

# Effective pressure boundary condition for the filtration through porous medium via homogenization

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## Abstract

We present homogenization of the viscous incompressible porous media flows under stress boundary conditions at the outer boundary. In addition to Darcy's law describing filtration in the interior of the porous medium, we derive rigorously the effective pressure boundary condition at the outer boundary. It is a linear combination of the outside pressure and the applied shear stress. We use the two-scale convergence in the sense of boundary layers, introduced by Allaire and Conca [*SIAM J. Math. Anal.*, 29 (1997), pp. 343-379] to obtain the boundary layer structure next to the outer boundary. The approach allows establishing the strong  $L^2$ -convergence of the velocity corrector and identifica-

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tion of the effective boundary velocity slip jump. The theoretical results are confirmed through numerical experiments.

*Keywords:* homogenization, stationary Navier-Stokes equations, stress boundary conditions, effective tangential velocity jump, porous media

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## 1. Introduction

The porous media flows are of interest in a wide range of engineering disciplines including environmental and geological applications, flows through filters etc. They take place in a material which consists of a solid skeleton and billions of interconnected fluid filled pores. The flows are characterised by large spatial and temporal scales. The complex geometry makes direct computing of the flows, and also reactions, deformations and other phenomena, practically impossible. In the applications, the mesoscopic modeling is privileged and one search for effective models where the information on the geometry is kept in the coefficients and which are valid everywhere. The technique which allows replacing the physical models posed at the microstructure level by equations valid globally, is called upscaling. Its mathematical variant, which gives also the rigorous relationship between the upscaled and the microscopic models is the homogenization technique.

It has been applied to a number of porous media problems, starting from the seminal work of Tartar [29] and the monograph [28]. Many subjects are reviewed in the book [16]. See also the references therein.

Frequently, one has processes on multiple domains and model-coupling approaches are needed. Absence of the statistical homogeneity does not allow direct use of the homogenization techniques. Examples of situations where the presence of an interface breaks the statistical homogeneity are

- the flow of a viscous fluid over a porous bed,
- the forced infiltration into a porous medium.

The tangential flow of an unconfined fluid over a porous bed is described by  
 25 the law of Beavers and Joseph [5] and it was rigorously derived in [19] and [24]  
 using a combination of the homogenization and boundary layer techniques. The  
 forced injection problem was introduced in [21] and the interface conditions were  
 rigorously established and justified in [10].

A particular class of the above problems is derivation of the homogenized  
 30 external boundary conditions for the porous media flows. In the case of the  
 zero velocity at the external boundary of the porous medium, one would impose  
 zero normal component of the Darcy velocity as the homogenized boundary  
 condition. The behavior of the velocity and pressure field close to the flat  
 external boundary, with such boundary condition, has been studied in [17], using  
 35 the technique from [18]. The error estimate in 2D, for an arbitrary geometry  
 has been established in [25].

The case of the velocity boundary conditions could be considered as "intu-  
 itively" obvious. Other class of problems arises when we have a contact of the  
 porous medium with another fluid flow and the normal contact force is given at  
 40 the boundary. It describes the physical situation when the upper boundary of  
 the porous medium is exposed to the atmospheric pressure and wind (see e.g.  
 [12]). Or, more generally, when the fluid that we study is in contact with an-  
 other given fluid. Assuming that the motion in porous medium is slow enough  
 that the interface  $\Sigma$  between two fluids can be seen as immobile. Intuitively, it  
 45 is expected that the homogenized pressure will take the prescribed value at the  
 boundary.

In this article we study the homogenization of the stationary Navier-Stokes  
 equations with the given normal contact force at the external boundary and we  
 will find out that the result is more rich than expected.

The novelty of the result is in the boundary condition on  $\Sigma$ . The value of  
 the Darcy pressure on the upper boundary  $\Sigma$  is now prescribed and its value  
 depends not only on the given applied pressure force  $Q$  but also on the shear  
 force  $P$ . Thus the macroscopic pressure is affected by the tangential component  
 of the contact force and not only by the normal component. As for the velocity,

it shows that

$$\mathbf{u}^\varepsilon(x) \approx \varepsilon P(x_1) \mathbf{w}\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 \mathbf{V}(x) .$$

50 Thus, in interior of the domain, the velocity is plain Darcean, while in vicinity of the upper boundary, a boundary layer term  $\varepsilon P(x_1) \mathbf{w}(x/\varepsilon)$  dominates.

The result can be used for the development of the model-coupling strategies, see [15] and [26].

## 2. Setting of the problem

55 We start by defining the geometry. For simplicity, we restrict our research to the 2D case, but the results should hold in the 3D case as well. Nevertheless, in that case the solid skeleton intersects the upper boundary and complications with regularity of solutions to the boundary layer problems arise. Hence the proofs are likely to be more technical.

Let  $\ell$  and  $d$  be two positive constants and let  $\Omega = (0, \ell) \times (-d, 0) \subset \mathbb{R}^2$  be a rectangle. We denote the upper boundary by

$$\Sigma = \{(x_1, 0) \in \mathbb{R}^2 ; x_1 \in (0, \ell)\} .$$

The bottom of the domain is denoted by

$$\Gamma = \{(x_1, -d) \in \mathbb{R}^2 ; x_1 \in (0, \ell)\} .$$

Let  $A \subset \subset \mathbb{R}^2$  be a smooth domain such that  $\bar{A} \subset (0, 1)^2 \equiv Y$ . The unit pore is  $Y^* = Y \setminus \bar{A}$ . Now we choose the small parameter  $\varepsilon \ll 1$  such that  $\varepsilon = \ell/m$ , with  $m \in \mathbb{N}$  and define

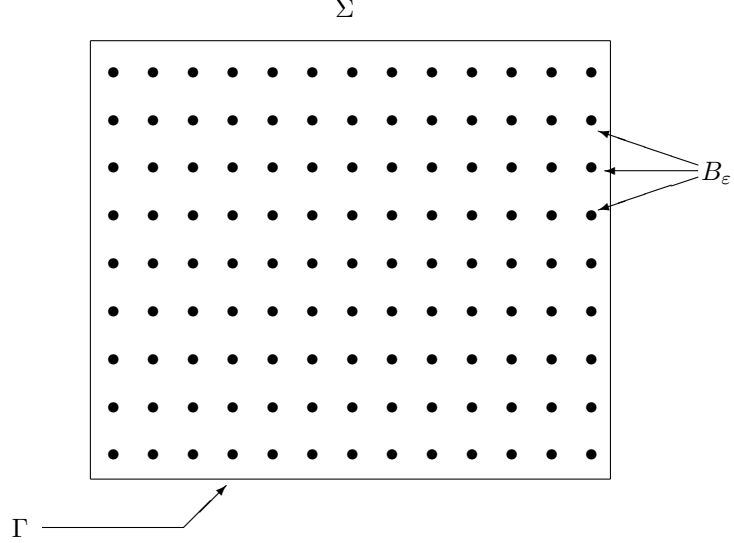
$$T_\varepsilon = \{\mathbf{k} \in \mathbb{Z}^2 ; \varepsilon(\mathbf{k} + A) \subset \Omega\} , Y_{\varepsilon, k}^* = \varepsilon(\mathbf{k} + Y^*) , A_k^\varepsilon = \varepsilon(\mathbf{k} + A).$$

60 The fluid part of the porous medium is now  $\Omega_\varepsilon = \Omega \setminus \overline{\bigcup_{\mathbf{k} \in T_\varepsilon} \varepsilon(\mathbf{k} + A)}$ . Finally,  $B_\varepsilon = \overline{\bigcup_{\mathbf{k} \in T_\varepsilon} \varepsilon(\mathbf{k} + A)}$  is the solid part of the porous medium and its boundary is  $S_\varepsilon = \partial B_\varepsilon$ . As it can be seen from the definition, the solid part of the porous medium is disconnected. That assumption is reasonable since we are in 2D.

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On  $\Sigma$  we prescribe the normal stress and  $\Gamma$  is an impermeable boundary. In  
 80 the dimensionless form, the Stokes problem that we study reads

$$-\mu \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{F} \quad , \quad \operatorname{div} \mathbf{u}^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \quad (1)$$

$$\mathbf{T}(\mathbf{u}^\varepsilon, p^\varepsilon) \mathbf{e}_2 = \mathbf{H} = (P, Q) \quad \text{on } \Sigma, \quad \mathbf{u}^\varepsilon = 0 \quad \text{on } S_\varepsilon \cup \Gamma, \quad (2)$$

$$(\mathbf{u}^\varepsilon, p^\varepsilon) \text{ is } \ell\text{-periodic in } x_1. \quad (3)$$

Here  $\mathbf{T}(\mathbf{v}, q)$  denotes the stress tensor and  $\mathbf{D} \mathbf{v}$  the rate of strain tensor

$$\mathbf{T}(\mathbf{v}, q) = -2\mu \mathbf{D} \mathbf{v} + q \mathbf{I} \quad , \quad \mathbf{D} \mathbf{v} = \frac{1}{2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^t)$$

and  $\mu$  is a positive constant.

**Assumption 1.** We suppose  $\partial A \in C^3$ ,  $\mathbf{F} \in C^1(\overline{\Omega})^2$  is  $\ell$ -periodic in  $x_1$  and  
 $P = P(x_1), Q = Q(x_1)$  being elements of  $C_{\text{per}}^1[0, \ell]$ .

For the existence, uniqueness and regularity of solutions to Stokes problem (1)-  
 85 (3), under Assumption 1, we refer e.g. to [7], Sec. 4.7.

Furthermore, we consider the full stationary incompressible Navier-Stokes system

$$-\mu \Delta \mathbf{u}^{1,\varepsilon} + (\mathbf{u}^{1,\varepsilon} \nabla) \mathbf{u}^{1,\varepsilon} + \nabla p^{1,\varepsilon} = \mathbf{F} \quad , \quad \operatorname{div} \mathbf{u}^{1,\varepsilon} = 0 \quad \text{in } \Omega_\varepsilon \quad (4)$$

$$\mathbf{T}(\mathbf{u}^{1,\varepsilon}, p^{1,\varepsilon}) \mathbf{e}_2 = \mathbf{H} = (P, Q) \quad \text{on } \Sigma, \quad \mathbf{u}^{1,\varepsilon} = 0 \quad \text{on } S_\varepsilon \cup \Gamma \quad (5)$$

$$(\mathbf{u}^{1,\varepsilon}, p^{1,\varepsilon}) \quad \text{is } \ell\text{-periodic in } x_1. \quad (6)$$

Existence of a solution for problem (4)-(6) is discussed in Sec. 5.

Our goal is to study behavior of solutions to (1)-(3) and (4)-(6) in the limit  
 90 when the small parameter  $\varepsilon \rightarrow 0$ .

### 3. The main result

Our goal is to describe the effective behavior of the fluid flow in the above described situation. The filtration in the bulk is expected to be described by Darcy's law and we are looking for the effective boundary condition on the upper  
 95 boundary  $\Sigma$ . To do so, we apply various homogenization techniques, such as two-scale convergence ([27], [1]) and the two-scale convergence for boundary layers ([3]). We prove the following result:

**Theorem 1.** *Let us suppose Assumption 1 and let  $(\mathbf{u}^\varepsilon, p^\varepsilon)$  be the solution of problem (1)-(3).*

*Then there exists an extension of  $p^\varepsilon$  to the whole  $\Omega$ , denoted again by the same symbol, such that*

$$p^\varepsilon \rightarrow p^0 \quad \text{strongly in } L^2(\Omega), \quad \text{as } \varepsilon \rightarrow +0, \quad (7)$$

100 where  $p^0$  is the solution of problem

$$\operatorname{div} \mathbf{K}(\nabla p^0 - \mathbf{F}) = 0 \quad \text{in } \Omega, \quad (8)$$

$$p^0 \quad \text{is } \ell\text{-periodic in } x_1, \quad \mathbf{n} \cdot \mathbf{K}(\nabla p^0 - \mathbf{F}) = 0 \quad \text{on } \Gamma, \quad (9)$$

$$p^0 = C_\pi P + Q \quad \text{on } \Sigma, \quad (10)$$

with  $\mathbf{K}$  the permeability tensor, defined by (83), and  $C_\pi$  the boundary layer pressure stabilisation constant given by (41).

