MODELLING AND HOMOGENIZING A PROBLEM OF SORPTION/DESORPTION IN POROUS MEDIA *

ANDRO MIKELIĆ

Institut Camille Jordan,
UFR Mathématiques, Site de Gerland, Bât. A
Université Claude Bernard Lyon 1,
50, Avenue Tony Garnier,
69367 Lyon Cedex 07, FRANCE
Andro.Mikelic@univ-lyon1.fr

MARIO PRIMICERIO

Dipartimento di Matematica " Ulisse Dini "
Viale Morgagni 67/A
I-50134 Firenze, ITALY
mario.primicerio@math.unifi.it

Received (Day Month Year)
Revised (Day Month Year)
Communicated by (xxxxxxxxxx)

We consider a convection-diffusion problem in a porous medium saturated by a solution of a chemical substance A in water. A nonlinear non-equilibrium kinetics of sorption/desorption of A on the porous matrix is assumed. We assume that the chemical substance can be transported by ionic exchange through the walls of an array of parallel tubes in which the solution flows at a prescribed velocity. The well-posedness of the problem is proved under different boundary conditions. If the array of tubes is periodic, we homogenize the problem and we prove that there exists a unique solution to the homogenized problem, in which the terms of interaction due to chemical exchange through the walls of the tubes are cast in the differential equation.

Keywords: sorption in porous media; homogenization; non-linear chemical kinetics; non-linear parabolic PDEs.

AMS Subject Classification: 35B27, 35K20, 76S05

1. Introduction

In a previous paper [18] we made a preliminary analysis of a mathematical problem modeling ionic exchange in a porous medium, saturated by a liquid solution, through the injection of a liquid in an array of parallel pipes whose walls are permeable to the

*The research of A.M. was supported in part by the GDR MOMAS (Modélisation Mathématique et Simulations numériques liées aux problèmes de gestion des déchets nucléaires: 2439 - ANDRA, BRGM, CEA, EDF, CNRS)

chemical substance to be extracted from (or to be added to) the porous medium. In the present paper we discuss a more general model and present a complete analysis of its mathematical aspects.

Let the array $P \subset I\!R^3$ be the set

$$P = \bigcup_{k=1}^{N} P_k, \quad P_k \equiv \{(x-x_k)^2 + (y-y_k)^2 \le R_k^2, \quad 0 < z < H\}$$

for given positive $N, H, R_1, \dots, R_N$. We assume that the porous medium occupies the domain $K \setminus P$, where K is a cylinder $Q \times (0,H)$, containing P , and Q is a domain in $I\!R^2$ having smooth boundary.

We suppose that the porous medium is saturated by a solution of a chemical substance $\mathcal{A}$ in water. If $c(\underline{x},t)$ is the concentration of $\mathcal{A}$ in the solution (i.e. the mass of chemical per unit volume of water) and n is the porosity, the mass balance equation reads

$$\frac{\partial(nc)}{\partial t} = -\mathrm{div}\,\big(c\underline{q} - nD\nabla c\big) + n\Gamma + f, \quad \underline{x} \in K \setminus P,\, t > 0, \tag{1.1}$$

where $\underline{q}$ (a given divergence-free vector, since the porous medium is rigid and the fluid incompressible) is the volume of liquid flowing per unit time through a unit surface normal to it, Γ (mass per unit volume of liquid) is the rate at which the substance is produced/destroyed within the solution e.g. because of internal chemical reaction, decay etc. and f is the quantity of pollutant entering the solution (per unit bulk volume and per unit time), because of desorption from the solid matrix; of course $f < 0$ means that the chemical is leaving the solution because it is adsorbed on the grains of the porous matrix.

Conversely, balance of the same substance bound to the matrix has the following expression

$$\frac{\partial}{\partial t}\big((1-n)\rho_s F\big) = (1-n)\Gamma_s - f, \quad \underline{x} \in K \setminus P,\, t > 0, \tag{1.2}$$

where F is the mass ratio between the chemical adsorbed and the solid grains, ρ_s is the density of the latter and Γ_s has the same meaning as Γ . We assume that the adsorption does not affect porosity n significantly.

In addition to (1.1),(1.2) a law regulating the dynamics of adsorption/desorption has to be specified, i.e. f has to be prescribed.

