + \| \text{ div } \frac{\partial^3 \mathbf{u}^\varepsilon}{\partial t^3}\|_{L^\infty(0,T;L^2(\Omega_f^\varepsilon)^9)} \leq C, \tag{4.12}$$

$$\|e\left(\frac{\partial^3 \mathbf{u}^\varepsilon}{\partial t^3}\right)\|_{L^2(0,T;L^2(\Omega_f^\varepsilon)^9)} \leq \frac{C}{\varepsilon}. \tag{4.13}$$

Proof. We take $\varphi = \partial_t \mathbf{u}^\varepsilon$ as test function in (3.21). This yields

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \varepsilon \kappa^\varepsilon |\partial_t \mathbf{u}^\varepsilon(t)|^2\, dx + \int_{\tilde{\Omega_s^\varepsilon}} Ae(\mathbf{u}^\varepsilon(t)) : e(\mathbf{u}^\varepsilon(t))\, dx + \int_{\Omega_f^\varepsilon} \frac{|\text{ div } \mathbf{u}^\varepsilon|^2}{\kappa_{co}}\, dx\right) +$$
$$\int_{\Omega_f^\varepsilon} 2\varepsilon^2 |e(\partial_t \mathbf{u}^\varepsilon(t))|^2\, dx \;=\; \int_\Omega \psi^\varepsilon \mathbf{F} \partial_t \mathbf{u}^\varepsilon(t)\, dx \tag{4.14}$$

Let $\hat{\mathbf{u}^\varepsilon}$ be the H^1-extension of $\mathbf{u}^\varepsilon$ as in [11], [1] , [21] and [27]. We apply Proposition 2 to obtain

$$|\int_0^t \int_\Omega \psi^\varepsilon \mathbf{F} \partial_\tau \mathbf{u}^\varepsilon(\tau)\, dx d\tau| \leq |\int_\Omega \psi^\varepsilon \mathbf{F} \mathbf{u}^\varepsilon(t)\, dx| + |\int_0^t \int_\Omega \psi^\varepsilon \partial_\tau \mathbf{F} \mathbf{u}^\varepsilon(\tau)\, dx d\tau|$$
$$\leq C\left(\|e(\mathbf{u}^\varepsilon(t))\|_{L^2(\tilde{\Omega_s^\varepsilon})^9} + \varepsilon \int_0^t \|e\left(\frac{\partial \mathbf{u}^\varepsilon}{\partial \tau}\right)\|_{L^2(\Omega_f^\varepsilon)^9}\, dx d\tau + C_1 \int_0^t \|e(\mathbf{u}^\varepsilon(t))\|_{L^2(\tilde{\Omega_s^\varepsilon})^9}\, dx d\tau\right). \tag{4.15}$$

The equality (4.15) implies the estimate

$$\|e(\mathbf{u}^\varepsilon)\|_{L^\infty(0,T;L^2(\tilde{\Omega_s^\varepsilon})^9)} + \| \text{ div } \frac{\partial \mathbf{u}^\varepsilon}{\partial t}\|_{L^\infty(0,T;L^2(\Omega_f^\varepsilon)^9)} + \varepsilon \|e\left(\frac{\partial \mathbf{u}^\varepsilon}{\partial t}\right)\|_{L^2(0,T;L^2(\Omega_f^\varepsilon)^9)} \leq C. \tag{4.16}$$

Next let $\omega(t) = \hat{\mathbf{u}^\varepsilon}(t) - \mathbf{u}^\varepsilon(t)$ on Ω_f^ε and 0 elsewhere. Then for every $t \in (0,T)$, we have $\omega(t) \in H^1(\Omega_f^\varepsilon)^3$ and $\omega(t)$ is periodic in (x_1, x_2). Due to the periodicity, only possible rigid body motion is translation by a constant vector. Thus Poincare's inequality (4.1) together with the Second Korn Inequality imply

$$||\omega(t)||_{L^2(\Omega_f^\varepsilon)^3} \leq C\varepsilon ||\nabla \omega(t)||_{L^2(\Omega_f^\varepsilon)^9} \leq C\varepsilon ||e(\omega(t))||_{L^2(\Omega_f^\varepsilon)^9}, \tag{4.17}$$

for all $t \in (0,T)$. The estimate (4.17) gives

$$\|\frac{\partial \mathbf{u}^\varepsilon}{\partial t}\|_{L^\infty(0,T;L^2(\Omega)^3)} \leq C. \tag{4.18}$$

It remains to consider the time derivatives of one order higher.

At $t = 0$ we have

$$\begin{cases} \varepsilon \kappa_f^0 \dfrac{\partial^2 \mathbf{u}^\varepsilon}{\partial t^2}|_{t=0} = \varepsilon^2 \Delta \dfrac{\partial \mathbf{u}^\varepsilon}{\partial t}|_{t=0} + \psi_f \mathbf{F}|_{t=0} - \dfrac{1}{\kappa_{co}} \nabla \text{ div } \mathbf{u}^\varepsilon|_{t=0} = 0 \text{ in } \Omega_f^\varepsilon, \\ \varepsilon \kappa_s^0 \dfrac{\partial^2 \mathbf{u}^\varepsilon}{\partial t^2}|_{t=0} = \text{ div}(Ae(\mathbf{u}^\varepsilon))|_{t=0} + \psi_s \mathbf{F}|_{t=0} = 0 \text{ in } \tilde{\Omega_s^\varepsilon}, \end{cases} \tag{4.19}$$

implying that

$$\frac{\partial^2 \mathbf{u}^\varepsilon}{\partial t^2}|_{t=0} = 0 \quad \text{in} \quad \Omega.$$

After taking a derivative with respect to time in (3.21) and repeating the above calculations with $\mathbf{F}$ replaced by $\partial_t \mathbf{F}$, we obtain the estimate (4.13) for the intermediate time derivatives. After iterating once more the procedure we get (4.13). □

4.2 Strong and two-scale convergence for the solution to the ε-problem

The fluid displacement corresponding to the velocity field is oscillatory and the appropriate convergence is *the two-scale convergence*, developed in [3] and [25]. We recall its definition and basic properties.

Definition 1 A bounded sequence $\{w^\varepsilon\} \subset L^2(\Omega)$ is said to two-scale converge to a limit $w \in L^2(\Omega \times \mathscr{Y})$ if for any $\sigma \in C^\infty(\Omega; C^\infty_{\mathrm{per}}(\mathscr{Y}))$ ("per" denotes 1-periodicity) one has

$$\lim_{\varepsilon \to 0} \int_\Omega w^\varepsilon(x)\sigma(x, \frac{x}{\varepsilon})\, dx = \int_\Omega \int_{\mathscr{Y}} w(x,y)\sigma(x,y)\, dy\, dx.$$

Next, we give various useful properties of two-scale convergence.