Next, let  $(\mathbf{w}, \pi)$  be the solution of the boundary layer problem (36)-(38). Then, after extending  $\mathbf{u}^\varepsilon$  and  $\mathbf{w}$  by zero to the perforations, we have

$$\frac{\mathbf{u}^\varepsilon(x) - \varepsilon P(x_1) \mathbf{w}(x/\varepsilon)}{\varepsilon^2} \rightharpoonup \mathbf{V} \text{ weakly in } L^2(\Omega), \quad (11)$$

$$\frac{\mathbf{u}^\varepsilon(x)}{\varepsilon^2} \rightharpoonup P(x_1) \left( \int_{G^*} w_1(y) dy \right) \delta_\Sigma \mathbf{e}_1 + \mathbf{V} \text{ weak* in } \mathcal{M}(\Omega), \quad (12)$$

$$\frac{\mathbf{u}^\varepsilon - \varepsilon P(x_1) \mathbf{w}\left(\frac{x}{\varepsilon}\right)}{\varepsilon^2} - \sum_{k=1}^2 \mathbf{w}^k\left(\frac{x}{\varepsilon}\right) \left( F_k - \frac{\partial p^0}{\partial x_k} \right) \rightarrow 0 \text{ strongly in } L^2(\Omega), \quad (13)$$

as  $\varepsilon \rightarrow +0$ .  $G^*$  is the semi-infinite strip of unit cells given by (28),  $\mathbf{V}$  satisfies the Darcy law

$$\mathbf{V} = \mathbf{K}(\mathbf{F} - \nabla p^0),$$

$\mathcal{M}(\Omega)$  denotes the set of Radon measures on  $\Omega$  and  $\delta_\Sigma$  is the Dirac measure concentrated on  $\Sigma$ , i.e.  $\langle \delta_\Sigma | \psi \rangle = \int_\Sigma \psi(x_1, 0) dx_1$ .

**Remark 1.** Let us suppose that the boundary layer geometry has the mirror symmetry with respect to the axis  $\{y_1 = 1/2\}$ . Then  $w_2$  and  $\pi$  are uneven functions with respect to the axis and  $C_\pi = 0$ . In particular, this result applies to the case of circular inclusions.

An analogous result holds for the homogenization of the stationary Navier-Stokes equations (4)-(6) and the homogenized problem is the same as in the linearized case:

**Theorem 2.** Under the assumptions on the geometry and the data from Theorem (1), there exist solutions  $(\mathbf{u}^{1,\varepsilon}, p^{1,\varepsilon})$  of problem (4)-(6) such that convergences (7), (11)-(13) take place.

#### 4. Proof of Theorem 1

The proof is divided in several steps. First we derive the a priori estimates. Then we pass to the two-scale limit for boundary layers, in order to determine the local behavior of the solution in vicinity of the boundary. Once it is achieved,

we subtract the boundary layer corrector from the original solution and use the classical two-scale convergence to prove that the residual converges towards the limit that satisfies the Darcy law. At the end we prove the strong convergences.

#### 4.1. Step one: A priori estimates

125 We first recall that in  $\Omega_\varepsilon$  Poincaré and trace constants depend on  $\varepsilon$  in the following way

$$|\phi|_{L^2(\Omega_\varepsilon)} \leq C \varepsilon |\nabla \phi|_{L^2(\Omega_\varepsilon)} \quad (14)$$

$$|\phi|_{L^2(\Sigma)} \leq C \sqrt{\varepsilon} |\nabla \phi|_{L^2(\Omega_\varepsilon)} , \quad \forall \phi \in H^1(\Omega_\varepsilon) , \phi = 0 \text{ on } S_\varepsilon \quad (15)$$

The second inequality is a consequence of the trace estimate in  $L^2(\Sigma)$  through interpolation between  $L^2(\Omega_\varepsilon)$  and  $H^1(\Omega_\varepsilon)$ . It is needed for application of the two-scale convergence in the sense of the boundary layers.

130 We also recall that the norms  $|\mathbf{D}\mathbf{v}|_{L^2(\Omega_\varepsilon)}$  and  $|\nabla \mathbf{v}|_{L^2(\Omega_\varepsilon)}$  are equivalent, due to the Korn's inequality, which is independent of  $\varepsilon$  (see e.g. [7]).

Here and in the sequel we assume that  $\mathbf{u}^\varepsilon$  is extended by zero to the whole  $\Omega$ . In order to extend the pressure  $p^\varepsilon$  we need Tartar's construction from his seminal paper [29]. It relies on the related construction of the restriction operator, acting

135 from the whole domain  $\Omega$  to the pore space  $\Omega_\varepsilon$ . In our setting we deal with the functional spaces

$$\begin{aligned} X_2 &= \{ \mathbf{z} \in H^1(\Omega)^2 ; \mathbf{z} = 0 \text{ for } x_2 = -d \} \\ X_2^\varepsilon &= \{ \mathbf{z} \in X_2 ; \mathbf{z} = 0 \text{ on } S_\varepsilon \} . \end{aligned}$$

Then, after [29] and the detailed review in [2], there exists a continuous restriction operator  $R_\varepsilon \in \mathcal{L}(X_2, X_2^\varepsilon)$ , such that

$$\begin{aligned} \operatorname{div} (R_\varepsilon \mathbf{z}) &= \operatorname{div} \mathbf{z} + \sum_{\mathbf{k} \in T_\varepsilon} \frac{1}{|Y_{\varepsilon, \mathbf{k}}^*|} \chi_{\varepsilon, \mathbf{k}} \int_{A_{\mathbf{k}}^\varepsilon} \operatorname{div} \mathbf{z} \, dx, \quad \forall \mathbf{z} \in X_2, \\ |R_\varepsilon \mathbf{z}|_{L^2(\Omega_\varepsilon)} &\leq C (\varepsilon |\nabla \mathbf{z}|_{L^2(\Omega)} + |\mathbf{z}|_{L^2(\Omega)}) , \quad \forall \mathbf{z} \in X_2, \\ |\nabla R_\varepsilon \mathbf{z}|_{L^2(\Omega_\varepsilon)} &\leq \frac{C}{\varepsilon} (\varepsilon |\nabla \mathbf{z}|_{L^2(\Omega)} + |\mathbf{z}|_{L^2(\Omega)}) , \quad \forall \mathbf{z} \in X_2, \end{aligned}$$

where  $\chi_{\varepsilon, \mathbf{k}}$  denotes the characteristic function of the set  $Y_{\varepsilon, \mathbf{k}}^*$ ,  $\mathbf{k} \in T_\varepsilon$ . Through a duality argument, it gives an extension of the pressure gradient and it was



found in [23] that the pressure extension  $\tilde{p}$  is given by the explicit formula

$$\tilde{p}^\varepsilon = \begin{cases} p^\varepsilon & \text{in } \Omega_\varepsilon \\ \frac{1}{|Y_{\varepsilon,k}^*|} \int_{Y_{\varepsilon,k}^*} p^\varepsilon dx & \text{in } Y_{\varepsilon,k}^* \text{ for each } \mathbf{k} \in T_\varepsilon. \end{cases} \quad (16)$$

For details we refer to [2]. In addition, a direct computation yields

$$\int_{\Omega_\varepsilon} p^\varepsilon \operatorname{div} (R_\varepsilon \mathbf{z}) dx = \int_{\Omega} \tilde{p}^\varepsilon dx \operatorname{div} \mathbf{z} dx, \quad \forall \mathbf{z} \in X_2. \quad (17)$$

Both the velocity and the pressure extensions are, for simplicity, denoted by  
140 the same symbols as the original functions  $(\mathbf{u}^\varepsilon, p^\varepsilon)$ .

It is straightforward to see that:

**Lemma 1.** *Let  $(\mathbf{u}^\varepsilon, p^\varepsilon)$  be the solution to problem (1), (2). Then there exists some constant  $C > 0$ , independent of  $\varepsilon$ , such that*

$$|\nabla \mathbf{u}^\varepsilon|_{L^2(\Omega)} \leq C\sqrt{\varepsilon} \quad (18)$$

$$|\mathbf{u}^\varepsilon|_{L^2(\Omega)} \leq C\varepsilon^{3/2} \quad (19)$$

$$|p^\varepsilon|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\varepsilon}}. \quad (20)$$

**Proof.** We start from the variational formulation of problem (1), (2)

$$\mu \int_{\Omega_\varepsilon} \mathbf{D}\mathbf{u}^\varepsilon : \mathbf{D}\mathbf{v} dx = \int_{\Sigma} \mathbf{H} \cdot \mathbf{v} dS + \int_{\Omega_\varepsilon} \mathbf{F} \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in V(\Omega_\varepsilon), \quad (21)$$

$$V(\Omega_\varepsilon) = \{\mathbf{v} \in H^1(\Omega_\varepsilon)^2; \operatorname{div} \mathbf{v} = 0, \mathbf{v} = 0 \text{ on } S_\varepsilon \cup \Gamma, \mathbf{v} \text{ is } \ell\text{-periodic in } x_1\}$$

Using  $\mathbf{u}^\varepsilon$  as the test function and applying (14)-(15) yield

$$\mu \int_{\Omega_\varepsilon} |\mathbf{D}\mathbf{u}^\varepsilon|^2 dx = \int_{\Sigma} \mathbf{H} \cdot \mathbf{u}^\varepsilon dS + \int_{\Omega_\varepsilon} \mathbf{F} \cdot \mathbf{u}^\varepsilon dx \leq C\sqrt{\varepsilon} |\mathbf{D}\mathbf{u}^\varepsilon|_{L^2(\Omega_\varepsilon)}.$$

Now (14) implies (18) and (19). Since we have extended the pressure to the solid part of  $\Omega$ , using Tartar's construction, (18) and (17) imply

$$|p^\varepsilon|_{L^2(\Omega)/\mathbb{R}} = \sup_{g \in L^2(\Omega)/\mathbb{R}} \frac{\int_{\Omega} p^\varepsilon g dx}{|g|_{L^2(\Omega)/\mathbb{R}}} = \sup_{\mathbf{z} \in X_2} \frac{\int_{\Omega_\varepsilon} p^\varepsilon \operatorname{div} (\mathbf{R}_\varepsilon \mathbf{z}) dx}{|\mathbf{z}|_{H^1(\Omega)^2}} \leq \frac{C}{\varepsilon} |\nabla \mathbf{u}^\varepsilon|_{L^2(\Omega)},$$

giving the pressure estimate (20).  $\square$

We recall the definition and some basic compactness results for two-scale convergence for boundary layers due to Allaire and Conca [3]. In the sequel, if the index  $y$  is added to the differential operators  $\mathbf{D}_y$ ,  $\nabla_y$ ,  $\text{div}_y$ , then the derivatives are taken with respect to the fast variables  $y_1, y_2$  instead of  $x_1, x_2$ . Let  $G = (0, 1) \times (-\infty, 0)$  be an infinite band. The bounded sequence  $(\phi_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega)$  is said to two-scale converge in the sense of the boundary layers if there exists  $\phi_0(x_1, y) \in L^2(\Sigma \times G)$  such that

$$\frac{1}{\varepsilon} \int_{\Omega} \phi_\varepsilon(x) \psi\left(x_1, \frac{x}{\varepsilon}\right) dx \rightarrow \int_{\Sigma} \int_G \phi_0(x_1, y) \psi(x_1, y) dx_1 dy, \quad \text{as } \varepsilon \rightarrow +0, \quad (22)$$

for all smooth functions  $\psi(x_1, y)$  defined in  $\Sigma \times \overline{G}$ , with bounded support, such that  $y_1 \mapsto \psi(x_1, y_1, y_2)$  is 1-periodic.