As discussed e.g. in [3] there are two classes of laws that can be applied:

(i) **equilibrium isotherms**, when the quantities on the solid and in the adjacent solution are in equilibrium; and
(ii) **non-equilibrium isotherms**, when it is assumed that equilibrium is approached at a rate depending on the local values of c and of F .

Of course the use of laws of type (i) or (ii) depends on the time scale of the phenomenon we are studying. For general considerations about the "sufficiently fast" and reversible, and about the "insufficiently fast" and/or irreversible chemical reactions in solute transport analysis, see [23].

From a mathematical point of view, in case (i) the relation is monotone and F can be expressed in terms of c or vice-versa. Thus (1.1)-(1.2) reduce to a single (nonlinear) parabolic equation. Case (ii) is more general and more interesting, as the relation between c and F turns out to be a differential equation whose form depends on the nature of the chemical and of the porous matrix.

Among the forms that are found in the literature the most common (see [3]) are the non-equilibrium Langmuir isotherm (see [11])

$$\frac{\partial F}{\partial t} = \frac{1}{\tau}\big(\frac{\alpha c}{1+\beta c} - F\big) \tag{1.3}$$

and the non-equilibrium Freundlich isotherm (see [24])

$$\frac{\partial F}{\partial t} = \frac{1}{\tau}\big(\alpha c^{\beta} - F\big), \tag{1.4}$$

where α and β are experimental constants and $\tau > 0$ represents the time scale of the adsorption/desorption dynamics so that the case of vanishing τ takes us back to situation (i).

As far as Γ and Γ_s are concerned, they are assumed to be known and depend possibly on c and F respectively. For instance, in case of a substance undergoing radioactive (or any other type of linear) decay, we have

$$\Gamma = -\tilde{\lambda} c, \quad \Gamma_s = -\tilde{\mu} F, \tag{1.5}$$

for some positive constants $\tilde{\lambda}, \tilde{\mu}$. Upon normalization, we have that the following two equations hold in $K \setminus P$ and for $t > 0$

$$\frac{\partial U}{\partial t} - D\Delta U + \underline{q}\cdot\nabla U + \lambda U = S(V - \Phi(U)), \tag{1.6}$$

$$\frac{\partial V}{\partial t} = -S(V - \Phi(U)) - \mu V, \tag{1.7}$$

where the function f, according to (1.3),(1.4) has been expressed in a general form through two increasing functions S and Φ, such that $\Phi(0) = S(0) = 0$.

Equations (1.6) and (1.7) will be supplemented by initial conditions

$$U(\underline{x}, 0) = U_0(\underline{x}), \quad \underline{x} \in K \setminus P, \tag{1.8}$$

$$V(\underline{x}, 0) = V_0(\underline{x}), \quad \underline{x} \in K \setminus P, \tag{1.9}$$

and by suitable conditions on the external boundary Σ of $K \setminus P$. Let $\underline{n}_e$ be the normal to Σ pointing outwards. We write $\Sigma = \Sigma^+ \cup \Sigma^-$ where $\Sigma^- \equiv \big\{\underline{x} \in \Sigma : \underline{q}\cdot\underline{n}_e < 0\big\}$ and we assume that chemical $\mathcal{A}$ does not cross Σ^-, whereas on the seepage face it leaves the domain with the fluid. Thus

$$\begin{cases} -D\dfrac{\partial U}{\partial \underline{n}_e}(\underline{x}, t) + U(\underline{x}, t)\underline{q}\cdot\underline{n}_e = 0,\ x \in \Sigma^-,\ t > 0, \\ \\ \dfrac{\partial U}{\partial \underline{n}_e}(\underline{x}, t) = 0,\ x \in \Sigma^+,\ t > 0. \end{cases} \tag{1.10}$$

Note that in the special case $\Sigma^- = \emptyset$ and thus $\underline{q} \cdot \underline{n}_e = 0$ on Σ, the condition (1.10) reduces to the homogeneous Neumann condition.

For a reason that will be made clear later, we will also consider conditions

$$\begin{cases} -D \dfrac{\partial U}{\partial \underline{n}_e}(\underline{x}, t) + U(\underline{x}, t)\underline{q} \cdot \underline{n}_e - \vartheta U(\underline{x}, t) = 0,\ x \in \Sigma^-,\ t > 0, \\ \dfrac{\partial U}{\partial \underline{n}_e}(\underline{x}, t) = 0,\ x \in \Sigma^+,\ t > 0, \end{cases} \tag{1.11}$$

for some $\vartheta > 0$.