Proposition 4 ([3])

1. From each bounded sequence $\{w^\varepsilon\}$ in $L^2(\Omega)$ one can extract a subsequence which two-scale converges to a limit $w \in L^2(\Omega \times \mathscr{Y})$.
2. Let w^ε and $\varepsilon \nabla w^\varepsilon$ be bounded sequences in $L^2(\Omega)$. Then there exists a function $w \in L^2\big(\Omega; H^1_{\mathrm{per}}(\mathscr{Y})\big)$ and a subsequence such that both w^ε and $\varepsilon \nabla w^\varepsilon$ two-scale converge to w and $\nabla_y w$, respectively.
3. Let w^ε two-scale converge to $w \in L^2(\Omega \times \mathscr{Y})$. Then w^ε converges weakly in $L^2(\Omega)$ to $\int_Y w(x,y)\, dy$.
4. Let $\lambda \in L^\infty_{\mathrm{per}}(\mathscr{Y}), \lambda^\varepsilon = \lambda(x/\varepsilon)$ and let a sequence $\{w^\varepsilon\} \subset L^2(\Omega)$ two-scale converge to a limit $w \in L^2(\Omega \times \mathscr{Y})$. Then $\lambda^\varepsilon w^\varepsilon$ two-scale converges to the limit λw.

Using the a priori estimates and the notion of two-scale convergence, we are able to prove our main convergence result for the solutions of the system (3.21). For this, we need to refine the basic properties of the 2-scale convergence.

If we have two different estimates for gradients in the solid and in the fluid part, then we need the corresponding two-scale compactness result. It reads

Proposition 5 Let $\{\mathbf{w}^\varepsilon\} \subset H^1(\Omega_1)^3$ be a sequence such that

$$||\mathbf{w}^\varepsilon||_{L^2(\Omega_1)^3} \leq C,\ ||\nabla \mathbf{w}^\varepsilon||_{L^2(\Omega_s^\varepsilon)^9} \leq C,\ ||\ \mathrm{div}\ \mathbf{w}^\varepsilon||_{L^2(\Omega_f^\varepsilon)} \leq C \quad \text{and} \quad ||\nabla \mathbf{w}^\varepsilon||_{L^2(\Omega_f^\varepsilon)^9} \leq \frac{C}{\varepsilon}.$$

Then there exist functions $\mathbf{w} \in H^1(\Omega_1)^3$, $\mathbf{v} \in L^2(\Omega_1; H^1_{\mathrm{per}}(\mathscr{Y}_f)^3)$, $\mathbf{v} = 0$ on $\Omega_1 \times \overline{\mathscr{Y}}_s$, $\mathrm{div}_y \mathbf{v} = 0$ in $\Omega_1 \times \mathscr{Y}_f$, $\mathrm{div}_x \int_{\mathscr{Y}_f} \mathbf{v}\, dy \in L^2(\Omega_1)$ and $\mathbf{u}^1 \in L^2(\Omega_1; H^1_{\mathrm{per}}(\mathscr{Y}_s)^3/\mathbb{R})$ such that, up to a subsequence,

$$\mathbf{w}^\varepsilon \to \mathbf{w}(x) + \chi_{\mathscr{Y}_f}(y)\mathbf{v}(x,y) \quad \text{in the 2-scale sense,} \tag{4.20}$$

$$\chi_{\Omega_s^\varepsilon} \nabla \mathbf{w}^\varepsilon \to \chi_{\mathscr{Y}_s}(y)[\nabla_x \mathbf{w}(x) + \nabla_y \mathbf{u}^1(x,y)] \quad \text{in the 2-scale sense,} \tag{4.21}$$

$$\varepsilon \chi_{\Omega_f^\varepsilon} \nabla \mathbf{w}^\varepsilon \to \chi_{\mathscr{Y}_f}(y) \nabla_y \mathbf{v}(x,y) \quad \text{in the 2-scale sense.} \tag{4.22}$$

If in addition for every $\varphi \in C^\infty(\overline{\Omega}_1)$ and every $\zeta \in H^1_{per}(\mathscr{Y})^3$, such that supp $\zeta \subset \mathscr{Y}_f$, we have

$$|\int_{\Omega_1} \mathrm{div}\ \mathbf{w}^\varepsilon \varphi(x) \mathrm{div}\,_y \zeta(\frac{x}{\varepsilon})\, dx| \leq C\varepsilon, \tag{4.23}$$

then the following convergence hold:

$$\chi_{\Omega_f^\varepsilon} \mathrm{div}\ \mathbf{w}^\varepsilon \to \chi_{\mathscr{Y}_f}(y)\big(\mathrm{div}\,_x \mathbf{w}(x) + \frac{1}{|\mathscr{Y}_f|} \mathrm{div}\,_x \int_{\mathscr{Y}_f} \mathbf{v}\, dy - \frac{1}{|\mathscr{Y}_f|} \int_{\mathscr{Y}_s} \mathrm{div}\,_y\ \mathbf{u}^1(x,y)\, dy\big) \quad \text{in the 2-scale sense.} \tag{4.24}$$

Finally, if $\mathbf{w}^\varepsilon|_{\{x_3 = L/L_{obs}\}} = 0$, then $\mathbf{w}|_{\{x_3 = L/L_{obs}\}} = 0$ and $\int_{\mathscr{Y}_f} \mathbf{v}|_{\{x_3 = L/L_{obs}\}}\, dy\, \mathbf{e}^3 = 0$.

Proof. Most of the assertions are known and for proof we refer to [3] (Lemma 4.7),[13] and [12]. It remains to establish that $\text{div}_x \int_{\mathscr{Y}_f} \mathbf{v}\, dy \in L^2(\Omega_1)$, the convergence (4.24) and to prove the last assertion. We start with the former.

We know that div $\mathbf{w}^\varepsilon$ has a 2-scale limit $G \in L^2(\Omega_1)$. Next we have

$$G|_{\mathscr{Y}_s} = \text{div}_{\,x}\mathbf{w}(x) + \text{div}_{\,y}\mathbf{u}^1(x,y) \quad \text{and} \quad \text{div}_{\,x}\mathbf{w}^\varepsilon \rightharpoonup \int_{\mathscr{Y}} G\, dy = \text{div}_{\,x}\mathbf{w}(x) + \text{div}_{\,x}\int_{\mathscr{Y}_f} \mathbf{v}\, dy$$

and we conclude that $\text{div}_x \int_{\mathscr{Y}_f} \mathbf{v}\, dy \in L^2(\Omega_1)$.

Next using the estimate (4.23) we observe that G does not depend on y on $\mathscr{Y}_f$ and

$$G|_{\mathscr{Y}_f} = \text{div}_{\,x}\mathbf{w}(x) - \frac{1}{|\mathscr{Y}_f|}\int_{\mathscr{Y}_s} \text{div}_{\,y}\mathbf{u}^1(x,y)\, dy + \frac{1}{|\mathscr{Y}_f|}\text{div}_{\,x}\int_{\mathscr{Y}_f} \mathbf{v}(x,y)\, dy,$$

thus establishing (4.24).