We need the following functional space

$$\begin{aligned} \mathcal{D}^1 &= \{\psi \in C^\infty(G) ; \psi \text{ is 1-periodic in } y_1 \\ &\text{and compactly supported in } y_2 \in (-\infty, 0]\} \end{aligned}$$

Now  $D_{\#}^1(G)$  is the closure of  $\mathcal{D}^1$  in the norm  $|\psi|_{D_{\#}^1(G)} = |\nabla \psi|_{L^2(G)}$ . It should be noticed that such functions do not necessarily vanish as  $y_2 \rightarrow -\infty$ . For that kind of convergence we have the following compactness result from [3]:

**Theorem 3.** 1. *Let us suppose*

$$\frac{1}{\sqrt{\varepsilon}} |\phi_\varepsilon|_{L^2(\Omega)} \leq C. \quad (23)$$

*Then there exists  $\phi_0 \in L^2(\Sigma \times G)$  and a subsequence, denoted by the same indices, such that*

$$\phi_\varepsilon \rightarrow \phi_0 \text{ two-scale in the sense of boundary layers, as } \varepsilon \rightarrow +0. \quad (24)$$

2. *Let us suppose*

$$\frac{1}{\sqrt{\varepsilon}} (|\phi_\varepsilon|_{L^2(\Omega)} + \varepsilon |\nabla \phi_\varepsilon|_{L^2(\Omega)}) \leq C. \quad (25)$$

Then there exists  $\phi_0 \in L^2(\Sigma; D_{\#}^1(G))$  and a subsequence, denoted by the same indices, such that in the limit  $\varepsilon \rightarrow +0$

$$\phi_\varepsilon \rightarrow \phi_0 \quad \text{two-scale in the sense of boundary layers} \quad (26)$$

$$\varepsilon \nabla \phi_\varepsilon \rightarrow \nabla_y \phi_0 \quad \text{two-scale in the sense of boundary layers.} \quad (27)$$

Using the *a priori* estimates, we now undertake our first passing to the limit.

Before we start we define

$$\mathcal{C} = \bigcup_{j=0}^{-\infty} (j \mathbf{e}_2 + \partial A) \quad , \quad M = \bigcup_{j=0}^{-\infty} (j \mathbf{e}_2 + \overline{A}) \quad , \quad G^* = G \setminus \bigcup_{j=0}^{-\infty} (j \mathbf{e}_2 + A). \quad (28)$$

We introduce the space  $D_{\#0}^1(G^*)$  defined similarly as  $D_{\#}^1(G)$  but on  $G^*$  and such that its elements have zero trace on  $\mathcal{C}$ . Thus, we take

$$\begin{aligned} \mathcal{D}^1 &= \{ \psi \in C^\infty(G^*) ; \psi|_{\mathcal{C}} = 0, \psi \text{ is } 1\text{-periodic in } y_1, \\ &\text{and compactly supported in } y_2 \in (-\infty, 0] \}. \end{aligned}$$

Then  $D_{\#0}^1(G^*)$  is its closure in the norm  $|\psi|_{D_{\#0}^1(G^*)} = |\nabla \psi|_{L^2(G^*)}$ . Those func-  
150 tions do vanish as  $y_2 \rightarrow -\infty$  due to the zero trace on  $\mathcal{C}$  that prevents them to  
tend to a constant.

**Lemma 2.** Let  $(\mathbf{v}^0, q^0) \in L^2(\Sigma; D_{\#0}^1(G^*)) \times L^2(\Sigma; L_{loc}^2(G^*))$  be given by the boundary layer problem

$$-\mu \Delta_y \mathbf{v}^0 + \nabla_y q^0 = 0, \quad \operatorname{div}_y \mathbf{v}^0 = 0 \quad \text{in } G^*, \quad (29)$$

$$(-2\mu \mathbf{D}_y \mathbf{v}^0 + q^0 \mathbf{I}) \mathbf{e}_2 = \mathbf{H} \quad \text{for } y_2 = 0, \quad \mathbf{v}^0 = 0 \quad \text{on } \mathcal{C}, \quad (30)$$

$$(\mathbf{v}^0, q^0) \text{ is } 1\text{-periodic in } y_1, \quad \mathbf{v}^0 \rightarrow 0 \quad \text{as } y_2 \rightarrow -\infty. \quad (31)$$

Then

$$\frac{1}{\varepsilon} \mathbf{u}^\varepsilon \rightarrow \mathbf{v}^0 \quad \text{two-scale in the sense of boundary layers} \quad (32)$$

$$\nabla \mathbf{u}^\varepsilon \rightarrow \nabla_y \mathbf{v}^0 \quad \text{two-scale in the sense of boundary layers, as } \varepsilon \rightarrow +0. \quad (33)$$

**Proof.** The a priori estimates (19) and (18) and the compactness theorem 3 imply the existence of some  $\mathbf{v}^0 \in L^2(\Sigma; D_{\#0}^1(G^*))$  such that  $\mathbf{v}^0 = 0$  on  $M$  and

$$\frac{1}{\varepsilon} \mathbf{u}^\varepsilon \rightarrow \mathbf{v}^0 \quad \text{two-scale in the sense of boundary layers} \quad (34)$$

$$\nabla \mathbf{u}^\varepsilon \rightarrow \nabla_y \mathbf{v}^0 \quad \text{two-scale in the sense of boundary layers, as } \varepsilon \rightarrow +0. \quad (35)$$

Now we take the test function  $\mathbf{z}^\varepsilon(x) = \mathbf{z}(x_1, \frac{x}{\varepsilon}) \in D_{\#0}^1(G^*)^2$  such that  $\operatorname{div}_y \mathbf{z} = 0$  and  $\mathbf{z}(x_1, \cdot) = 0$  in  $M$  in the variational formulation for (1), (2)

$$\frac{2\mu}{\varepsilon} \int_{\Omega_\varepsilon} \varepsilon \mathbf{D} \frac{\mathbf{u}^\varepsilon}{\varepsilon} : \varepsilon \mathbf{D} \mathbf{z}^\varepsilon dx - \int_{\Omega_\varepsilon} p^\varepsilon \operatorname{div} \mathbf{z}^\varepsilon dx = \int_\Sigma \mathbf{H} \cdot \mathbf{z}^\varepsilon dS + \int_{\Omega_\varepsilon} \mathbf{F} \cdot \mathbf{z}^\varepsilon dx.$$

Since

$$\frac{\partial \mathbf{z}^\varepsilon}{\partial x_j} = \varepsilon^{-1} \frac{\partial \mathbf{z}}{\partial y_j} + \delta_{1j} \frac{\partial \mathbf{z}}{\partial x_1}$$

we get in the limit

$$2\mu \int_\Sigma \left( \int_G \mathbf{D}_y \mathbf{v}^0(x_1, y) : \mathbf{D}_y \mathbf{z}(x_1, y) dy \right) dx_1 = \int_\Sigma \mathbf{H} \cdot \left( \int_0^1 \mathbf{z}(x_1, y_1, 0) dy_1 \right) dx_1.$$

Furthermore, since  $\operatorname{div} \mathbf{u}^\varepsilon = 0$  it easily follows that  $\operatorname{div}_y \mathbf{v}^0 = 0$ . Thus there  
155 exists  $q^0 \in L^2(\Sigma; L_{loc}^2(G^*))$  such that  $(\mathbf{v}^0, q^0)$  satisfy (29)-(31).  $\square$

The boundary layer corrector  $(\mathbf{v}^0, q^0)$  can be decomposed as  $\mathbf{v}^0 = P(x_1) \mathbf{w}(y)$  and  $q^0 = P(x_1) \pi(y) + Q(x_1)$ , where

$$-\mu \Delta_y \mathbf{w} + \nabla_y \pi = 0, \quad \operatorname{div}_y \mathbf{w} = 0 \quad \text{in } G^*, \quad (36)$$

$$(-2\mu \mathbf{D}_y \mathbf{w} + \pi \mathbf{I}) \mathbf{e}_2 = \mathbf{e}_1 \quad \text{for } y_2 = 0, \quad \mathbf{w} = 0 \text{ on } \mathcal{C}, \quad (37)$$

$$(\mathbf{w}, \pi) \text{ is 1-periodic in } y_1, \quad \mathbf{w} \rightarrow 0 \text{ as } y_2 \rightarrow -\infty. \quad (38)$$

Problem (36), (38) is of the boundary layer type. Existence of the solution and  
160 exponential decay can be proved as in [18] (See also [22] for the scalar case).

We have

**Theorem 4.** *Problem (36), (38) has a unique solution  $(\mathbf{w}, \pi) \in D_{\#0}^1(G^*) \times L_{loc}^2(G^*)$ . Furthermore, there exists a constant  $C_\pi$  such that*

$$|e^{\alpha|y_2|} (\pi - C_\pi)|_{L^2(G^*)} \leq C \quad (39)$$

$$|e^{\alpha|y_2|} \mathbf{w}|_{L^2(G^*)} + |e^{\alpha|y_2|} \nabla \mathbf{w}|_{L^2(G^*)} \leq C. \quad (40)$$

for some constants  $C, \alpha > 0$ .

In the sense of (39) we write

$$C_\pi = \lim_{y_2 \rightarrow -\infty} \pi(y) . \quad (41)$$

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**Remark 2.** If  $\partial A \in C^3$  then the regularity theory for the Stokes operator applies and (39), (40) hold pointwise. For more details on the regularity see e.g. [7].

**Remark 3.** Let the solution  $\mathbf{w}$  to system (36)- (38) be extended by zero to  $M$ . Let  $b > a > 0$  be arbitrary constants. Then we have

$$0 = \int_a^b \int_0^1 \operatorname{div} \mathbf{w} \, dy = \int_0^1 w_2(y_1, b) \, dy_1 - \int_0^1 w_2(y_1, a) \, dy_1.$$

Using (40) yields

$$\int_0^1 w_2(y_1, y_2) \, dy_1 = 0, \quad \forall y_2 \leq 0. \quad (42)$$

(42) implies

$$\int_{G^*} w_2 \, dy = 0. \quad (43)$$

**Remark 4.** Integrating (37) with respect to  $y_1$  yields

$$\mathbf{e}_1 = \int_0^1 \left[ -\mu \left( \frac{\partial w_1}{\partial y_2} \mathbf{e}_1 + \frac{\partial w_2}{\partial y_1} \mathbf{e}_1 + 2 \frac{\partial w_2}{\partial y_2} \mathbf{e}_2 \right) + \pi \mathbf{e}_2 \right] (y_1, 0) \, dy_1 .$$

Equating the second components gives

$$\begin{aligned} 0 &= \int_0^1 \left( -2\mu \frac{\partial w_2}{\partial y_2} + \pi \right) (y_1, 0) \, dy_1 = \int_0^1 \left( 2\mu \frac{\partial w_1}{\partial y_1} + \pi \right) (y_1, 0) \, dy_1 = \\ &= \int_0^1 \pi(y_1, 0) \, dy_1 . \end{aligned}$$

If we test (36) with  $\mathbf{w}^k$  and (80) by  $\mathbf{w}$  and combine, we get

$$C_\pi = K_{22}^{-1} \left( \int_0^1 w_1^2(y_1, 0) \, dy_1 + \int_0^1 \left( -2\mu \frac{\partial \mathbf{w}^k}{\partial y_2} + \pi^k \mathbf{e}_2 \right) (y_1, 0) \mathbf{w}(y_1, 0) \, dy_1 \right) .$$

Finally, we denote  $J = \{y_2 \in ]-\infty, 0] ; (y_1, y_2) \notin M, y_1 \in ]0, 1[ \}$ . Denoting

$$m_A = \min\{y_2 \in [0, 1] ; (y_1, y_2) \in \overline{A}\} , \quad M_A = \max\{y_2 \in [0, 1] : (y_1, y_2) \in \overline{A}\} .$$

170 The set  $J$  is then a union of disjoint intervals  $J_0 = ]0, m_A[$ ,  $J_i = ]i - 1 + M_A, i + m_A[$ ,  $i = 1, 2, \dots$ . It is easy to see that the mapping  $t \mapsto \int_0^1 \pi(y_1, t) dy_1$  is constant on each of the intervals  $J_i$ . If those constants are denoted by  $c_i$  then  $c_0 = 0$  and  $\lim_{i \rightarrow \infty} c_i = C_\pi$ .