In addition, we have to prescribe the conditions on the walls of the pipes .

There, we assume that water can not cross the boundary ($\underline{q} \cdot \underline{n}_e = 0$, $\forall k,$ on the boundary $\partial P_k \cap K$) and natural conditions for ionic exchange suggest that flux of $\mathcal{A}$ is proportional to the jump in concentrations, or more generally that, for $k = 1, \dots, N$,

$$D \frac{\partial U}{\partial \underline{n}_k} = \gamma[U(\underline{x}, t) - \delta c_k(\underline{x}, t)], \quad \underline{x} \in \partial P_k \cap K, \quad t > 0, \tag{1.12}$$

where γ is an increasing function from $I\!R$ to $I\!R, \gamma[0] = 0$, and $\underline{n}_k$ is the unit outward normal vector to the cylinder P_k, while c is the concentration at the inner wall.

We will also consider the condition

$$D \frac{\partial U}{\partial \underline{n}_k} - \vartheta U = \gamma[U(\underline{x}, t) - \delta c_k(\underline{x}, t)], \quad \underline{x} \in \partial P_k \cap K, \quad t > 0. \tag{1.13}$$

Next, we have to write the mass balance for c inside each tube.

Assume $R_k << \text{diam } Q$ for $k = 1, \dots, N$ so that, for any $t > 0$, the concentration c can be thought to depend on position through the z coordinate only. Moreover, we assume incompressibility of water and suppose that walls are rigid and impermeable to water so that a bulk velocity $v_k(t)$ directed along the z-axis can be defined. For simplicity, we suppose $v_k(t) = v(t) > 0, \forall k$.

Thus, putting $\delta c(\underline{x}, t) = u_k(\underline{x}, t)$ for each $\underline{x} \in P_k$, we have that, at any time, u_k depends on the coordinate z only and we write

$$\begin{aligned} \frac{\partial u_k}{\partial t} + v(t) \frac{\partial u_k}{\partial z} - d \frac{\partial^2 u_k}{\partial z^2} = \frac{2}{R_k} \int_0^{2\pi} \gamma\big[U(x_k + R_k \cos\phi, \\ y_k + R_k \sin\phi, z, t) - u_k(z, t)\big]\ d\phi, \quad z \in (0, H),\ t > 0,\ k = 1, \dots, N. \end{aligned} \tag{1.14}$$

We will have initial conditions

$$u_k(z, 0) = u_{k0}(z), \quad z \in (0, H), \quad k = 1, 2...N, \tag{1.15}$$

and boundary conditions at $z = 0$ and $z = H$.

We suppose e.g. that clear water is injected at $z = 0$, so that we can essentially assume

$$u_k(0, t) = 0, \quad t > 0, \quad k = 1, 2...N. \tag{1.16}$$

At $z = H$, we may prescribe several type of boundary conditions. The simplest is to suppose that $z = H$ is a "seepage surface" in the sense that the liquid (together with the chemicals dissolved in it) is instantaneously removed as it leaves P_k . This means

$$\frac{\partial u_k}{\partial z}(H,t) = 0, \quad t > 0,\, k = 1, \ldots, H. \tag{1.17}$$

Again, we write a modified condition that will be useful in the sequel

$$\frac{\partial u_k}{\partial z}(H,t) + \vartheta u_k(H,t) = 0, \quad t > 0,\, k = 1, \ldots, H. \tag{1.18}$$

A less standard condition consists in assuming that all tube discharge in the same reservoir of volume V that can be considered instantaneously mixed, so that the concentration of $\mathcal{A}$ in the reservoir can be considered as a space-independent unknown function $\Upsilon(t)$.