Now let us prove the last assertion. We extend $\mathbf{w}^\varepsilon|_{\Omega_s^\varepsilon}$ to Ω_f^ε using the operator P_ε from Lemma 2. $P_\varepsilon \mathbf{w}^\varepsilon = 0$ on $\{x_3 = L/L_{obs}\} \cap \overline{\Omega_s^\varepsilon}$ and by the inequality (4.2) from Lemma 1 we have that $||P_\varepsilon \mathbf{w}^\varepsilon||_{L^2(\{x_3=L/L_{obs}\})^3} \leq \sqrt{\varepsilon}$. Since $P_\varepsilon \mathbf{w}^\varepsilon \to \mathbf{w}$ weakly in $H^1(\Omega_1)$, we conclude that $\mathbf{w} = 0$ on $\{x_3 = L/L_{obs}\}$. Finally $\mathbf{w}^\varepsilon|_{\{x_3=L/L_{obs}\}}\mathbf{e}^3$ is uniformly bounded in $H^{-1/2}(\{x_3 = L/L_{obs}\})$ and we obtain $\int_{\mathscr{Y}_f} \mathbf{v}|_{\{x_3=L/L_{obs}\}}\, dy\, \mathbf{e}^3 = 0$. □

In Theorem 2 the reader is referred to Figures 3 and 4 and the geometry and variables are rescaled in (3.2).

Theorem 2 Let $\mathbf{u}^\varepsilon$ be the variational solution of (3.21), let $\bar{\psi} = |\mathscr{Y}_f|\psi_f + |\mathscr{Y}_s|\psi_s$ and let $-\kappa_{co}p^\varepsilon = \chi_{\Omega_f^\varepsilon}$ div $\mathbf{u}^\varepsilon$. Then there exist limits $(\mathbf{u}^0, \mathbf{u}^1, \mathbf{v}, p^0, \mathbf{u}^0_{el}) \in H^2(0,T;H^1(\Omega_1)^3) \times H^2(0,T;L^2(\Omega_1;H^1_{\text{per}}(\mathscr{Y}_s)^3/\mathbb{R}))$ $\times H^3(0,T;L^2(\Omega_1;\, H^1_{\text{per}}(\mathscr{Y}_f)^3)) \times H^3(0,T;L^2(\Omega_1)) \times H^1(0,T;H^1(\Omega_{el})^3)$, which are periodic in (x_1,x_2) with period L/L_{obs}, such that for all $t \in (0,T)$ we have

$$\mathbf{v} = 0 \quad \text{on} \quad \Omega_1 \times \overline{\mathscr{Y}}_s, \quad \text{div}_{\,y}\mathbf{v} = 0 \;\text{ in } \quad \Omega_1 \times \mathscr{Y}_f, \quad \text{div}_{\,x}\int_{\mathscr{Y}_f} \mathbf{v}\, dy \in L^2(\Omega_1) \tag{4.25}$$

and the convergence of the following holds

$$(\mathbf{u}^\varepsilon,\ \partial_t\mathbf{u}^\varepsilon) \to (\mathbf{u}^0 + \chi_{\mathscr{Y}_f}(y)\mathbf{v},\ \partial_t\mathbf{u}^0 + \chi_{\mathscr{Y}_f}(y)\partial_t\mathbf{v}) \quad \text{in the 2-scale sense in } \Omega_1; \tag{4.26}$$

$$\chi_{\Omega_s^\varepsilon}(\nabla\mathbf{u}^\varepsilon,\ \nabla\partial_t\mathbf{u}^\varepsilon) \to \chi_{\mathscr{Y}_s}(y)(\nabla_x\mathbf{u}^0 + \nabla_y\mathbf{u}^1,\ \nabla_x\partial_t\mathbf{u}^0 + \nabla_y\partial_t\mathbf{u}^1) \quad \text{in the 2-scale sense in } \Omega_1; \tag{4.27}$$

$$\varepsilon\chi_{\Omega_f^\varepsilon}\nabla\partial_t\mathbf{u}^\varepsilon \to \chi_{\mathscr{Y}_f}(y)\nabla_y\partial_t\mathbf{v}(x,y,t) \quad \text{in the 2-scale sense in } \Omega_1; \tag{4.28}$$

$$\chi_{\Omega_f^\varepsilon}\,\text{div}\,\mathbf{u}^\varepsilon = -\kappa_{co}p^\varepsilon \to \chi_{\mathscr{Y}_f}(y)\big(\text{div}_{\,x}\mathbf{u}^0(x,t) + \frac{1}{|\mathscr{Y}_f|}\text{div}_{\,x}\int_{\mathscr{Y}_f}\mathbf{v}(x,y,t)\, dy - \frac{1}{|\mathscr{Y}_f|}\int_{\mathscr{Y}_s}\text{div}_{\,y}\,\mathbf{u}^1(x,y,t)\, dy\big)$$

$$= -\chi_{\mathscr{Y}_f}(y)\kappa_{co}p^0(x,t) \qquad \text{in the 2-scale sense in } \Omega_1; \tag{4.29}$$

$$\chi_{\Omega_f^\varepsilon}\partial_t p^\varepsilon \to \chi_{\mathscr{Y}_f}(y)\partial_t p^0(x,t) \qquad \text{in the 2-scale sense in } \Omega_1; \tag{4.30}$$

$$\chi_{\Omega_{el}}\mathbf{u}^\varepsilon \to \mathbf{u}^0_{el}(x,t) \quad \text{weakly in } H^2((0,T)\times\Omega_{el})^3. \tag{4.31}$$

Furthermore, $(\mathbf{u}^0, \mathbf{u}^1, \mathbf{v}, p^0, \mathbf{u}^0_{el})$ satisfies the system

$$\kappa_{co}|\mathscr{Y}_f|\partial_t p^0(x,t) + |\mathscr{Y}_f|\,\text{div}_x\mathbf{u}^0(x,t) + \int_{\mathscr{Y}_f}\partial_t\mathbf{v}(x,y,t)\, dy - \int_{\mathscr{Y}_s}\text{div}_{\,y}\partial_t\mathbf{u}^1(x,y,t)\, dy = 0$$

$$\text{in } \Omega_1 \times (0,T) \tag{4.32}$$

$$\int_{\Omega_{el}} Ae_x(\mathbf{u}^0_{el}) : e_x(\varphi)\, dx + \int_{\Omega_1}\int_{\mathscr{Y}_s} A(e_x(\mathbf{u}^0)+e_y(\mathbf{u}^1)) : e_x(\varphi)\, dydx - |\mathscr{Y}_f| \int_{\Omega_1} p^0 \operatorname{div}_x\varphi\, dx =$$

$$\int_{\Omega_1} \bar{\psi}\mathbf{F}\varphi\, dx + \int_{\Omega_{el}} \psi_s\mathbf{F}\varphi\, dx\ ,\ \ \forall\varphi\in V; \tag{4.33}$$

$$\int_{\Omega_1}\int_{\mathscr{Y}_s} A(e_x(\mathbf{u}^0)+e_y(\mathbf{u}^1)) : e_y(\psi)\, dydx - \int_{\Omega_1}\int_{\mathscr{Y}_f} p^0 \operatorname{div}_y\psi\, dydx = 0,\ \forall\psi\in L^2(\Omega_1, H^1_{\text{per}}(\mathscr{Y}))^3; \tag{4.34}$$