#### 4.3. Step three: Derivation of the Darcy law via classical two-scale convergence

We now know the behavior of  $(\mathbf{u}^\varepsilon, p^\varepsilon)$  in vicinity of  $\Sigma$ . To get additional information of the behavior far from the boundary we deduce the boundary layer corrector from  $(\mathbf{u}^\varepsilon, p^\varepsilon)$  and define

$$\mathbf{U}^\varepsilon(x) = \mathbf{u}^\varepsilon(x) - \varepsilon P(x_1) \mathbf{w}(x/\varepsilon) \quad , \quad P^\varepsilon(x) = p^\varepsilon(x) - [P(x_1) \pi(x/\varepsilon) + Q(x_1)] \quad .$$

The stress tensor  $\mathbf{T}(\mathbf{v}, q) = 2\mu \mathbf{D}\mathbf{v} - q \mathbf{I}$  for such approximation satisfies

$$\begin{aligned} \mathbf{T}(\mathbf{U}^\varepsilon, P^\varepsilon) &= \mathbf{T}(\mathbf{u}^\varepsilon, p^\varepsilon) - P(x_1) (2\mu \mathbf{D}_y \mathbf{w} - \pi \mathbf{I}) - \\ &\quad - 2\mu \varepsilon \frac{dP}{dx_1} \begin{bmatrix} w_1 & w_2/2 \\ w_2/2 & 0 \end{bmatrix} = \mathbf{T}(\mathbf{u}^\varepsilon, p^\varepsilon) - \\ &\quad - \begin{bmatrix} P(x_1) \left( 2\mu \frac{\partial w_1}{\partial y_1} - \pi \right) + 2\mu \varepsilon \frac{dP}{dx_1} w_1 - Q & \mu \left( P(x_1) \left( \frac{\partial w_1}{\partial y_2} + \frac{\partial w_2}{\partial y_1} \right) + \varepsilon \frac{dP}{dx_1} w_2 \right) \\ \mu \left( P(x_1) \left( \frac{\partial w_1}{\partial y_2} + \frac{\partial w_2}{\partial y_1} \right) + \varepsilon \frac{dP}{dx_1} w_2 \right) & P(x_1) \left( 2\mu \frac{\partial w_2}{\partial y_2} - \pi \right) - Q(x_1) \end{bmatrix} \end{aligned}$$

By direct computation we get

$$-\operatorname{div} \mathbf{T}(\mathbf{U}^\varepsilon, P^\varepsilon) = \mathbf{f}^\varepsilon, \quad (44)$$

$$\mathbf{f}^\varepsilon \equiv \mathbf{F} + \mu \varepsilon \frac{d^2 P}{dx_1^2} (\mathbf{w} + w_1 \mathbf{e}_1) + \frac{dP}{dx_1} (2\mu \frac{\partial \mathbf{w}}{\partial y_1} - \pi \mathbf{e}_1 + \mu \nabla_y w_1) - \frac{dQ}{dx_1} \mathbf{e}_1, \quad (45)$$

$$\operatorname{div} \mathbf{U}^\varepsilon = -\varepsilon \frac{dP}{dx_1} w_1 \quad \text{in } \Omega_\varepsilon, \quad (46)$$

$$\mathbf{U}^\varepsilon = 0 \quad \text{on } S_\varepsilon, \quad \mathbf{U}^\varepsilon = -\varepsilon P(x_1) \mathbf{w}(x/\varepsilon) \quad \text{on } \Gamma, \quad (47)$$

$$(-2\mu \mathbf{D} \mathbf{U}^\varepsilon + P^\varepsilon \mathbf{I}) \mathbf{e}_2 = 0 \quad \text{on } \Sigma. \quad (48)$$

175 We want to derive appropriate a priori estimates for  $(\mathbf{U}^\varepsilon, P^\varepsilon)$ . However, according to (46), the divergence of  $\mathbf{U}^\varepsilon$  is still too large for our purpose. Thus we need to compute the additional divergence corrector.

**Lemma 3.** *There exists  $\Phi \in H^2(G^*)^2$  such that*

$$\operatorname{div}_y \Phi = w_1 \text{ in } G^*, \quad (49)$$

$$\Phi \text{ is 1-periodic in } y_1, \quad \Phi = 0 \text{ on } \mathcal{C}, \quad \Phi(y_1, 0) = C \mathbf{e}_2, \quad (50)$$

$$e^{\gamma|y_2|} \Phi \in L^2(G^*)^4 \text{ and } |\Phi(y_1, y_2)| \leq C e^{-\gamma|y_2|}, \text{ for some } \gamma > 0. \quad (51)$$

**Proof.** We follow [18] and search for  $\Phi$  in the form

$$\Phi = \nabla_y \psi + \operatorname{curl}_y h = \left( \frac{\partial \psi}{\partial y_1} - \frac{\partial h}{\partial y_2}, \frac{\partial \psi}{\partial y_2} + \frac{\partial h}{\partial y_1} \right).$$

The function  $\psi$  solves

$$-\Delta_y \psi = w_1(y) \text{ in } G^*, \quad \frac{\partial \psi}{\partial \mathbf{n}} = 0 \text{ on } \mathcal{C}, \quad (52)$$

$$\frac{\partial \psi}{\partial y_2} = d_0 = \text{const. for } y_2 = 0, \quad \psi \text{ is 1-periodic in } y_1, \quad (53)$$

with  $\mathbf{n} = (n_1, n_2)$  being the exterior unit normal on  $\mathcal{C}$  and  $\mathbf{t} = (-n_2, n_1)$  the tangent. The constant  $d_0$  is chosen in a way that problem (52)-(53) admits a solution. By simple integration it turns out that  $d_0 = -\int_{G^*} w_1(y) dy$ . Since the right-hand side is in  $H^1(G^*)$ , the problem has a solution  $\psi \in H^3(G^*)$  that can be chosen to have an exponential decay

$$|\psi|_{H^1(G^* \cap \{|y_2| > s\})} \leq C e^{-\gamma s}. \quad (54)$$

Next we use the trace theorem and construct a  $y_1$ -periodic function  $h \in H^3(G^*)$  such that

$$\frac{\partial h}{\partial \mathbf{t}} = \operatorname{curl} h \cdot \mathbf{n} = 0, \quad \frac{\partial h}{\partial \mathbf{n}} = \operatorname{curl} h \cdot \mathbf{t} = -\frac{\partial \psi}{\partial \mathbf{t}} \text{ on } \mathcal{C}$$

and

$$\frac{\partial h}{\partial y_2}(y_1, 0) = \frac{\partial \psi}{\partial y_1}(y_1, 0), \quad \frac{\partial h}{\partial y_1}(y_1, 0) = 0 \text{ (achieved if we take } h(y_1, 0) = \text{const. )}.$$

<sup>180</sup> The function  $\Phi$ , constructed above, satisfies (49) and (50). Exponential decay (54) of  $\psi$  implies exponential decay of  $h$  in the same sense and, finally, gives (51).  $\square$

Again, assuming that  $\mathbf{U}^\varepsilon$  is extended by zero to the pores  $B_\varepsilon$  we extend  $P^\varepsilon$  using the formula (16) to prove:

**Lemma 4.**

$$|\nabla \mathbf{U}^\varepsilon|_{L^2(\Omega)} \leq C \varepsilon \quad (55)$$

$$|\mathbf{U}^\varepsilon|_{L^2(\Omega)} \leq C \varepsilon^2 \quad (56)$$

$$|P^\varepsilon|_{L^2(\Omega)} \leq C \quad (57)$$

**Proof.** It is straightforward to see that for the right-hand side, we have

$$|\mathbf{f}^\varepsilon|_{L^2(\Omega)} \leq C \quad .$$

Furthermore

$$\mathbf{f}^\varepsilon = \mathbf{F} - \left( \frac{dP}{dx_1} C_\pi + \frac{dQ}{dx_1} \right) \mathbf{e}_1 + \mathbf{g}^\varepsilon \quad ,$$

with  $|\mathbf{g}^\varepsilon|_{L^2(\Omega)} = O(\sqrt{\varepsilon})$ . The idea is to test the system (44) with

$$\tilde{\mathbf{U}}^\varepsilon = \mathbf{U}^\varepsilon + \varepsilon^2 \frac{dP}{dx_1}(x_1) \Phi\left(\frac{x}{\varepsilon}\right) \quad , \quad (58)$$

where  $\Phi$  is constructed in lemma 3. By the construction

$$\operatorname{div} \tilde{\mathbf{U}}^\varepsilon = \varepsilon^2 \frac{d^2 P}{dx_1^2} \Phi_1^\varepsilon$$

with  $\Phi^\varepsilon(x) = \Phi(x/\varepsilon)$  . Thus

$$|\operatorname{div} \tilde{\mathbf{U}}^\varepsilon|_{L^2(\Omega)} \leq C \varepsilon^{5/2} \quad .$$

The weak form of (44) reads

$$2\mu \int_{\Omega_\varepsilon} \mathbf{D} \mathbf{U}^\varepsilon : \mathbf{D} \mathbf{z} \, dx - \int_{\Omega_\varepsilon} P^\varepsilon \operatorname{div} \mathbf{z} \, dx = \int_{\Omega_\varepsilon} \mathbf{f}^\varepsilon \cdot \mathbf{z} \, dx \quad , \quad \forall \mathbf{z} \in X_2^\varepsilon \quad (59)$$

so that

$$\left| \int_{\Omega_\varepsilon} P^\varepsilon \operatorname{div} \mathbf{z} \, dx \right| \leq C (|\mathbf{D} \mathbf{U}^\varepsilon|_{L^2(\Omega_\varepsilon)} + \varepsilon) \|\mathbf{z}\|_{H^1(\Omega_\varepsilon)} \quad , \quad \forall \mathbf{z} \in X_2^\varepsilon. \quad (60)$$

Next we use identity (17) to obtain the estimate

$$\left| \int_{\Omega} \tilde{P}^\varepsilon \operatorname{div} \mathbf{z} \, dx \right| = \left| \int_{\Omega_\varepsilon} P^\varepsilon \operatorname{div} (R_\varepsilon \mathbf{z}) \, dx \right| \leq \frac{C}{\varepsilon} (|\mathbf{D} \mathbf{U}^\varepsilon|_{L^2(\Omega_\varepsilon)} + \varepsilon) \|\mathbf{z}\|_{H^1(\Omega)} \quad , \quad (61)$$

$\forall \mathbf{z} \in X_2$  . Since  $\operatorname{div} : X_2 \rightarrow L^2(\Omega)$  is a surjective continuous operator, (61)

yields

$$|\tilde{P}^\varepsilon|_{L^2(\Omega)} \leq C (\varepsilon^{-1} |\mathbf{D} \mathbf{U}^\varepsilon|_{L^2(\Omega_\varepsilon)} + 1) \quad . \quad (62)$$



Now we take  $\mathbf{z} = \tilde{\mathbf{U}}^\varepsilon$  as a test function in (59). To be precise, we observe that  $\tilde{\mathbf{U}}^\varepsilon$  is not exactly in  $X_2^\varepsilon$  since it is not equal to zero for  $x_2 = -d$ . But, that value is exponentially small, of order  $e^{-\gamma/\varepsilon}$ , so it can be easily corrected by lifting its boundary value by a negligibly small function. Thus, slightly abusing the notation, we consider it as an element of  $X_2^\varepsilon$ . Then, due to the (58)

$$\left| \int_{\Omega_\varepsilon} P^\varepsilon \operatorname{div} \tilde{\mathbf{U}}^\varepsilon dx \right| = \left| \varepsilon^2 \int_{\Omega_\varepsilon} P^\varepsilon \frac{d^2 P}{dx_1^2} \Phi_1^\varepsilon dx \right| \leq C \varepsilon |\mathbf{D} \mathbf{U}^\varepsilon|_{L^2(\Omega_\varepsilon)} + C \varepsilon^2. \quad (63)$$

Consequently, we get (55)-(57).  $\square$

At this point we use the classical two-scale convergence (see e.g. [27], [1]). For readers' convenience we recall basic definitions and compactness results.

Let  $Y = [0, 1]^2$  and let  $C_\#^\infty(Y)$  be the set of all  $C^\infty$  functions defined on  $Y$  and periodic with period 1. We say that a sequence  $(v^\varepsilon)_{\varepsilon>0}$ , from  $L^2(\Omega)$ , two-scale converges to a function  $v^0 \in L^2(\Omega)$  if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v^\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx \rightarrow \int_{\Omega} \int_Y v^0(x, y) \psi(x, y) dx dy, \quad \text{as } \varepsilon \rightarrow +0,$$

185 for any  $\psi \in C_0^\infty(\Omega; C_\#^\infty(Y))$ .

For such convergence we have the following compactness result from [1] and [27] that we shall need in the sequel

**Theorem 5.** • Let  $(v^\varepsilon)_{\varepsilon>0}$  be a bounded sequence in  $L^2(\Omega)$ . Then we can extract a subsequence that two-scale converges to some  $v^0 \in L^2(\Omega \times Y)$ .