The mass balance can be written as follows

$$V\frac{d\Upsilon(t)}{dt} = -\pi \sum_{j=1}^{N} R_j^2 \big(d\frac{\partial u_j}{\partial z}(H,t) - v(t)u_j(H,t)\big) - v(t)\Upsilon(t)\pi \sum_{j=1}^{N} R_j^2,\, t > 0 \tag{1.19}$$

$$\Upsilon(0) = u_0 \geq 0. \tag{1.20}$$

In addition we should specify a relationship between $u_k(H,t)$, $\frac{\partial u_k}{\partial z}(H,t)$ and $\Upsilon(t)$ introducing a sort of impedance of the boundary layer between each tube and the reservoir. To simplify we can assume that concentration is continuously changing from the pipes to the reservoirs so that

$$u_k(H,t) = \Upsilon(t), \quad t > 0, \quad k = 1, 2 ... N. \tag{1.21}$$

Summing up, we have

$$\frac{\partial u_k(H,t)}{\partial t} + \frac{d\pi}{V} \sum_{j=1}^{N} R_j^2 \frac{\partial u_j}{\partial z}(H,t) = 0, \quad t > 0,\, k = 1, \ldots, N, \tag{1.22}$$

$$u_k(H,0) = u_0 \geq 0, \quad h = 1, \ldots, N. \tag{1.23}$$

Once again, we will consider a modified condition

$$\frac{\partial u_k(H,t)}{\partial t} + \vartheta u_k(H,t) + \frac{d\pi}{V} \sum_{j=1}^{N} R_j^2 \frac{\partial u_j}{\partial z}(H,t) = 0, \quad t > 0,\, k = 1, \ldots, N. \tag{1.24}$$

The plan of the paper is the following. We start by considering the problem as it is stated above, i.e. at a scale that can be seen as mesoscopic. As a matter of fact, the porous medium is considered in the framework of continuum so that Darcy's law is assumed to hold; moreover, the radii of the pipes are supposed to be "small" with respect to the dimensions of the domain Q . For such a problem (with arbitrary number and spacing of pipes, and without periodicity or assumptions on values of R_k) we deduce a-priori bounds (Section 2) and we prove uniqueness (Section 3) and existence (Section 4) of solutions. In the proof of existence given in [18] linearity of S, Φ and Γ is essential, whereas here we deal with the general case.

In the last section we present the homogenization of our mesoscopic model. We suppose that the array of parallel tubes is periodic, that their radius is tending to zero but their number tends to infinity, with constant porosity of the system. For the homogenization of PDE's in perforated domains, with non-homogeneous conditions at the interfacial boundaries or even surface chemical reactions, classical references are the article [6], by D. Cioranescu and P. Donato, and [14] by U. Hornung and W. Jäger. We note that the reference [14] was first mathematically rigorous paper treating homogenization of the adsorption and surface diffusion effects. There is even a notion of 2-scale convergence, by M. Neuss-Radu, G. Allaire, U. Hornung and A. Damlamian, in [2] and [21], adapted to the problems with reactions on surfaces. The mentioned results allow to homogenize efficiently **linear** problems. Homogenization of the linear version of our problem is in [18], and it was achieved by directed application of two classical references. For homogenization of bio-medical problems with the surface bio-heat transfer condition we refer to [13].

For non-linear problems situation is more delicate. Our basic reference is the paper [15], where a combination of the two-scale convergence and compactness methods was used to homogenize a huge class of non-linear parabolic problems, with non-linear interface conditions. In addition to the reference [15], we mention recent papers [7] and [8] on the homogenization of non-linear adsorption problems. Characteristic of all these references is that compactness of the concentrations in a porous medium surrounding grains was enough to pass to the limit in the non-linear surface terms. E.g. in [8] it was possible to write the surface concentrations as function of the volume concentrations, using an ordinary differential equation. Similarly, in [15], the monotonicity results for 2-scale convergence were used. Our setting is quite different. If we make a smooth extension of the boundary concentrations, then the gradient would behave as $1/\varepsilon$. Such estimate wouldn't be useful and we were obliged to develop the new compactness results, in order to pass to the limit. We use the equation on the surfaces of the cylinders and the PDE is the porous part to prove a Frchet-Kolmogorov equicontinuity estimate in L^2. We believe that this approach could be applied to other homogenization problems involving non-linear terms on the surfaces.