$$2\int_{\Omega_1}\int_{\mathscr{Y}_f} e_y\left(\frac{\partial\mathbf{v}}{\partial t}\right) : e_y(\zeta)\, dydx - \int_{\Omega_1}\int_{\mathscr{Y}_f} p^0 \operatorname{div}_x\zeta\, dydx = \int_{\Omega_1}\int_{\mathscr{Y}_f} \psi_f\mathbf{F}\zeta\, dydx,\ \forall\zeta\in L^2(\Omega_1, H^1_{\text{per}}(\mathscr{Y}))^3,$$

$$\text{such that }\ \zeta=0\ \text{ on }\ \Omega_1\times\bar{\mathscr{Y}}_s,\ \operatorname{div}_x\int_{\mathscr{Y}_f}\zeta\, dy\in L^2(\Omega_1),\ \int_{\mathscr{Y}_f}\zeta\, dy\cdot\mathbf{e}^3|_{\Sigma\cup\{x_3=L/L_{obs}\}}=0$$

$$\text{and }\ \operatorname{div}_y\zeta=0\ \text{ on }\ \Omega_1\times\mathscr{Y}_f, \tag{4.35}$$

with the interface, boundary and initial conditions

$$\mathbf{u}^0=\mathbf{u}^0_{el}\ \text{a.e on}\ \Sigma\times(0,T);\quad \mathbf{u}^0=0\ \text{ on }\ \{x_3=L/L_{obs}\}\times(0,T);\ \mathbf{u}^0_{el}=0\ \text{ on }\ \{x_3=-L/L_{obs}\}\times(0,T); \tag{4.36}$$

$$\int_{\mathscr{Y}_f} v\, dy\cdot\mathbf{e}^3=0\quad\text{on }\ (\Sigma\cup\{x_3=L/L_{obs}\})\times(0,T);\quad v|_{\{t=0\}}=0,\ \ p^0|_{\{t=0\}}=0. \tag{4.37}$$

Thus $(\mathbf{u}^0,\mathbf{u}^1,\mathbf{v},p^0,\mathbf{u}^0_{el})$ is the unique solution of the two-scale homogenized problem (4.33)-(4.37), with periodic boundary conditions in (x_1,x_2); Finally the whole sequence converges.

Remark 3 As well-known in the homogenization theory, the results do not depend on the outer boundary conditions. The periodicity in (x_1,x_2) was chosen for the sake of simplicity.

Proof. The convergences (4.26)-(4.31), respectively, are direct consequences of the a priori estimates (4.12)-(4.13) from Proposition 3 and Proposition 5 .

Passing to the limit $\varepsilon\to 0$ in the variational formulation (3.21) follows the proof of analogous result in [12]. Here we assume that all test functions are periodic in (x_1,x_2) with period L/L_{obs} and $\varphi\in C^\infty(\bar{\Omega})^3$, $\varphi|_{\{x_3=\pm\frac{L}{L_{obs}}\}}=0$, $\psi\in C^\infty_0(\Omega_1;C^\infty_{\text{per}}(\mathscr{Y}))^3$ and $\zeta\in C^\infty_0(\Omega_1;C^\infty_{\text{per}}(\mathscr{Y}))^3$, where $\operatorname{supp}\zeta\subset\Omega_1\times\mathscr{Y}_f$ and $\operatorname{div}_y\zeta=0$ in $\Omega_1\times\mathscr{Y}_f$.

Then we have for every $t\in(0,T)$

$$\varepsilon\frac{d^2}{dt^2}\int_{\Omega_1}\kappa^\varepsilon\mathbf{u}^\varepsilon\left(\varphi(x)+\zeta(x,\frac{x}{\varepsilon})+\varepsilon\psi(x,\frac{x}{\varepsilon})\right)dx\to 0; \tag{4.38}$$

$$\varepsilon\int_\Omega 2\chi_{\Omega^\varepsilon_f} e\left(\frac{\partial\mathbf{u}^\varepsilon}{\partial t}\right) : e\left(\varepsilon\varphi+\varepsilon^2\psi(x,\frac{x}{\varepsilon})+\varepsilon\zeta(x,\frac{x}{\varepsilon})\right)dx\to\int_{\Omega_1}\int_{\mathscr{Y}_f} 2e_y\left(\frac{\partial\mathbf{v}}{\partial t}\right) : e_y(\zeta)\, dydx; \tag{4.39}$$

$$\int_\Omega \chi_{\tilde{\Omega}^\varepsilon_s} Ae(\mathbf{u}^\varepsilon) : e\left(\varphi+\varepsilon\psi\left(x,\frac{x}{\varepsilon}\right)+\zeta(x,\frac{x}{\varepsilon})\right)dx\to\int_{\Omega_1}\int_{\mathscr{Y}_s} A\big(e_x(\mathbf{u}^0)+$$

$$e_y(\mathbf{u}^1)\big) : (e_x(\varphi)+e_y(\psi))\, dydx+\int_{\Omega_{el}} Ae_x(\mathbf{u}^0_{el}) : e_x(\varphi)\, dx; \tag{4.40}$$

$$\int_{\Omega_1}\chi_{\Omega^\varepsilon_f}\frac{\operatorname{div}\mathbf{u}^\varepsilon}{\kappa_{co}}\operatorname{div}(\varphi+\varepsilon\psi(x,\frac{x}{\varepsilon})+\zeta(x,\frac{x}{\varepsilon}))\to-\int_\Omega\int_{\mathscr{Y}} p^0(\operatorname{div}_x\varphi+\operatorname{div}_y\psi+\operatorname{div}_x\zeta)\, dydx \tag{4.41}$$

$$\int_\Omega \psi^\varepsilon\mathbf{F}\left(\varphi+\varepsilon\psi\left(x,\frac{x}{\varepsilon}\right)+\zeta\left(x,\frac{x}{\varepsilon}\right)\right)\to\int_\Omega\bar{\psi}\mathbf{F}\varphi\, dx+\int_\Omega\int_{\mathscr{Y}_f}\psi_f\mathbf{F}\zeta\, dydx, \tag{4.42}$$

Hence, the variational formulation (3.21), in the limit leads to the equations (4.32)-(4.35), where the test functions $\varphi(x)$, $\varepsilon\psi\left(x,\frac{x}{\varepsilon}\right)$ or $\zeta\left(x,\frac{x}{\varepsilon}\right)$ are used respectively.