190 • Let  $(v^\varepsilon)_{\varepsilon>0}$  be a sequence in  $H^1(\Omega)$  such that  $v^\varepsilon$  and  $\varepsilon \nabla v^\varepsilon$  are bounded in  $L^2(\Omega)$ . Then, there exists a function  $v^0 \in L^2(\Omega; H_\#^1(Y))$  and a subsequence for which

$$v^\varepsilon \rightarrow v^0 \quad \text{in two-scales}, \quad (64)$$

$$\varepsilon \nabla v^\varepsilon \rightarrow \nabla_y v^0 \quad \text{in two-scales, as } \varepsilon \rightarrow +0. \quad (65)$$

**Lemma 5.** Let  $(\mathbf{U}^\varepsilon, P^\varepsilon)$  be the solution of the residual problem (46)-(48). Then

$$\varepsilon^{-2} \mathbf{U}^\varepsilon \rightarrow \mathbf{U}^0 \text{ in two-scales,} \quad (66)$$

$$\varepsilon^{-1} \nabla \mathbf{U}^\varepsilon \rightarrow \nabla_y \mathbf{U}^0 \text{ in two-scales,} \quad (67)$$

$$P^\varepsilon \rightarrow P^0 \text{ in two-scales, as } \varepsilon \rightarrow +0, \quad (68)$$

where  $(\mathbf{U}^0, P^0, Q^0) \in L^2(\Omega; H_\#^1(Y^*)) \times H^1(\Omega) \times L^2(\Omega; L^2(Y^*)/\mathbb{R})$  is the solution of the two-scale problem

$$-\mu \Delta_y \mathbf{U}^0 + \nabla_y Q^0 + \nabla_x P^0 = \mathbf{F} - \left( \frac{dQ}{dx_1} + C_\pi \frac{dP}{dx_1} \right) \mathbf{e}_1 \text{ in } Y^* \times \Omega, \quad (69)$$

$$\operatorname{div}_y \mathbf{U}^0 = 0 \text{ in } Y^* \times \Omega, \quad (70)$$

$$\mathbf{U}^0 = 0 \text{ on } S \times \Omega, \text{ } (\mathbf{U}^0, Q^0) \text{ is 1-periodic in } y, \quad (71)$$

$$\operatorname{div}_x \left( \int_Y \mathbf{U}^0 dy \right) = 0 \text{ in } \Omega, \left( \int_Y \mathbf{U}^0 dy \right) \cdot \mathbf{n} = 0 \text{ on } \Gamma, P^0 = 0 \text{ on } \Sigma. \quad (72)$$

**Proof.** Using the estimates (55)-(57) we get that there exist  $\mathbf{U}^0 \in L^2(\Omega; H_\#^1(Y))$

and  $P^0 \in L^2(\Omega \times Y)$  such that

$$\varepsilon^{-2} \mathbf{U}^\varepsilon \rightarrow \mathbf{U}^0 \text{ in two-scales,}$$

$$\varepsilon^{-1} \nabla \mathbf{U}^\varepsilon \rightarrow \nabla_y \mathbf{U}^0 \text{ in two-scales,}$$

$$P^\varepsilon \rightarrow P^0 \text{ two-scale.}$$

It follows directly that  $\mathbf{U}^0(x, y) = 0$  for  $y \in \overline{A}$ .

First, for  $\psi(x, y) \in C^\infty(Y \times \Omega)$ , periodic in  $y$ , such that  $\psi = 0$  for  $y \in \overline{A}$

$$\begin{aligned} 0 &\leftarrow \int_\Omega \frac{dP}{dx_1}(x_1) w_1 \left( x, \frac{x}{\varepsilon} \right) \psi \left( x, \frac{x}{\varepsilon} \right) dx = \varepsilon^{-1} \int_\Omega \operatorname{div} \mathbf{U}^\varepsilon \psi \left( x, \frac{x}{\varepsilon} \right) dx \\ &= - \int_\Omega \left( \varepsilon \nabla_x \psi \left( x, \frac{x}{\varepsilon} \right) + \nabla_y \psi \left( x, \frac{x}{\varepsilon} \right) \right) \cdot \frac{\mathbf{U}^\varepsilon(x)}{\varepsilon^2} dx \rightarrow \\ &\rightarrow \int_\Omega \int_Y \mathbf{U}^0 \cdot \nabla_y \psi(x, y) dy dx \Rightarrow \boxed{\operatorname{div}_y \mathbf{U}^0 = 0}. \end{aligned} \quad (73)$$

We then test equations (44)-(48) with  $\mathbf{m}^\varepsilon(x) = \mathbf{m}\left(x, \frac{x}{\varepsilon}\right)$ , where  $\mathbf{m} \in H^1(\Omega; H_\#^1(Y))$ ,  $\mathbf{m} = 0$  for  $y \in \overline{M}$ .

$$\begin{aligned} 0 \leftarrow \varepsilon \int_{\Omega} \mathbf{f}^\varepsilon \mathbf{m}^\varepsilon \, dx &= 2\mu \int_{\Omega} \mathbf{D} \mathbf{U}^\varepsilon(x) \left[ \mathbf{D}_y \mathbf{m}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \mathbf{D}_x \mathbf{m}\left(x, \frac{x}{\varepsilon}\right) \right] dx - \\ &\int_{\Omega} P^\varepsilon(x) \left( \varepsilon \operatorname{div}_x \mathbf{m}(x, x/\varepsilon) + \operatorname{div}_y \mathbf{m}(x, x/\varepsilon) \right) dx \rightarrow \\ &= - \int_{\Omega} \int_Y P^0(x, y) \operatorname{div}_y \mathbf{m}(x, y) \, dy \, dx. \end{aligned} \quad (74)$$

Thus  $\nabla_y P^0 = 0$  implying  $\boxed{P^0 = P^0(x)}$ .

205 Next we test system (44)-(48) with  $\mathbf{Z}^\varepsilon(x) = \mathbf{Z}\left(x, \frac{x}{\varepsilon}\right)$ , where  $\mathbf{Z} \in H^1(\Omega; H_\#^1(Y))$ , such that  $\operatorname{div}_y \mathbf{Z} = 0$  and  $\mathbf{Z} = 0$  for  $y \in \overline{A}$ . It yields

$$\begin{aligned} \int_{\Omega} \left[ \mathbf{F} - \frac{dP(x_1)C_\pi + Q(x_1)}{dx_1} \mathbf{e}_1 \right] \int_Y \mathbf{Z} \, dy \leftarrow \int_{\Omega} \mathbf{f}^\varepsilon \mathbf{Z}^\varepsilon &= - \int_{\Omega} P^\varepsilon(x) \operatorname{div}_x \mathbf{Z}(x, x/\varepsilon) \, dx \\ + \frac{2\mu}{\varepsilon} \int_{\Omega} \mathbf{D} \mathbf{U}^\varepsilon(x) \left( \mathbf{D}_y \mathbf{Z}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \mathbf{D}_x \mathbf{Z}\left(x, \frac{x}{\varepsilon}\right) \right) dx &\rightarrow \\ \rightarrow 2\mu \int_{\Omega} \int_Y \mathbf{D}_y \mathbf{U}^0(x, y) \mathbf{D}_y \mathbf{Z}(x, y) \, dy \, dx &- \int_{\Omega} \int_Y P^0(x) \operatorname{div}_x \mathbf{Z}(x, y) \, dy \, dx. \end{aligned} \quad (75)$$

We conclude that  $\nabla_x P^0 \in L^2(\Omega)$  and  $(\mathbf{U}^0, P^0)$  satisfies equations (69)-(71).

The effective filtration velocity boundary conditions are determined by picking a smooth test-function  $\psi \in C^\infty(\Omega)$ , periodic in  $x_1$ ,  $\psi = 0$  on  $\Sigma$ , and testing

$$\operatorname{div} \tilde{\mathbf{U}}^\varepsilon = \varepsilon^2 \frac{dP'}{dx_1} \Phi_1^\varepsilon$$

with it. It gives

$$\begin{aligned} - \int_{\Omega_\varepsilon} \frac{dP'}{dx_1}(x_1) \Phi_1\left(\frac{x}{\varepsilon}\right) \psi(x) \, dx &= \varepsilon^{-2} \int_{\Omega_\varepsilon} \operatorname{div} \tilde{\mathbf{U}}^\varepsilon(x) \psi(x) \, dx = \\ &= - \int_{\Omega_\varepsilon} \varepsilon^{-2} \tilde{\mathbf{U}}^\varepsilon(x) \cdot \nabla \psi(x) \, dx - \int_0^\ell \tilde{U}_2^\varepsilon(x_1, -d) \psi(x_1, -d) \, dx_1. \end{aligned} \quad (76)$$

210 The last integral on the right hand side is negligible due to the exponential decay of  $\mathbf{w}$  and  $\Phi$ . The first integral on the right hand side, due to (66), converges

and, due to the construction of  $\tilde{\mathbf{U}}^\varepsilon$ ,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varepsilon^{-2} \tilde{\mathbf{U}}^\varepsilon(x) \cdot \nabla \psi(x) dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varepsilon^{-2} \mathbf{U}^\varepsilon(x) \cdot \nabla \psi(x) dx = \\ &= \int_{\Omega} \int_Y \mathbf{U}^0(x, y) dy \cdot \nabla \psi(x) dx . \end{aligned}$$

For the left-hand side in (76) we get

$$\left| \int_{\Omega} \frac{d^2 P}{dx_1^2}(x_1) \Phi_1\left(\frac{x}{\varepsilon}\right) \psi(x) dx \right| \leq C \sqrt{\varepsilon} .$$

Thus

$$\int_{\Omega} \left( \int_Y \mathbf{U}^0 dy \right) \cdot \nabla \psi dx = 0$$

meaning that

$$\operatorname{div}_x \left( \int_Y \mathbf{U}^0 dy \right) = 0 \text{ in } \Omega , \quad \left( \int_Y \mathbf{U}^0 dy \right) \cdot \mathbf{n} = 0 \text{ on } \Gamma .$$

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We still need to determine the boundary condition for  $P^0$  on  $\Sigma$ .

Let  $\mathbf{b}$  be a smooth function defined on  $\Omega \times Y$ , such that  $\operatorname{div}_y \mathbf{b} = 0$  and  $\mathbf{b} = 0$  on  $\Gamma$  and  $\mathbf{b} = 0$  for  $y \in \overline{A}$ . We now use  $\mathbf{b}^\varepsilon(x) = \mathbf{b}(x, x/\varepsilon)$  as a test function in (44)-(48). We obtain in the limit  $\varepsilon \rightarrow +0$

$$\begin{aligned} \int_{\Omega} \mathbf{f}^\varepsilon \cdot \mathbf{b}^\varepsilon dx &= 2\mu \int_{\Omega} \mathbf{D} \mathbf{U}^\varepsilon \left[ \mathbf{D}_x \mathbf{b} \left( \cdot, \frac{\cdot}{\varepsilon} \right) + \varepsilon^{-1} \mathbf{D}_y \mathbf{b} \left( \cdot, \frac{\cdot}{\varepsilon} \right) \right] dx - \\ \int_{\Omega} P^\varepsilon \operatorname{div}_x \mathbf{b} \left( \cdot, \frac{\cdot}{\varepsilon} \right) dx &\rightarrow 2\mu \int_{\Omega} \int_Y \mathbf{D}_y \mathbf{U}^0 \mathbf{D}_y \mathbf{b} dy dx - \int_{\Omega} P^0 \operatorname{div}_x \left( \int_Y \mathbf{b} dy \right) dx . \end{aligned} \quad (77)$$

As for the left-hand side, we have

$$\int_{\Omega} \mathbf{f}^\varepsilon \cdot \mathbf{b}^\varepsilon dx \rightarrow \int_{\Omega} \left[ \mathbf{F} - \frac{d(P(x_1)C_\pi + Q(x_1))}{dx_1} \mathbf{e}_1 \right] \left( \int_Y \mathbf{b} dy \right) dx$$

so that

$$\begin{aligned} 2\mu \int_{\Omega} \int_Y \mathbf{D}_y \mathbf{U}^0 \mathbf{D}_y \mathbf{b} dy dx - \int_{\Omega} P^0 \operatorname{div}_x \left( \int_Y \mathbf{b} dy \right) dx &= \\ \int_{\Omega} \int_Y \mathbf{b} \left[ \mathbf{F} - \frac{d(P(x_1)C_\pi + Q(x_1))}{dx_1} \mathbf{e}_1 \right] dy dx . \end{aligned}$$

Using (69)-(72) yields

$$\int_{\Omega} P^0 \operatorname{div} \left( \int_Y \mathbf{b} \, dy \right) dx = - \int_{\Omega} \nabla P^0 \cdot \left( \int_Y \mathbf{b} \, dy \right) dx.$$