2. Assumptions on data and a priori bounds

We consider the following problems:

We prescribe the nonnegative bounded functions $U_0(\underline{x}), V_0(\underline{x}), \underline{x} \in K \setminus P$, and $u_{k0}(z), z \in (0,H)$, $k = 1, 2...N$, and we look for $N+2$ functions $U(\underline{x},t), V(\underline{x},t)$, $\underline{x} \in K \setminus P, t > 0$, and $u_k(z,t), z \in (0,H), t > 0$, such that equations (1.6), (1.7) and (1.14) are satisfied and for given $D > 0$, q, $\lambda \geq 0$, $\mu \geq 0$, $v \geq 0$, $d > 0$ conditions (1.8)-(1.10), (1.12) and (1.15)-(1.16) are fulfilled, together with

either (i) (1.17)

or (ii) (1.22)-(1.23)

The problem will be called **Problem** ($\mathcal{P}$) in case (i) and **Problem** ($\mathcal{P}'$) in case (ii).

Moreover, we consider similar problems with (1.10), (1.12), (1.17), (1.22) replaced by (1.11), (1.13), (1.18), (1.24) respectively and with $\tilde{\lambda}$ and $\tilde{\mu}$ in (1.6), (1.7) replaced by $\lambda+\vartheta$ and $\mu+\vartheta$.

We will call the corresponding problems **Problem** ($\mathcal{P}_\vartheta$) and **Problem** ($\mathcal{P}'_\vartheta$).

We will use the following assumptions on the data

(A) S, Φ, γ are continuous increasing functions, such that $\Phi(0) = 0 = \gamma(0) = S(0)$.

(A1) In addition to (**A**), we suppose that the functions S, Φ, γ are locally Lipschitz and that Φ is strictly monotone.

(B) $\underline{q}$ is a continuous divergence-free vector field on $\overline{K} \times [0,T]$.

(B1) In addition to (**B**), we suppose that $\underline{q} \in W^{1,\infty}(K \times (0,T))^3$

(C) v is a continuous non-negative function on $[0,T]$

(D) $U_0, V_0 \in H^1(K \setminus P)$, $\underline{u}_0 \in H^1(0,H)^N$ and $\underline{u}_0(0) = \underline{0}$.

(D1) $U_0, V_0 \in H^2(K \setminus P)$, $\underline{u}_0 \in H^2(0,H)^N$ and $\underline{u}_0(0) = \underline{0}$.

Then we have the following L^∞ -a priori limitations:

Theorem 2.1. *Let the assumption **(A)** be satisfied and let M be such that*

$$0 \le U_0(\underline{x}) \le M,\ \underline{x} \in K \setminus P, \tag{2.1}$$

$$0 \le V_0(\underline{x}) \le \Phi(M),\ \underline{x} \in K \setminus P, \tag{2.2}$$

$$0 \le u_{k0}(z) \le M,\ z \in (0,H),\ \ k = 1,2...N. \tag{2.3}$$

*Then for any classical solution of **Problem** $\mathcal{P}_\vartheta$ we have*

$$0 \le U(\underline{x},t) \le M, \quad \underline{x} \in K \setminus P,\ t > 0, \tag{2.4}$$

$$0 \le V(\underline{x},t) \le \Phi(M), \quad \underline{x} \in K \setminus P,\ t > 0, \tag{2.5}$$

$$0 \le u_k(z,t) \le M, \quad z \in (0,H), t > 0,\ \ k = 1,2...N, \tag{2.6}$$

*For **Problem** $\mathcal{P}'_\vartheta$, (2.4)-(2.6) hold under the conditions (2.1)-(2.3) and*

$$0 \le u_0 \le M, \quad z \in (0,H). \tag{2.7}$$

To prove the theorem, we need the following

Lemma 2.1. *Fix $\varepsilon > 0$ and let the assumptions of Theorem 2.1 be satisfied. Let us suppose that there is a $t_0 > 0$ such that for $t \in (0,t_0)$ we have*

$$U(\underline{x},t) > -\varepsilon, \quad \underline{x} \in K \setminus P, \tag{2.8}$$

then on the same time interval we also have

$$V(\underline{x},t) > \Phi(-\varepsilon) \quad \underline{x} \in K \setminus P, \tag{2.9}$$

$$u_k(z,t) > -\varepsilon, \quad z \in (0,H),\ k = 1,\dots,N. \tag{2.10}$$

Proof. Assume (2.9) is violated for the first time at some point $(\underline{\tilde{x}}, \tilde{t})$, $\underline{\tilde{x}} \in K \backslash P$, $\tilde{t} \in (0, t_0)$. Then

$$\partial_t V(\underline{\tilde{x}}, \tilde{t}) = -S(\Phi(-\varepsilon) - \Phi(U)) - (\mu + \vartheta)\Phi(-\varepsilon). \tag{2.11}$$

But then, according to (2.8), the argument of S is negative; moreover $\Phi(-\varepsilon) < 0$. Thus $\partial_t V(\underline{\tilde{x}}, \tilde{t})$ would be positive yielding a contradiction.