In order to get the first equality in (4.36), we extend $\mathbf{u}^\varepsilon|_{\Omega_{el}}$ to Ω_1 using the H^1-extension from [27], satisfying the properties from Lemma 2. Let us denote the extension $\hat{\mathbf{u}}^\varepsilon_{el}$. Then $\mathbf{u}^\varepsilon - \hat{\mathbf{u}}^\varepsilon_{el} = 0$ on Σ. We extend $(\mathbf{u}^\varepsilon - \hat{\mathbf{u}}^\varepsilon_{el})|_{\Omega^\varepsilon_s}$ to Ω^ε_f again using the operator P_ε from Lemma 2. Here $P_\varepsilon(\mathbf{u}^\varepsilon - \hat{\mathbf{u}}^\varepsilon_{el})(t) = 0$ on $\Sigma \cap \overline{\Omega^\varepsilon_s}$ and by the inequality (4.2) from Lemma 1 we have $||P_\varepsilon(\mathbf{u}^\varepsilon - \hat{\mathbf{u}}^\varepsilon_{el})(t)||_{L^2(\Sigma)^3} \leq \sqrt{\varepsilon}$. Since $P_\varepsilon \mathbf{u}^\varepsilon(t) \rightharpoonup \mathbf{u}^0(t)$ weakly in $H^1(\Omega_1)$ and $\hat{\mathbf{u}}^\varepsilon_{el}(t) \rightharpoonup \mathbf{u}^0_{el}(t)$ weakly in $H^1(\Omega_{el})$, we conclude that $\mathbf{u}^0(t) = \mathbf{u}^0_{el}(t)$ on Σ. Finally $[\mathbf{u}^\varepsilon(t)]|_\Sigma \mathbf{e}^3$ is uniformly bounded in $H^{-1/2}(\Sigma)$ and we get $\int_{\mathscr{Y}_f} \mathbf{v}(t)|_\Sigma \, dy \, \mathbf{e}^3 = 0$. The traces on the boundaries $\{x_3 = \pm L/L_{obs}\}$ are calculated analogously to the above calculation and as in the proof of Proposition 3.

Concerning the initial values, we have $p^\varepsilon|_{t=0} = 0$ and p^ε is uniformly bounded in $H^3(0,T;L^2(\Omega_1))$. Hence $p^0|_{t=0} = 0$. Using (4.33) and (4.34), we conclude that $\mathbf{u}^0 = 0 = \mathbf{u}^1$ and $\mathbf{u}^0_{el} = 0$. Since $\chi_{\Omega^\varepsilon_f} \mathbf{u}^\varepsilon|_{t=0} = 0 \to \chi_{\mathscr{Y}_f}(y)(\mathbf{u}^0|_{t=0} + \mathbf{v}|_{t=0}) = 0$, we obtain that $\mathbf{v}|_{t=0} = 0$, which proves (4.37).

In order to prove uniqueness, it is enough to study the problem (4.32)-(4.37) with $\mathbf{F} = 0$. Let $\mathbf{u}(t) = \chi_{\Omega_1} \mathbf{u}^0(t) + \chi_{\Omega_{el}} \mathbf{u}^0_{el}(t) \in H^1(\Omega)^3$. Now we take $\varphi = \mathbf{u}(t)$, $\psi = \mathbf{u}^1(t)$ and $\zeta = \mathbf{v}(t)$. We add the resulting equations (4.33) and (4.34), apply (4.32) and obtain

$$\int_{\Omega_{el}} A e_x(\mathbf{u}^0_{el}(t)) : e_x(\mathbf{u}^0_{el}(t)) \, dx + \int_{\Omega_1} \int_{\mathscr{Y}_s} A(e_x(\mathbf{u}^0(t)) + e_y(\mathbf{u}^1(t))) : (e_x(\mathbf{u}^0(t)) + e_y(\mathbf{u}^1(t))) \, dydx +$$
$$2 \int_{\Omega_1} \int_{\mathscr{Y}_f} e_y\left(\frac{\partial \mathbf{v}(t)}{\partial t}\right) : e_y(\mathbf{v}(t)) \, dydx + |\mathscr{Y}_f| \kappa_{co} \int_{\Omega_1} |p^0|^2 \, dx = 0. \tag{4.43}$$

Since $\mathbf{v}(0) = 0$, we get $\mathbf{v}(t) = 0$ and $p^0(t) = 0$. Next $e_x(\mathbf{u}^0_{el}) = 0$ implies $\mathbf{u}^0_{el} = 0$ and $e_x(\mathbf{u}^0) = 0$ implies $\mathbf{u}^0 = 0$. In the last step, $e_y(\mathbf{u}^1) = 0$ implies that $\mathbf{u}^1$ is a constant vector. With the uniqueness proof we have achieved the proof of the Theorem. □

Remark 4 Passing from the pore level fluid/structure problem to the macroscopic quasi-static Biot system, involves appearance of an initial time layer. Initial conditions for the displacement $\mathbf{u}$ and for the velocity $\partial_t \mathbf{u}$ are lost. We get the globally defined initial condition for the pressure. For simplicity, we start with zero initial conditions and highly compatible forcing term at $t = 0$. These assumptions guarantee that the initial time layer does not appear.

Corollary 1 Let $(\mathbf{u}^0, \mathbf{u}^1, \mathbf{v}, p^0, \mathbf{u}^0_{el})$ be the unique solution of the two-scale homogenized problem (4.33)-(4.37), with periodic boundary conditions in (x_1, x_2). Then it satisfies the following differential system

$$\kappa_{co} |\mathscr{Y}_f| \partial_t p^0(x,t) + |\mathscr{Y}_f| \operatorname{div}_x \mathbf{u}^0(x,t) + \int_{\mathscr{Y}_f} \partial_t \mathbf{v}(x,y,t) \, dy - \int_{\mathscr{Y}_s} \operatorname{div}_y \partial_t \mathbf{u}^1(x,y,t) \, dy = 0$$
$$\text{in } \Omega_1 \times (0,T) \tag{4.44}$$

$$-\operatorname{div}_x \left(\int_{\mathscr{Y}_s} A(e_x(\mathbf{u}^0(t)) + e_y(\mathbf{u}^1(t))) \, dy \right) + |\mathscr{Y}_f| \nabla_x p^0 = \bar{\psi} \mathbf{F}(t) \quad \text{in } \Omega_1 \times (0,T) \tag{4.45}$$

$$-\operatorname{div}_x \left(A(e_x(\mathbf{u}^0_{el}(t))) \right) = \psi_s \mathbf{F}(t) \quad \text{in } \Omega_{el} \times (0,T) \tag{4.46}$$

$$-\operatorname{div}_y \left(A(e_x(\mathbf{u}^0(t)) + e_y(\mathbf{u}^1(t))) \right) = 0 \quad \text{in } \Omega_L \times \mathscr{Y}_s \times (0,T) \tag{4.47}$$

$$A(e_x(\mathbf{u}^0(t)) + e_y(\mathbf{u}^1(t))\mathbf{n} + p^0(t)\mathbf{n} = 0 \quad \text{on } \Omega_L \times (\partial \mathscr{Y}_s \setminus \partial \mathscr{Y}) \times (0,T) \tag{4.48}$$

$$\left(\int_{\mathscr{Y}_s} A(e_x(\mathbf{u}^0(t)) + e_y(\mathbf{u}^1(t))) \, dy - |\mathscr{Y}_f| p^0 I \right) \mathbf{e}^3 = A(e_x(\mathbf{u}^0_{el}(t)) \mathbf{e}^3 \quad \text{on } \Sigma \times (0,T) \tag{4.49}$$