It implies

$$2\mu \int_{\Sigma} \left( \int_Y \mathbf{b} \cdot \mathbf{e}_2 \, dy \right) P^0 dx = 0$$

and, finally,  $\boxed{P^0 = 0 \text{ on } \Sigma.}$

220 Proving uniqueness of a weak solution for problem (69)-(72) is straightforward.  $\square$

#### 4.4. Step four: Strong convergence

We start by proving the strong convergence for the pressure. We follow the approach from [28]. Let  $\{\mathbf{z}^\varepsilon\}_{\varepsilon>0}$  be a sequence in  $X_2$  such that

$$\mathbf{z}^\varepsilon \rightharpoonup \mathbf{z}^0 \text{ weakly in } H^1(\Omega) .$$

Then we have

$$\int_{\Omega} \tilde{P}^\varepsilon \operatorname{div} \mathbf{z}^\varepsilon dx - \int_{\Omega} P^0 \operatorname{div} \mathbf{z} dx = \int_{\Omega} \tilde{P}^\varepsilon \operatorname{div} (\mathbf{z}^\varepsilon - \mathbf{z}) dx + \int_{\Omega} (\tilde{P}^\varepsilon - P^0) \operatorname{div} \mathbf{z} dx.$$

For two integrals on the right-hand side we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (\tilde{P}^\varepsilon - P^0) \operatorname{div} \mathbf{z} dx &= 0 \quad \text{and} \\ \int_{\Omega} \tilde{P}^\varepsilon \operatorname{div} (\mathbf{z}^\varepsilon - \mathbf{z}) dx &= \int_{\Omega_\varepsilon} P^\varepsilon \operatorname{div} R_\varepsilon (\mathbf{z}^\varepsilon - \mathbf{z}) dx = \\ 2\mu \int_{\Omega_\varepsilon} \mathbf{D} \left( \frac{\mathbf{U}^\varepsilon}{\varepsilon} \right) \varepsilon \mathbf{D}(R_\varepsilon (\mathbf{z}^\varepsilon - \mathbf{z})) dx &\rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 . \end{aligned}$$

Using surjectivity of the operator  $\operatorname{div} : X_2 \rightarrow L^2(\Omega)$  we conclude that

225  $\boxed{\tilde{P}^\varepsilon \rightarrow P^0 \text{ strongly in } L^2(\Omega).}$

Next we prove the strong convergence for the velocity. We define

$$\mathbf{U}^{0,\varepsilon}(x) = \sum_{k=1}^2 \mathbf{w}^k(x/\varepsilon) \left[ F_k(x) - \frac{\partial}{\partial x_k} (P^0(x) + C_\pi P(x_1) + Q(x_1)) \right] .$$

Then for the  $L^2$ -norms we have

$$\begin{aligned} \int_{\Omega_\varepsilon} \left| \frac{\mathbf{U}^\varepsilon}{\varepsilon^2} - \mathbf{U}^{0,\varepsilon} \right|^2 dx &\leq C 2\mu \varepsilon^2 \int_{\Omega_\varepsilon} \left| \mathbf{D} \left( \frac{\mathbf{U}^\varepsilon}{\varepsilon^2} - \mathbf{U}^{0,\varepsilon} \right) \right|^2 dx = \\ &= C \left\{ 2\mu \varepsilon^{-2} \int_{\Omega_\varepsilon} |\mathbf{D} \mathbf{U}^\varepsilon|^2 dx + 2\mu \varepsilon^2 \int_{\Omega_\varepsilon} |\mathbf{D} \mathbf{U}^{0,\varepsilon}|^2 dx - \right. \\ &\quad \left. - 4\mu \int_{\Omega_\varepsilon} \mathbf{D} \left( \frac{\mathbf{U}^\varepsilon}{\varepsilon} \right) \varepsilon \mathbf{D} \mathbf{U}^{0,\varepsilon} dx \right\}. \end{aligned}$$

Using the smoothness of  $\mathbf{U}^0$  we get, as  $\varepsilon \rightarrow 0$

$$\begin{aligned} \text{(i)} \quad &\varepsilon^2 \int_{\Omega_\varepsilon} |\mathbf{D} \mathbf{U}^{0,\varepsilon}|^2 dx = \int_{\Omega_\varepsilon} |\mathbf{D}_y \mathbf{U}^{0,\varepsilon}|^2 dx + O(\varepsilon) \rightarrow \int_{\Omega \times Y^*} |\mathbf{D}_y \mathbf{U}^0|^2 dx dy. \\ \text{(ii)} \quad &2\mu \int_{\Omega \times Y^*} |\mathbf{D}_y \mathbf{U}^0|^2 dx = \int_{\Omega} \left( \mathbf{F} - \frac{d(P(x_1)C_\pi + Q(x_1))}{dx_1} \mathbf{e}_1 \right) \int_{Y^*} \mathbf{U}^0 dy dx. \\ \text{(iii)} \quad &2\mu \varepsilon^{-2} \int_{\Omega_\varepsilon} |\mathbf{D} \mathbf{U}^\varepsilon|^2 dx = 2\mu \varepsilon^{-2} \int_{\Omega_\varepsilon} \mathbf{D} \mathbf{U}^\varepsilon \mathbf{D} \tilde{\mathbf{U}}^\varepsilon dx + O(\sqrt{\varepsilon}). \\ \text{(iv)} \quad & \end{aligned}$$

$$\begin{aligned} &2\mu \varepsilon^{-2} \int_{\Omega_\varepsilon} \mathbf{D} \mathbf{U}^\varepsilon \mathbf{D} \tilde{\mathbf{U}}^\varepsilon dx - \varepsilon^{-2} \int_{\Omega_\varepsilon} P^\varepsilon \operatorname{div} \tilde{\mathbf{U}}^\varepsilon dx = \\ &\int_{\Omega_\varepsilon} \left( \mathbf{F} - \frac{d(P(x_1)C_\pi + Q(x_1))}{dx_1} \mathbf{e}_1 \right) \frac{\mathbf{U}^\varepsilon}{\varepsilon^2} dx + O(\sqrt{\varepsilon}). \end{aligned}$$

$$\text{(v)} \quad \varepsilon^{-2} \int_{\Omega_\varepsilon} P^\varepsilon \operatorname{div} \tilde{\mathbf{U}}^\varepsilon dx = \int_{\Omega_\varepsilon} P^\varepsilon \frac{d^2 P}{dx_1^2} \Phi^\varepsilon dx \rightarrow 0.$$

(vi)

(iii), (iv) and (v)  $\Rightarrow$

$$2\mu \varepsilon^{-2} \int_{\Omega_\varepsilon} |\mathbf{D} \mathbf{U}^\varepsilon|^2 dx \rightarrow \int_{\Omega} \left[ \mathbf{F} - \frac{d(P(x_1)C_\pi + Q(x_1))}{dx_1} \mathbf{e}_1 \right] \int_{Y^*} \mathbf{U}^0 dy dx.$$

$$\text{(vii)} \quad \int_{\Omega_\varepsilon} \mathbf{D} \left( \frac{\mathbf{U}^\varepsilon}{\varepsilon} \right) \varepsilon \mathbf{D} \mathbf{U}^{0,\varepsilon} dx \rightarrow \int_{\Omega \times Y^*} |\mathbf{D}_y \mathbf{U}^0|^2 dx dy.$$

Thus

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} \left| \frac{\mathbf{U}^\varepsilon}{\varepsilon^2} - \mathbf{U}^{0,\varepsilon} \right|^2 dx = 0.$$

235 4.5. Step five: Weak\* convergence of the boundary layer corrector

To prove convergence (12) we need to show that

$$\varepsilon^{-1} P(x_1) \mathbf{w}(x/\varepsilon) \rightharpoonup P(x_1) \left( \int_{G^*} \mathbf{w}(y) dy \right) \delta_\Sigma \text{ weak* in } \mathcal{M}(\Omega), \text{ as } \varepsilon \rightarrow +0.$$

Thus we take the test function  $\mathbf{z} \in C(\overline{\Omega})^2$  and, using the exponential decay of  $\mathbf{w}$ , we get

$$\begin{aligned} \frac{1}{\varepsilon} \int_{\Omega} P(x_1) \mathbf{w} \left( \frac{x}{\varepsilon} \right) \mathbf{z}(x) dx &= \frac{1}{\varepsilon} \int_0^\ell P(x_1) \int_{\varepsilon \log \varepsilon}^0 \mathbf{w} \left( \frac{x}{\varepsilon} \right) \mathbf{z}(x) dx_2 dx_1 + O(\varepsilon) = \\ &= \int_0^\ell P(x_1) \mathbf{z}(x_1, 0) \int_{-\infty}^0 \mathbf{w} \left( \frac{x_1}{\varepsilon}, y_2 \right) dy_2 dx_1 + O(\varepsilon |\log \varepsilon|). \end{aligned}$$

Using the well known property of the mean of a periodic function (see e.g. [28]) yields

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^\ell P(x_1) \mathbf{z}(x_1, 0) \int_{-\infty}^0 \mathbf{w} \left( \frac{x_1}{\varepsilon}, y_2 \right) dy_2 dx_1 &= \\ &= \int_0^\ell P(x_1) \mathbf{z}(x_1, 0) \left( \int_{-\infty}^0 \int_0^1 \mathbf{w}(y) dy_1 dy_2 \right) dx_1 = \\ &= \int_0^\ell P(x_1) \mathbf{z}(x_1, 0) \int_{G^*} \mathbf{w}(y) dy dx_1 = \left( \int_{G^*} \mathbf{w}(y) dy \right) P(x_1) \langle \delta_\Sigma | \mathbf{z} \rangle. \end{aligned}$$

240 4.6. Step six: Separation of scales and the end of the proof of Theorem 1

We can separate the variables in (69)-(72) by setting

$$\mathbf{U}^0(x, y) = \sum_{k=1}^2 \mathbf{w}^k(y) \left[ F_k(x) - \frac{\partial}{\partial x_k} (Q(x_1) + C_\pi P(x_1) + P^0(x)) \right], \quad (78)$$

$$Q^0(x, y) = \sum_{k=1}^2 \pi^k(y) \left[ F_k(x) - \frac{\partial}{\partial x_k} (Q(x_1) + C_\pi P(x_1) + P^0(x)) \right], \quad (79)$$

with

$$-\mu \Delta \mathbf{w}^k + \nabla \pi^k = \mathbf{e}_k, \quad \operatorname{div} \mathbf{w}^k = 0 \quad \text{in } Y^*, \quad (80)$$

$$\mathbf{w}^k = 0 \quad \text{on } \partial A, \quad (\mathbf{w}^k, \pi^k) \text{ is } 1 - \text{periodic}. \quad (81)$$

Inserting the separation of scales formulas (78)-(79) into (69)-(72) yields

$$\begin{cases} \operatorname{div} \mathbf{K} [\mathbf{F} - \nabla (P^0 + C_\pi P + Q)] = 0 & \text{in } \Omega, \\ P^0 = 0 & \text{on } \Sigma, \quad P^0 \text{ is } \ell - \text{periodic in } x_1, \\ \mathbf{n} \cdot \mathbf{K} [\mathbf{F} - \nabla (P^0 + C_\pi P + Q)] = 0 & \text{on } \Gamma. \end{cases} \quad (82)$$

Here

$$\mathbf{K} = [K_{ij}] = [\int_Y w_j^i dy] \quad (83)$$

stands for the positive definite and symmetric permeability tensor. System (82) is a well-posed mixed boundary value problem for a linear elliptic equation for  $P^0$ .

Nevertheless, it is important to note that  $P^0$  is not the limit or homogenized pressure since

$$p^\varepsilon(x) = P^\varepsilon(x) + \pi\left(\frac{x}{\varepsilon}\right) P(x_1) + Q(x_1) .$$

Obviously

$$p^\varepsilon \rightharpoonup p^0 \equiv P^0 + C_\pi P + Q .$$

This ends the proof of theorem 1 since the limit pressure is  $p^0$  and it satisfies the boundary value problem (8)-(10).

## 245 5. Proof of Theorem 2

We start by proving that problem (4)-(6) admits at least one solution satisfying estimates (18)-(20).

It is well known that in the case of the stress boundary conditions, the inertia term poses difficulties and existence results for the stationary Navier-Stokes  
250 system can be obtained only under conditions on data and/or the Reynolds number (see e.g. [11]). Presence of many small solid obstacles in the porous media flows corresponds to a small Reynolds number, expressed through the presence of  $\varepsilon$  in Poincaré's and trace estimates (14) and (15).