Now assume that (2.10) is violated for the first time for some $\tilde{k}$ and at some point $\tilde{z} \in [0, H]$, $\tilde{t} \in (0, t_0)$.

Of course, it cannot be $\tilde{z} = 0$, because of (1.16). If $\tilde{z} \in (0, H)$, we would have that the left hand side of (1.14), written for $k = \tilde{k}$ and at $(\tilde{z}, \tilde{t})$ would be non-positive, while the argument of γ in the integral on the right hand side is positive, yielding a contradiction.

We have to exclude that $\tilde{z} = H$. In the case of **Problem** ($\mathcal{P}_\vartheta$) , (1.17) would imply $\dfrac{\partial u_{\tilde{k}}}{\partial t}(H, t) = \vartheta\varepsilon > 0$, i.e. a contradiction.

The case of **Problem** ($\mathcal{P}'_\vartheta$) is more delicate. First we note that if $\tilde{z} = H$ we would have $u_k(H, \tilde{t}) = -\varepsilon$ for any k because of (1.21) and hence $\dfrac{\partial u_k}{\partial z} \leq 0$ for $t = \tilde{t}$ and for any k. Then, from (1.24) we have

$$\frac{\partial u_k}{\partial z}(H, \tilde{t}) \leq \vartheta\varepsilon > 0, \tag{2.12}$$

a contradiction. □

The same kind of argument enables us to prove the following

Lemma 2.2. *Fix $\varepsilon > 0$ and let the assumptions of Theorem 2.1 be satisfied. Let us suppose that there is a $t_0 > 0$ such that for $t \in (0, t_0)$ we have*

$$U(\underline{x}, t) < M + \varepsilon, \quad \underline{x} \in K \setminus P, \tag{2.13}$$

then on the same time interval we also have

$$V(\underline{x}, t) < \Phi(M + \varepsilon) \quad \underline{x} \in K \setminus P, \tag{2.14}$$

$$u_k(z, t) < M + \varepsilon, \quad z \in (0, H),\ k = 1, \ldots, N. \tag{2.15}$$

Now we are in situation to prove Theorem 2.1.

Proof of Theorem 2.1. By the preceding lemmas, if we prove that it cannot exist a first $\widehat{t}$ such that (2.8) and (2.13) are violated, then we have that (2.9), (2.10) and (2.14),(2.15) hold for any $t > 0$.

We assume that there exists $\underline{x} \in \overline{K \setminus P}$ such that $\widehat{t}$ is the first time for which

$$U(\underline{\tilde{x}}, \widehat{t}) = -\varepsilon \tag{2.16}$$

and we prove that this leads to a contradiction (the proof can be repeated to prove the upper estimate). We recall that Lemma 2.1 implies that (2.9) and (2.10) are satisfied for $t \in (0, \widehat{t})$.

First we exclude that $\underline{\tilde{x}} \in \Sigma$. Indeed in this case (1.11) implies $\dfrac{\partial U}{\partial \underline{n}_e} > 0$, a contradiction. If $\underline{\tilde{x}} \in K \setminus P$ the left hand side of (1.6) is $\le -(\lambda+\vartheta)\varepsilon$ while the right hand side is nonnegative since $V \ge \Phi(-\varepsilon)$.

We have to exclude that $\underline{\tilde{x}} \in \partial P_k$ for some $\widehat{k}$. But the right hand side of (1.13)would be non positive and hence $\dfrac{\partial U}{\partial \underline{n}_e} < 0$, i.e. a contradiction, since $\underline{n}_e$ is the normal to ∂P_k pointing out of the tube.

Since ε is arbitrary we conclude that (2.4), (2.5) and (2.6) hold under the assumptions of Theorem 2.1. □

Remark 2.1. It is easy to verify that the assumption on monotonicity of S, Φ and γ can be weakened. Indeed, adding a term ϑu_k on the left hand side of (1.14) yields the result also for nondecreasing γ. Monotonicity of S was never used and, concerning Φ it is sufficient to assume that it does not vanish identically in any neighborhood of the origin.