$$\mathbf{u}^0(t) = \mathbf{u}^0_{el}(t) \quad \text{on } \Sigma \times (0,T) \quad \text{and} \quad \mathbf{u}^0(t)|_{\{x_3 = \pm L/L_{obs}\}} = 0 \tag{4.50}$$

$$-\Delta_y \frac{\partial \mathbf{v}(t)}{\partial t} + \nabla_y \pi = \psi_f \mathbf{F}(t) - \nabla_x p^0(t) \quad \text{in } \Omega_L \times \mathscr{Y}_f \times (0,T) \tag{4.51}$$

$$\operatorname{div}_y \frac{\partial \mathbf{v}(t)}{\partial t} = 0 \quad \text{in } \Omega_L \times \mathscr{Y}_f \times (0,T) \text{ and } \frac{\partial \mathbf{v}(t)}{\partial t} = 0 \quad \text{on } \Omega_L \times (\partial \mathscr{Y}_f \setminus \partial \mathscr{Y}) \times (0,T) \tag{4.52}$$

$$\int_{\mathscr{Y}_f} \mathbf{v}(t)\, dy \cdot \mathbf{e}^3 = 0 \quad \text{on } (\Sigma \cup \{x_3 = L/L_{obs}\}) \times (0,T) \tag{4.53}$$

$$\mathbf{v}|_{t=0} = 0 \quad \text{in } \Omega_L \times \mathscr{Y}_f \quad \text{and} \quad p^0|_{t=0} = 0 \quad \text{in } \Omega_1. \tag{4.54}$$

5 Derivation of the effective equations for $\{\mathbf{u}, p^0\}$

The system (4.33)-(4.37) is too complicated to be used directly and it is important to separate the fast and slow scales, if possible. Scale separation for the dynamical diphasic system in Laplace's time domain was treated in [28]. We proceed as in [12], that is we seek for $\{u^1, v\}$ in the particular form, that we will be precisely defined below. Since our system is quasi-static, the decomposition calculations are simpler than in the dynamic case.

For the decomposition we need the following auxiliary problems:

For $i, j = 1, \dots, 3$, find $1-$periodic vector valued function $\mathbf{w}^{ij} \in H^1(\mathscr{Y}_s)^3$, $\int_{\mathscr{Y}_s} \mathbf{w}^{ij}(y)\, dy = 0$, satisfying

$$\begin{cases} \operatorname{div}_y \left\{ A \left(\dfrac{\mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^i}{2} + e_y(\mathbf{w}^{ij}) \right) \right\} = 0 & \text{in } \mathscr{Y}_s, \\ A(\dfrac{\mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^i}{2} + e_y(\mathbf{w}^{ij}))\mathbf{n} = 0 & \text{on } \partial \mathscr{Y}_s \setminus \partial \mathscr{Y}, \end{cases} \tag{5.1}$$

and find $1-$periodic vector valued function $\mathbf{w}^0 \in H^1(\mathscr{Y}_s)^3$, $\int_{\mathscr{Y}_s} \mathbf{w}^0(y)\, dy = 0$, satisfying

$$\begin{cases} -\operatorname{div}_y \left\{ A e_y(\mathbf{w}^0) \right\} = 0 & \text{in } \mathscr{Y}_s, \\ A e_y(\mathbf{w}^0)\mathbf{n} = -\mathbf{n} & \text{on } \partial \mathscr{Y}_s \setminus \partial \mathscr{Y}. \end{cases} \tag{5.2}$$

Due to the periodicity, the problems (5.1) and (5.2) have a unique solution with regularity depending only on the smoothness of the geometry. With the assumptions made, $\mathbf{w}^{ij}$, $\mathbf{w}^0$ are in $H^2(\mathscr{Y}_s)^3$.

Thus, we decompose $\mathbf{u}^1$ as

$$\mathbf{u}^1(x,y,t) = p^0(x,t)\mathbf{w}^0(y) + \sum_{i,j} \left(e_x(\mathbf{u}^0(x,t)) \right)_{ij} \mathbf{w}^{ij}(y). \tag{5.3}$$

Applying (5.3) we see that (4.47) and (4.48) are always satisfied.

The cell problem corresponding to $\mathbf{v}$ is

$$\begin{cases} -\Delta \mathbf{q}^i + \nabla \pi^i = \mathbf{e}^i & \text{in } \mathscr{Y}_f, \\ \operatorname{div}_y \mathbf{q}^i = 0 & \text{in } \mathscr{Y}_f, \\ \mathbf{q}^i|_{\partial \mathscr{Y}_f \setminus \partial \mathscr{Y}} = 0, & \{\mathbf{q}^i, \pi^i\} \text{ is 1-periodic.} \end{cases} \tag{5.4}$$

$\partial_t \mathbf{v}$ has the representation in terms of the $\{\mathbf{q}^i, \pi^i\}$:

$$\frac{\partial \mathbf{v}}{\partial t} = \sum_{j=1}^{3} \mathbf{q}^j(y)(\psi_f F_j(x,t) - \frac{\partial p^0(x,t)}{\partial x_j}). \tag{5.5}$$

Applying (5.5) into (4.51)-(4.53), we see that it is exactly satisfied.

The cell problems define the effective coefficients:

$$A^H_{klij} := \left(\int_{\mathscr{Y}_s} A \left(\frac{\mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^i}{2} + e_y(\mathbf{w}^{ij}) \right) \, dy \right)_{kl} \quad \text{(the dimensionless Gassman tensor)}, \tag{5.6}$$

$$\mathscr{B}^H := \int_{\mathscr{Y}_s} A e_y(\mathbf{w}^0) \, dy, \tag{5.7}$$

$$\mathscr{C}^H_{ij} := \int_{\mathscr{Y}_s} \operatorname{div}_y \mathbf{w}^{ij}(y) \, dy, \tag{5.8}$$

$$K_{ij} := \int_{\mathscr{Y}_f} q^j_i(y) dy \quad \text{(the permeability tensor)}. \tag{5.9}$$

Proposition 6 The tensors A^H, $\mathscr{B}^H$ and K, defined by (5.6), (5.7) and (5.9), respectively, are positive definite and symmetric. Furthermore, $\mathscr{C}^H = \mathscr{B}^H$ and $\int_{\mathscr{Y}_s} \operatorname{div}_y \mathbf{w}^0(y) \, dy < 0$.