In order to estimate the inertia term we need fractional order Sobolev spaces. we recall that

$$H^{1/2}(\Omega)^2 = \{\mathbf{z} \in L^2(\Omega)^2 \mid E\mathbf{z} \in H^{1/2}(\mathbb{R}^2)^2\},$$

where  $E : H^1(\Omega)^2 \rightarrow H^1(\mathbb{R}^2)^2$  is the classical Sobolev extension map. It is  
255 defined on the spaces  $H^\alpha(\Omega)$ ,  $\alpha \in (0, 1)$  through interpolation (see [13], Chapter 6).



Next, after [13], Chapter 6, one has

$$\left| \int_{\Omega_\varepsilon} (\mathbf{u}^{1,\varepsilon} \nabla) \mathbf{u}^{1,\varepsilon} \cdot \mathbf{v} \, dx \right| \leq C |\mathbf{u}^{1,\varepsilon}|_{H^{1/2}(\Omega)^2} |\nabla \mathbf{u}^{1,\varepsilon}|_{L^2(\Omega)^2} |\mathbf{v}|_{H^{1/2}(\Omega)^2}, \quad \forall \mathbf{v} \in V(\Omega_\varepsilon). \quad (84)$$

Using (14) in (84) yields

$$\left| \int_{\Omega_\varepsilon} (\mathbf{u}^{1,\varepsilon} \nabla) \mathbf{u}^{1,\varepsilon} \cdot \mathbf{u}^{1,\varepsilon} \, dx \right| \leq C\varepsilon |\nabla \mathbf{u}^{1,\varepsilon}|_{L^2(\Omega)^2}^3. \quad (85)$$

Now it is enough to have an a priori estimate for the  $H^1$ -norm. With such estimate the standard procedure would give existence of a solution. It consists of defining a finite dimensional Galerkin approximation and using the a priori estimate and Brouwer's theorem to show that it admits a solution satisfying a uniform  $H^1$ - a priori estimate. Finally, we let the number of degrees of freedom in the Galerkin approximation tend to infinity and obtain a solution through the elementary compactness. For more details we refer to the textbook of Evans [14], subsection 9.1.

We recall that the variational form of (4)-(6) is

$$\begin{aligned} \langle \mathcal{L}^\varepsilon \mathbf{u}^{1,\varepsilon}, \mathbf{v} \rangle &= 2\mu \int_{\Omega_\varepsilon} D\mathbf{u}^{1,\varepsilon} : D\mathbf{v} \, dx + \int_{\Omega_\varepsilon} (\mathbf{u}^{1,\varepsilon} \nabla) \mathbf{u}^{1,\varepsilon} \cdot \mathbf{v} \, dx - \\ &\quad - \int_{\Omega_\varepsilon} \mathbf{F} \cdot \mathbf{v} \, dx - \int_{\Sigma} \mathbf{H} \cdot \mathbf{v} \, dS = 0, \quad \forall \mathbf{v} \in V(\Omega_\varepsilon). \end{aligned} \quad (86)$$

Then, for  $\varepsilon \leq \varepsilon_0$ ,

$$\begin{aligned} \langle \mathcal{L}^\varepsilon \mathbf{u}^{1,\varepsilon}, \mathbf{u}^{1,\varepsilon} \rangle &\geq 2\mu |D\mathbf{u}^{1,\varepsilon}|_{L^2(\Omega_\varepsilon)^4}^2 - C\varepsilon |D\mathbf{u}^{1,\varepsilon}|_{L^2(\Omega_\varepsilon)^4}^3 - C\sqrt{\varepsilon} |D\mathbf{u}^{1,\varepsilon}|_{L^2(\Omega_\varepsilon)^4} \geq \\ &\geq \frac{C_1}{\varepsilon^2} > 0, \quad \text{if } |D\mathbf{u}^{1,\varepsilon}|_{L^2(\Omega_\varepsilon)^4} = \frac{1}{\sqrt{\varepsilon}}. \end{aligned} \quad (87)$$

As a direct consequence of (87), Brouwer's theorem implies existence of at least one solution for the  $N$  dimensional Galerkin approximation corresponding to (86) (see [14], subsection 9.1). After passing to the limit  $N \rightarrow +\infty$ , we obtain existence of at least one solution  $\mathbf{u}^\varepsilon$  for problem (86), such that  $|D\mathbf{u}^{1,\varepsilon}|_{L^2(\Omega_\varepsilon)^4} \leq 1/\sqrt{\varepsilon}$ . After plugging this information into estimate (85), equation (86) yields the energy estimate

$$2\mu |D\mathbf{u}^{1,\varepsilon}|_{L^2(\Omega_\varepsilon)^4}^2 \leq C\sqrt{\varepsilon} |D\mathbf{u}^{1,\varepsilon}|_{L^2(\Omega_\varepsilon)^4}^2 + C\sqrt{\varepsilon} |D\mathbf{u}^{1,\varepsilon}|_{L^2(\Omega_\varepsilon)^4},$$

265 implying estimates (18)-(20).

Now we have

$$\left| \int_{\Omega_\varepsilon} (\mathbf{u}^{1,\varepsilon} \nabla) \mathbf{u}^{1,\varepsilon} \cdot \mathbf{v} \right| \leq C\varepsilon |\nabla \mathbf{u}^{1,\varepsilon}|_{L^2(\Omega)^2}^2 |\nabla \mathbf{v}|_{L^2(\Omega)^2} \leq C\varepsilon^2 |\nabla \mathbf{v}|_{L^2(\Omega)^2}^2, \quad \forall \mathbf{v} \in V(\Omega_\varepsilon) \quad (88)$$

and we conclude that in the calculations from subsections 4.2-4.4 the inertia term does not play any role. Hence it does not contribute to the homogenized problem either. This observation concludes the proof of Theorem 2.  $\square$

## 6. Numerical confirmation of the effective model

In this section we use a direct computation of the boundary layer corrector (36-38) and the microscopic problem (1-3) to numerically confirm the estimate (39)

$$|\pi - C_\pi|_{L^2(G^*)} = O(\sqrt{\varepsilon})$$

and the strong convergence of the effective pressure (7). For the pressure we find out

$$|p^\varepsilon - p^0|_{L^2(\Omega)} = O(\sqrt{\varepsilon}),$$

270 which is consistent with the corrector type results from [18].

### 6.1. Confirmation of boundary layer estimate

We start with estimate (39). For this we need to compute the value  $C_\pi$  which is the limit value of the boundary layer pressure  $\pi$  for  $y_2 \rightarrow \infty$ , see (41). Since the boundary layer problem is defined on an unbounded domain, 275 we need to cut the domain and compute  $C_{\pi, cut}$ , which is the approximation of  $C_\pi$  on a cut-off domain with  $|y_2|$  large enough so that the difference  $|C_\pi - C_{\pi, cut}|$  is smaller than the machine precision. Since the value  $\pi(\mathbf{y})$  stabilizes to  $C_\pi$  exponentially fast, we expect that a boundary layer with a few unit cells gives an accurate approximation. Furthermore, the cut-off boundary layer is 280 computed by the finite element method. Thus, we compute  $C_{\pi, cut}^h$ , where the

superscript  $h$  indicates the Galerkin approximation, and we have to assure that the discretization error  $|C'_{\pi, \text{cut}} - C^h_{\pi, \text{cut}}|$  is small enough.

For the numerical approximation we first introduce the cut-off domain

$$G_l^* := G \setminus \bigcup_{j=0}^{-l} (j \mathbf{e}_2 + A)$$

and then consider the following cut-off boundary layer problem

**Problem 1 (Cut-off boundary layer problem).** *Find  $\mathbf{w}$  and  $\pi$ , both 1-periodic in  $y_1$ , such that it holds in the interior*

$$-\mu \Delta_y \mathbf{w}_l + \nabla_y \pi_l = 0 \quad \text{in } G_l^*, \quad (89)$$

$$\nabla \cdot \mathbf{w}_l = 0 \quad \text{in } G_l^*, \quad (90)$$

and on the boundaries

$$(-2\mu \mathbf{D}_y \mathbf{w}_l + \pi_l \mathbf{I}) = \mathbf{e}_1 \quad \text{for } y_2 = 0, \quad (91)$$

$$\mathbf{w}_l = 0 \quad \text{on } \mathcal{C} \quad (92)$$

$$w_{l,2} = \frac{\partial w_{l,1}}{\partial y_2} = 0 \quad \text{on } \Gamma_l, \quad (93)$$

where  $\Gamma_l = (0, 1) \times l$  is the lower boundary of the cut-off domain.

285 The inclusions are defined as in Figure 1. The solid domain  $A$  is

(a) circular in the isotropic case with radius  $r = 0.25$  and center  $(0.5, 0.5)$ , see Figure 1a;

(b) ellipsoidal in the anisotropic case with centre  $(0.5, 0.5)$  and semi-axes  $a = 0.357142857$  and  $b = 0.192307692$  rotated anti-clockwise by  $45^\circ$ , see Figure 1b.

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Problem (89)-(93) is approximated by the finite element method (FEM) using a Taylor-Hood element [30] with bi-quadratic elements for the velocity and bilinear for the pressure. Since the inclusions are curvilinear we use a quadratic description of the finite element boundaries (iso-parametric finite elements). The stabilized pressure value of the boundary layer is defined in our computations

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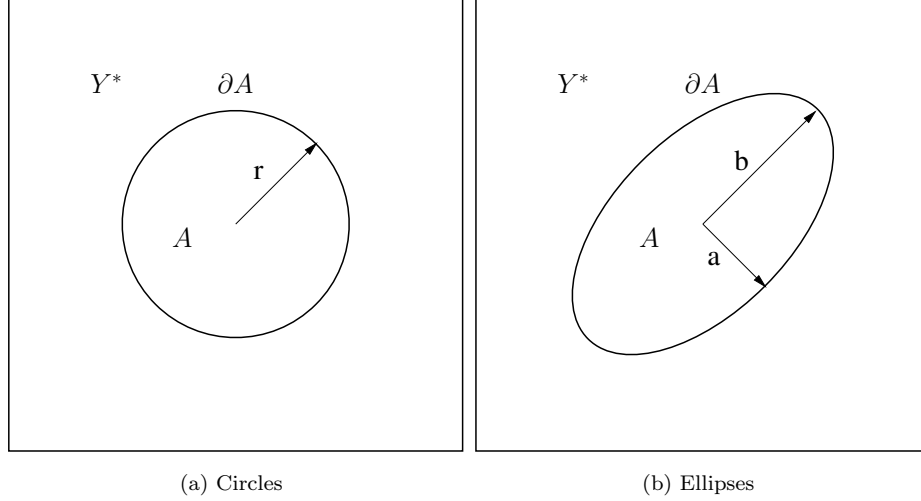


Figure 1: Inclusion geometry.

as  $C_{\pi, cut}^h := \pi_{l,h}(y_1, l)$ , i.e. it is the pressure value at the lower boundary of  $G_l^*$ . To define the value  $C_{\pi, cut}^h$  we have performed a test with increasing  $l$  to obtain the minimal length  $l$  of the cut-off domain for which the pressure value reaches convergence (up to machine precision). A shorter domain would introduce a numerical error and a longer domain would increase the computational costs without adding more accuracy.

In Table 1 the values of  $\pi_{l,h}(y_1, l)$  for increasing number of inclusions  $l$  are reported. It can be observed that one inclusion is enough to get the exact value  $C_\pi = 0$  for the circular inclusions. In case of elliptical inclusions the pressure is stabilized for  $l \geq 7$  and the effect of the cut-off domain can be seen only for smaller domains. Figure 2 shows a visualization of the boundary layer pressure  $\pi$  in the cut-off domain with seven inclusions. A convergence check with global refined meshes have shown that the discretization error is of the order  $O(10^{-8})$ . Therefore for the convergence study of the effective pressure, we consider as exact value for ellipses  $C_\pi = 0.2161642$ .