Next intrinsic property of the models are the **energy equalities**. We prove them for the strong solutions.

Proposition 2.1. *Let us suppose the assumptions on the data **(A1)** , **(B)**, **(C)** and **(D)**. Let* $\{U, V, \underline{u}\} \in H^1((K\setminus P)\times(0,T))^2 \times H^1((0,H)\times(0,T))^N$ *be a bounded solution for **Problem*** $(\mathcal{P}_\vartheta)$. *Then it satisfies the following energy equality*

$$\int_{K\setminus P} \frac{1}{2} U^2(\underline{x},t)\, d\underline{x} + D\int_0^t \int_{K\setminus P} |\nabla U|^2(\underline{x},\xi)\, d\underline{x} d\xi + \int_{K\setminus P}\int_0^{V(x,t)} \Phi^{-1}(\xi)\, d\xi dx +$$
$$\int_0^t\int_{K\setminus T} (\lambda+\vartheta) U^2(\underline{x},t)\, d\underline{x} d\xi + \int_0^t \int_{\Sigma^-} (\vartheta - \underline{q}\cdot\underline{n}_e)|U|^2\, dS d\xi + \sum_{k=1}^N \int_0^t \int_{\partial P_k} \vartheta U_k^2\, dS d\xi$$
$$+ \int_0^t \int_{K\setminus P} S(V-\Phi(U))(\Phi^{-1}(V) - U)(\underline{x},\xi)\, d\underline{x} d\xi + \sum_{k=1}^N \int_0^t \int_{\partial P_k} \gamma(U_k -$$
$$u_k)(U_k - u_k)\, dS d\xi + \sum_{k=1}^N \frac{R_k^2}{2} \Big\{ \int_0^H \frac{1}{2} u_k^2(z,t)\, dz + \int_0^t (\frac{v(\xi)}{2} + \vartheta) u_k^2(H,\xi)\, d\xi +$$
$$d\int_0^t \int_0^H |\partial_z u_k(z,t)|^2\, dz \Big\} + \int_0^t \int_{K\setminus P} (\mu+\vartheta) V\Phi^{-1}(V)(\underline{x},\xi)\, d\underline{x} d\xi = \int_{K\setminus P} \frac{1}{2} U_0^2(x)\, dx$$
$$+ \int_{K\setminus P} \int_0^{V_0(x)} \Phi^{-1}(\xi)\, d\xi dx + \sum_{k=1}^N \frac{R_k^2}{4} \int_0^H u_{k,0}^2(z)\, dz - \int_0^t \int_{K\setminus P} \underline{q} \nabla U U\, d\underline{x} d\xi, \tag{2.17}$$

where $U_k = U|_{\partial P_k}$.

Proof. We test the equation (1.6) with U, the equation (1.7) with $\Phi^{-1}(V)$ and add the resulting equalities. This yields

$$\partial_t \int_{K\setminus P} \frac{1}{2} U^2(\underline{x},t)\ d\underline{x} + D\int_{K\setminus P} |\nabla U|^2(\underline{x},t)\ d\underline{x} + \int_{K\setminus P} (\lambda+\vartheta)|U|^2(\underline{x},t)\ d\underline{x} +$$
$$\int_{K\setminus P} \underline{q}\nabla U U\ d\underline{x} + \int_{\Sigma^-} (\vartheta - \underline{q}\cdot\underline{n}_e)|U|^2\ dS + \partial_t \int_{K\setminus P}\int_0^{V(\underline{x},t)} \Phi^{-1}(\xi)\ d\xi d\underline{x} +$$
$$\int_{K\setminus P} S(V-\Phi(U))(\Phi^{-1}(V)-U)\ d\underline{x} + \sum_{k=1}^N \int_{\partial P_k} \gamma(U_k-u_k)(U_k-u_k)\ dS$$
$$+\sum_{k=1}^N \int_0^t \int_{\partial P_k} \vartheta U_k^2\ dS d\xi + \sum_{k=1}^N \int_{\partial P_k} \gamma(U_k-u_k)u_k\ dS = 0, \tag{2.18}$$

where U_k denotes the trace of U at ∂P_k. Next we test the equation (1.14) with u_k and get

$$\sum_{k=1}^N \int_{\partial P_k} \gamma(U_k-u_k)u_k\ dS = \sum_{k=1}^N \frac{R_k^2}{2}\Big\{ \partial_t \int_0^H \frac{1}{2} u_k^2(z,t)\ dz +$$
$$(\frac{1}{2}v(t)+\vartheta)u_k^2(H,t) + d\int_0^H |\partial_z u_k(z,t)|^2\ dz \Big\} \tag{2.19}$$