Proof. For the properties of the tensors A^H, $\mathscr{B}^H$ and K we refer to the book of Sanchez-Palencia [28], pages 129-190. Concerning the last assertion, it is easy to see that

$$\begin{aligned} \mathscr{B}^H_{ij} &= \int_{\mathscr{Y}_s} (A e_y(\mathbf{w}^0)(y))_{ij} \, dy = \frac{1}{2} \int_{\mathscr{Y}_s} A e_y(\mathbf{w}^0)(y) : (\mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^i) \, dy \\ &= - \int_{\mathscr{Y}_s} A e_y(\mathbf{w}^0)(y) : e_y(\mathbf{w}^{ij}) \, dy = \int_{\mathscr{Y}_s} \operatorname{div}_y \mathbf{w}^{ij}(y) \, dy = \mathscr{C}^H_{ij}. \end{aligned} \tag{5.10}$$

Finally,

$$M_0 = - \int_{\mathscr{Y}_s} \operatorname{div}_y \mathbf{w}^0(y) \, dy = \int_{\mathscr{Y}_s} A e_y(\mathbf{w}^0) : e_y(\mathbf{w}^0) > 0. \tag{5.11}$$

□

Now we use that

$$\operatorname{div}_x \{ (|\mathscr{Y}_f| I - \mathscr{B}^H) \partial_t \mathbf{u} \} = (|\mathscr{Y}_f| I - \mathscr{B}^H) : e_x(\partial_t \mathbf{u}), \tag{5.12}$$

and obtain the initial-boundary problem for $\{\mathbf{u}, p^0\}$:

$$- \operatorname{div}_x \{ A^H e_x(\mathbf{u}) \} + \operatorname{div}_x \{ (|\mathscr{Y}_f| I - \mathscr{B}^H) p^0 \} = \bar{\psi} \mathbf{F}(x,t) \quad \text{in } \Omega_1 \times (0,T), \tag{5.13}$$

$$- \operatorname{div}_x \{ A e_x(\mathbf{u}) \} = \psi_s \mathbf{F}(x,t) \quad \text{in } \Omega_{el} \times (0,T), \tag{5.14}$$

$$[\mathbf{u}]_\Sigma = 0 \quad \text{and} \quad \big(A^H e_x(\mathbf{u}) + (|\mathscr{Y}_f| I - \mathscr{B}^H) p^0 I \big) \mathbf{e}^3 |_\Sigma = A e_x(\mathbf{u}) \mathbf{e}^3 |_\Sigma \quad \text{for all } t \in (0,T), \tag{5.15}$$

$$M \partial_t p^0 + \operatorname{div}_x \{ K(\psi_f \mathbf{F} - \nabla_x p^0) + (|\mathscr{Y}_f| I - \mathscr{B}^H) \partial_t \mathbf{u} \} = 0 \quad \text{in } \Omega_1 \times (0,T), \tag{5.16}$$

$$K(\psi_f \mathbf{F} - \nabla_x p^0) \cdot \mathbf{e}^3 = 0 \quad \text{on } (\Sigma \cup \{x_3 = L/L_{obs}\}) \times (0,T), \tag{5.17}$$

$$p^0|_{t=0} = 0 \quad \text{on } \Omega_1; \quad \mathbf{u} = 0 \quad \text{on } \{x_3 = \pm L/L_{obs}\} \times (0,T), \tag{5.18}$$

$$\{\mathbf{u}, p^0\} \quad \text{is periodic in } (x_1,x_2) \quad \text{with period } L/L_{obs}. \tag{5.19}$$

In (5.16), $M = |\mathscr{Y}_f| \kappa_{co} + M_0 = |\mathscr{Y}_f| \kappa_{co} - \int_{\mathscr{Y}_s} \operatorname{div}_y \mathbf{w}^0(y) \, dy > 0$.

Proposition 7 Let $V_1 = \{ \varphi \in H^1(\Omega_1) \mid \varphi \text{ is periodic in } (x_1,x_2) \text{ with period } L/L_{obs} \}$. The homogenized equations given by (5.13)-(5.19) or in equivalent variational form

$$\text{Find } \{\mathbf{u}, p^0\} \in H^2(0,T;V) \times H^2(0,T;V_1) \text{ such that}$$

$$\int_{\Omega_1} \Big(A^H e_x(\mathbf{u}) - (|\mathscr{Y}_f| I - \mathscr{B}^H) p^0 I \Big) : e_x(\varphi)\, dx + \int_{\Omega_{el}} A e_x(\mathbf{u}) : e_x(\varphi)\, dx =$$

$$\int_{\Omega_1} \bar{\psi} \mathbf{F} \varphi\, dx + \int_{\Omega_{el}} \psi_s \mathbf{F} \varphi\, dx, \quad \varphi \in V; \tag{5.20}$$

$$\frac{\partial}{\partial t} \int_{\Omega_1} M p^0 \xi\, dx + \int_{\Omega_1} (|\mathscr{Y}_f| I - \mathscr{B}^H) \xi : e_x(\frac{\partial \mathbf{u}}{\partial t})\, dx - \int_{\Omega_1} K(\psi_f \mathbf{F} - \nabla_x p^0) \nabla \xi\, dx = 0,\ \forall \xi \in V_1 \tag{5.21}$$

$$p^0|_{t=0} = 0 \quad \text{on } \Omega_1. \tag{5.22}$$

System (5.20) through (5.22) has a unique solution, which defines through (5.3)-(5.5) the unique solution to the two-scale homogenized problem (4.33)-(4.37).

Proof. See the proof of Proposition 1. □

6 Strong convergence and correctors

Besides the standard convergences of the microscopic variables to the effective ones, we also prove the following convergences of the energies.

Proposition 8 We have the following convergences in energy,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_0^t \int_{\Omega_f^\varepsilon} |\nabla \partial_t \mathbf{u}^\varepsilon|^2\, dx d\tau = \int_0^t \int_{\Omega_1 \times \mathscr{Y}_f} |\partial_t \nabla_y \mathbf{v}(x,y,t)|^2\, dy dx d\tau, \tag{6.1}$$

$$\lim_{\varepsilon \to 0} \Big(\int_{\Omega_s^\varepsilon} A e(\mathbf{u}^\varepsilon(t)) : e(\mathbf{u}^\varepsilon(t))\, dx + \int_{\Omega_{el}} A e(\mathbf{u}^\varepsilon(t)) : e(\mathbf{u}^\varepsilon(t))\, dx \Big) =$$

$$\int_{\Omega_1} A^H e(\mathbf{u}(t)) : e(\mathbf{u}(t))\, dx + \int_{\Omega_{el}} A e(\mathbf{u}(t)) : e(\mathbf{u}(t))\, dx - \frac{\int_{\mathscr{Y}_s} \operatorname{div}_y \mathbf{w}^0(y)\, dy}{2} \int_{\Omega_1} (p^0(t))^2\, dx; \tag{6.2}$$

$$\lim_{\varepsilon \to 0} \int_{\Omega_f^\varepsilon} |\operatorname{div} \mathbf{u}^\varepsilon(t)|^2\, dx = \kappa_{co}^2 |\mathscr{Y}_f| \int_{\Omega_1} (p^0(t))^2\, dx \tag{6.3}$$