After computing the constant  $C_{\pi, cut}^h$  we proceed with the confirmation of

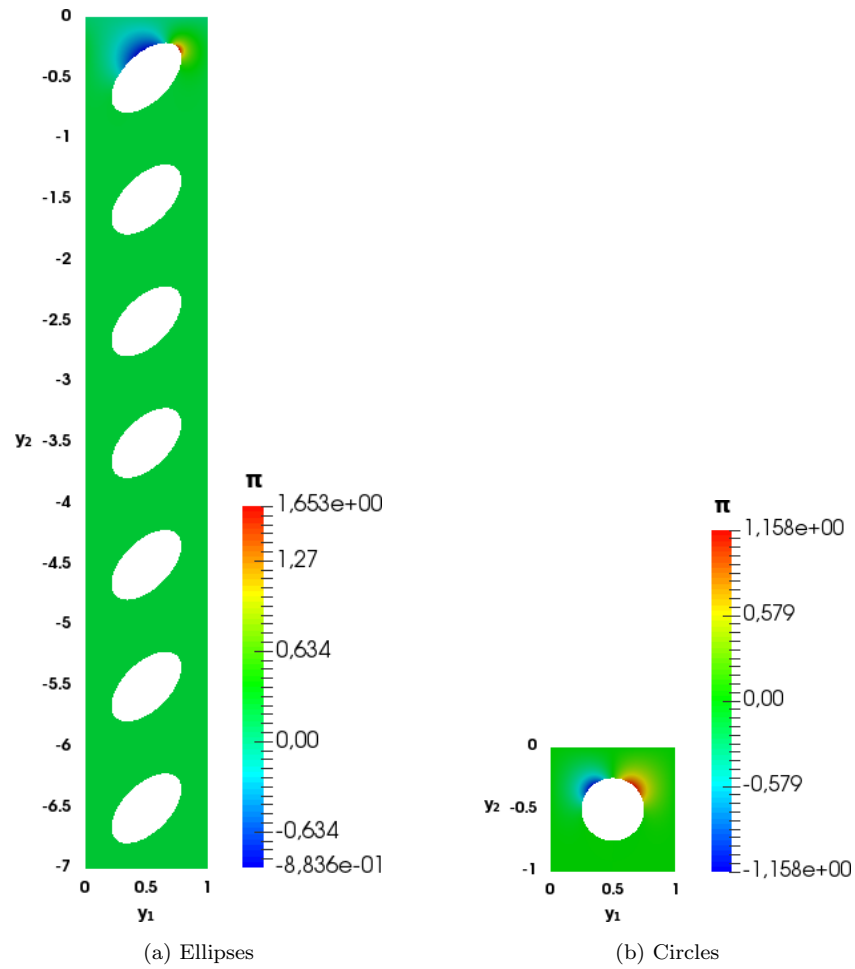


Figure 2: Visualization of boundary layer pressure and cut-off domain.

| # inclusions | $C_\pi$ ellipses          | $C_\pi$ circles |
|--------------|---------------------------|-----------------|
| 1            | 0.21632306467205          | 0               |
| 2            | 0. <b>2161</b> 6431545737 | 0               |
| 3            | 0. <b>216164</b> 22652809 | 0               |
| 4            | 0. <b>21616422</b> 675581 | 0               |
| 5            | 0. <b>2161642267</b> 5552 | 0               |
| 6            | 0. <b>216164226755</b> 55 | 0               |
| 7            | 0. <b>21616422675555</b>  | 0               |

Table 1: Stabilization of  $C_\pi$  in the cut-off domain with increasing number of inclusions.

the estimate (39) and plot in Figure 3 the convergence curves. We confirm the expected convergence rates

$$|\pi - C_\pi|_{L^1(G^*)} = O(\epsilon) \quad \text{and} \quad |\pi - C_\pi|_{L^2(G^*)} = O(\sqrt{\epsilon}).$$

## 6.2. Confirmation of effective pressure values

The next step is the confirmation of the estimate (7). For a stress tensor defined by the constant contact stress  $(P, Q)$  and a right hand side which depends only on  $x_2$  we have the analytical exact solution for the effective pressure

$$p^0(x_2) = C_\pi P + Q - \int_{x_2}^0 f_2(z) \, dz - \frac{K_{12}}{K_{22}} \int_{x_2}^0 f_1(z) \, dz. \quad (94)$$

To compute it we need the vales  $K_{12}$  and  $K_{22}$  of the permeability tensor. These are defined as follows

$$K_{ij} := \int_{Y^*} w_{c,j}^i \, dx,$$

with the 1-periodic solution  $\mathbf{w}^i$  ( $i = 1, 2$ ) of the  $i^{th}$  cell problem

$$\begin{aligned} -\Delta \mathbf{w}^i + \nabla \pi^i &= \mathbf{e}_i & \text{in } Y^*, \\ \nabla \cdot \mathbf{w}^i &= 0 & \text{in } Y^*, \\ \mathbf{w}^i &= 0 & \text{on } \partial A \end{aligned}$$

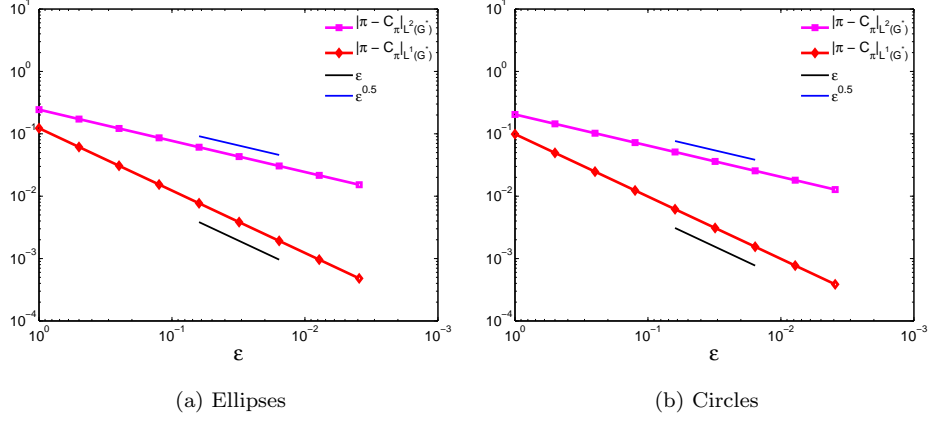


Figure 3: Confirmation of convergence for the boundary layer problem.

| Inclusion type | $K_{12}$         | $K_{11} = K_{22}$ |
|----------------|------------------|-------------------|
| Ellipses       | 0.00303449804138 | 0.0159787174788   |
| Circles        | 0.0              | 0.01990143534975  |

Table 2: Values of the permeability tensor components.

where  $Y^*$  is the unit pore domain of the cell problem with the corresponding inclusion  $A$  (see (80)-(81), (83)). The inclusions are defined as in our previous work [9]. They correspond to one cell of problem (89)-(93) and they are shown on Figure 1. Therefore, we use the values of the permeability tensor computed therein and reported in Table 2. We use the extension  $\tilde{p}_h^\varepsilon$  (16) for the microscopic pressure, where the subscript denotes the finite element approximation of the microscopic problem obtained with Taylor-Hood elements, as for the cut-off boundary layer.

With the expression of the effective pressure and the extension pressure we compute the convergence estimates. For the test case we use the values  $(P, Q)$  for the normal component of the stress tensor and  $\mathbf{f}(\mathbf{x})$  for the right hand side, needed in formula (94), as reported in Table 3. The results with the expected

| $\mu$ | $P(x_1)$ | $Q(x_1)$ | $f_1(\mathbf{x})$ | $f_2(\mathbf{x})$ | $C_\pi$ ellipses | $C_\pi$ circles |
|-------|----------|----------|-------------------|-------------------|------------------|-----------------|
| 1     | 1        | 1        | 1                 | 1                 | 0.2161642        | 0               |

Table 3: Values used for the computations.

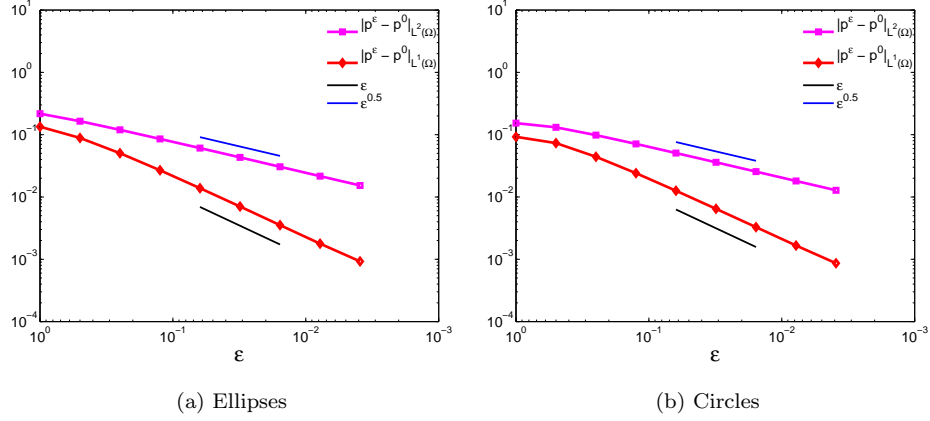


Figure 4: Confirmation of convergence for the microscopic problem.

convergence rates are depicted in Figure 4.

Considering the configuration described in Table 3, formula (94) gives the following effective pressure for the circular inclusions:

$$p^0(x_2) = 1 + x_2, \quad -1 \leq x_2 \leq 0, \quad (95)$$

and for the ellipses we have:

$$p^0(x_2) = 0.2161642 + 1 + x_2 + \frac{0.00303449804138}{0.0159787174788} x_2, \quad -1 \leq x_2 \leq 0. \quad (96)$$

325 In Figures 5 and 6 we show the pointwise difference  $|p^\epsilon(x_1, x_2) - p^0(x_2)|$  for the case of 8 inclusions corresponding to  $\epsilon = 0.125$  with circles and ellipses respectively. It is clearly visible that, as expected, the microscopic pressure  $p^\epsilon(x_1, x_2)$  stabilizes to the macroscopic value  $p^0(x_2)$  and that the largest difference is in the vicinity of the interface.



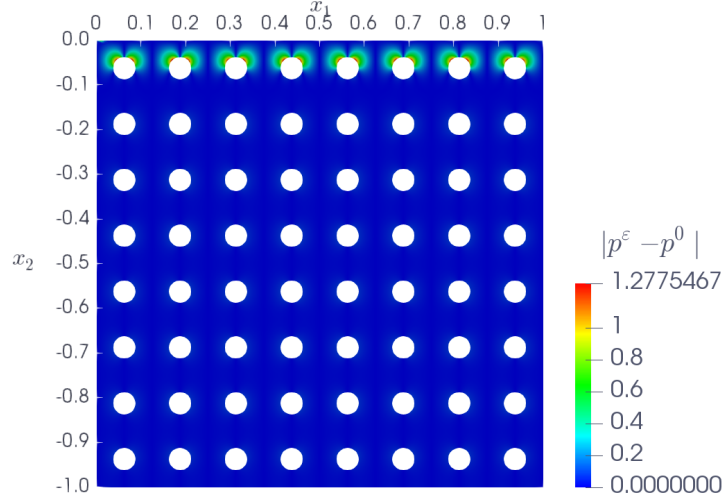


Figure 5: Visualization of the pointwise difference  $|p^\epsilon(x_1, x_2) - p^0(x_2)|$  for circular inclusions.

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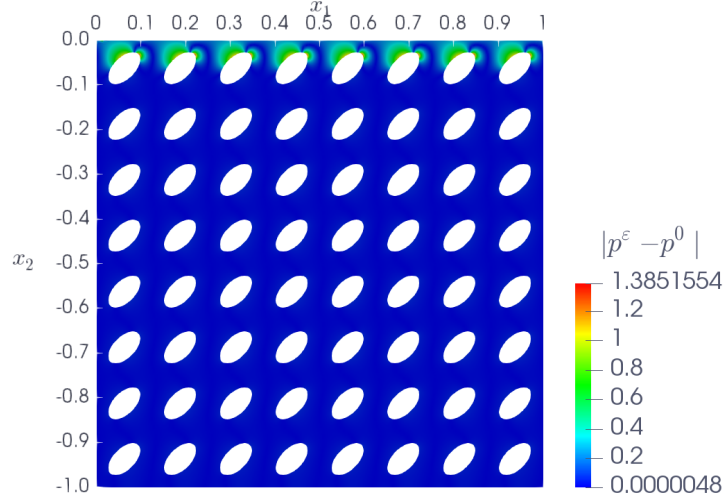


Figure 6: Visualization of the pointwise difference  $|p^\varepsilon(x_1, x_2) - p^0(x_2)|$  for elliptical inclusions.

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