Proof. The proof is standard (see Theorem 2.6 in [3]). We start from the energy equality corresponding to the variational equation (3.21):

$$\frac{\varepsilon}{2} \int_\Omega \kappa^\varepsilon |\partial_t \mathbf{u}^\varepsilon(t)|^2\, dx + \varepsilon^2 \int_0^t \int_{\Omega_f^\varepsilon} |\nabla \partial_t \mathbf{u}^\varepsilon|^2\, dx d\tau + \frac{1}{2} \int_{\Omega_s^\varepsilon} A e(\mathbf{u}^\varepsilon(t)) : e(\mathbf{u}^\varepsilon(t))\, dx$$

$$+ \frac{1}{2\kappa_{co}} \int_{\Omega_f^\varepsilon} |\operatorname{div} \mathbf{u}^\varepsilon(t)|^2\, dx + \frac{1}{2} \int_{\Omega_{el}} A e(\mathbf{u}^\varepsilon(t)) : e(\mathbf{u}^\varepsilon(t))\, dx =$$

$$\int_0^t \int_{\Omega_1} \psi^\varepsilon \mathbf{F}(\tau) \partial_t \mathbf{u}^\varepsilon(\tau)\, dx d\tau + \int_0^t \int_{\Omega_{el}} \psi_s \mathbf{F}(\tau) \partial_t \mathbf{u}^\varepsilon(\tau)\, dx d\tau. \tag{6.4}$$

For the homogenized variational problem (5.20)-(5.21) the energy equality reads

$$\frac{1}{2} \int_{\Omega_1} A^H e(\mathbf{u}(t)) : e(\mathbf{u}(t))\, dx + \frac{1}{2} \int_{\Omega_{el}} A e(\mathbf{u}(t)) : e(\mathbf{u}(t))\, dx + \frac{\kappa_{co} |\mathscr{Y}_f| - \int_{\mathscr{Y}_s} \operatorname{div}_y \mathbf{w}^0(y)\, dy}{2} \int_{\Omega_1} (p^0(t))^2\, dx +$$

$$\int_0^t \int_{\Omega_1 \times \mathscr{Y}_f} |\partial_t \nabla_y \mathbf{v}(x,y,t)|^2\, dy dx d\tau = \int_0^t \int_{\Omega_1} \bar{\psi} \mathbf{F}(t) \partial_t \mathbf{u}(\tau)\, dx d\tau +$$

$$\int_0^t \int_{\Omega_{el}} \psi_s \mathbf{F}(t) \partial_t \mathbf{u}(\tau)\, dx d\tau + \int_0^t \int_{\Omega_1} \psi_f \mathbf{F}(t) \Big(\int_{\mathscr{Y}_f} \partial_t \mathbf{v}(\tau)\, dy \Big)\, dx d\tau. \tag{6.5}$$

We note that

$$\int_{\Omega_1\times\mathscr{Y}_f} |\partial_t \nabla_y \mathbf{v}(x,y,t)|^2\, dydx = \sum_{i,j=1}^{3} \int_{\Omega_1} K_{ij}(\psi_f F_i - \frac{\partial p^0}{\partial x_i})(\psi_f F_j - \frac{\partial p^0}{\partial x_j})\, dx.$$

In (6.4) we observe the convergence of the right hand side to the right hand side of (6.5). Next we use the lower semicontinuity of the left hand side with respect to the two-scale convergence and the equality (6.5) to conclude (6.1)-(6.3). □

Theorem 3 The following strong two-scale convergences hold

$$\lim_{\varepsilon\to 0}\int_0^t \int_{\Omega_f^\varepsilon} \left| \varepsilon\partial_t \nabla \mathbf{u}^\varepsilon(x,\tau) - \sum_{j=1}^{3} \nabla_y \mathbf{q}^j(\frac{x}{\varepsilon})(F_j(x,\tau)\psi_f - \frac{\partial p^0(x,\tau)}{\partial x_j}) \right|^2 dxd\tau = 0; \tag{6.6}$$

$$\lim_{\varepsilon\to 0}\int_0^t \int_{\Omega_f^\varepsilon} \left| \partial_t \mathbf{u}^\varepsilon(x,\tau) - \partial_t \mathbf{u}(x,\tau) - \sum_{j=1}^{3} \mathbf{q}^j(\frac{x}{\varepsilon})(F_j(x,\tau)\psi_f - \frac{\partial p^0(x,\tau)}{\partial x_j}) \right|^2 dxd\tau = 0; \tag{6.7}$$

$$\lim_{\varepsilon\to 0}\int_0^t \int_{\Omega_f^\varepsilon} \left| \operatorname{div} \mathbf{u}^\varepsilon(x,\tau) + \kappa_{co} p^0(x,\tau) \right|^2 dxd\tau = 0; \tag{6.8}$$

and

$$\lim_{\varepsilon\to 0}\Bigg(\int_{\Omega_s^\varepsilon} \left| A^{1/2} e\big(\mathbf{u}^\varepsilon(x,t) - \mathbf{u}(x,t) - \varepsilon \sum_{i,j=1}^{3} e_{ij}(\mathbf{u}(x,t))\mathbf{w}^{ij}(\frac{x}{\varepsilon}) - \varepsilon p^0(x,t)\mathbf{w}^0(\frac{x}{\varepsilon})\big) \right|^2 dx + \int_{\Omega_{el}} \left| A^{1/2} e\left(\mathbf{u}^\varepsilon(x,t) - \mathbf{u}(x,t)\right) \right|^2 dx \Bigg) = 0. \tag{6.9}$$

Proof. We first remark that the regularity of the solutions of the cell problems (5.1), (5.2) and (5.4) implies that the functions $\mathbf{w}^0(x/\varepsilon)$, $\mathbf{w}^{ij}(x/\varepsilon)$, $\mathbf{q}^j(x/\varepsilon)$ and $\mathbf{v}^\varepsilon(x,t) = \mathbf{v}(x,x/\varepsilon,t)$ are measurable and well defined. We have

$$\int_0^t \int_{\Omega_f^\varepsilon} \varepsilon^2 |\nabla\partial_t \mathbf{v}^\varepsilon - \nabla\partial_t \mathbf{u}^\varepsilon|^2\, dxd\tau = \int_0^t \int_{\Omega_f^\varepsilon} |[\nabla_y \partial_t \mathbf{v}](x,\frac{x}{\varepsilon},\tau)|^2\, dxd\tau + \int_0^t \int_{\Omega^\varepsilon} \varepsilon^2 |\nabla\partial_t \mathbf{u}^\varepsilon|^2\, dxd\tau$$
$$-2\int_{\Omega_f^\varepsilon} \varepsilon [\nabla_y \partial_t \mathbf{v}](x,\frac{x}{\varepsilon},\tau)\cdot \nabla\partial_t \mathbf{u}^\varepsilon(x,\tau)\, dxd\tau + \mathscr{O}(\varepsilon). \tag{6.10}$$

Using Proposition 8 for the second term in the right hand side of (6.10) and passing to the two-scale limit in the third term in the right hand side of (6.10), we deduce (6.6).

Using the scaled Poincaré inequality (4.1) in Ω_f^ε (see Lemma 1) yields (6.7).

Proof of (6.8) goes along the same lines and is based on (6.3).

Finally, even if there is an effective coefficients jump on Σ, $\mathbf{u}$ is H^2 in space in Ω_1. Hence the proof of (6.9) is analogous and based on (6.2). □

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