After inserting (2.19) into (2.18) we get the energy equality (2.17). □

Proposition 2.2. *Let us suppose the assumptions on the data* ***(A1)*** , ***(B)***, ***(C)*** *and* ***(D)***. *Let* $\{U,V,\underline{u},\Upsilon\} \in H^1((K\setminus P)\times(0,T))^2 \times H^1((0,H)\times(0,T))^N \times H^1(0,T)$ *be a bounded solution for* ***Problem*** $(\mathcal{P}'_\vartheta)$. *Then it satisfies the following energy equality*

$$\int_{K\setminus P} \frac{1}{2} U^2(\underline{x},t)\ d\underline{x} + D\int_0^t \int_{K\setminus P} |\nabla U|^2(\underline{x},\xi)\ d\underline{x}d\xi + \int_{K\setminus P}\int_0^{V(x,t)} \Phi^{-1}(\xi)\ d\xi dx +$$
$$\int_0^t \int_{K\setminus P} (\lambda+\vartheta)U^2(\underline{x},t)\ d\underline{x}d\xi + \int_0^t \int_{\Sigma^-} (\vartheta-\underline{q}\cdot\underline{n}_e)|U|^2\ dSd\xi + \sum_{k=1}^N \int_0^t\int_{\partial P_k} \vartheta U_k^2\ dSd\xi +$$
$$\int_0^t \int_{K\setminus P} S(V-\Phi(U))(\Phi^{-1}(V)-U)(\underline{x},\xi)\ d\underline{x}d\xi + \sum_{k=1}^N \int_0^t \int_{\partial P_k} \gamma(U_k-$$
$$u_k)(U_k-u_k)\ dSd\xi + \sum_{k=1}^N \frac{R_k^2}{2}\Big\{ \int_0^H \frac{1}{2}u_k^2(z,t)\ dz + d\int_0^t\int_0^H |\partial_z u_k(z,t)|^2\ dz\Big\} +$$
$$\frac{V}{4\pi}\Upsilon^2(t) + \frac{1}{2}\int_0^t \{\frac{\vartheta}{\pi} + v(\tau)(\sum_{k=1}^N \frac{1}{2}R_k^2)\}\Upsilon^2(\tau)\ d\tau + \int_0^t\int_{K\setminus P} (\mu+\vartheta)V\Phi^{-1}(V)(\underline{x},\xi)\ d\underline{x}d\xi$$
$$= \int_{K\setminus P} \frac{1}{2}U_0^2(x)\ dx + \frac{V}{4\pi}u_0^2 + \int_{K\setminus P}\int_0^{V_0(x)} \Phi^{-1}(\xi)\ d\xi dx +$$
$$\sum_{k=1}^N \frac{R_k^2}{4}\int_0^H u_{k,0}^2(z)\ dz - \int_0^t\int_{K\setminus P} \underline{q}\nabla U U\ d\underline{x}d\xi \tag{2.20}$$

where $U_k = U|_{\partial P_k}$.

Proof. We test the equation (1.6) with U , the equation (1.7) with $\Phi^{-1}(V)$ and add the resulting equalities. This yields

$$\partial_t \int_{K\setminus P} \frac{1}{2} U^2(\underline{x},t)\ d\underline{x} + D\int_{K\setminus P} |\nabla U|^2(\underline{x},t)\ d\underline{x} + \int_{K\setminus P} (\lambda+\vartheta)|U|^2(\underline{x},t)\ d\underline{x} +$$

$$\int_{K\setminus P} \underline{q}\nabla U U\ d\underline{x} + \int_{\Sigma^-} (\vartheta - \underline{q}\cdot\underline{n}_e)|U|^2\ dS + \partial_t \int_{K\setminus P}\int_0^{V(\underline{x},t)} \Phi^{-1}(\xi)\ d\xi d\underline{x} +$$

$$\int_{K\setminus P} S(V-\Phi(U))(\Phi^{-1}(V)-U)\ d\underline{x} + \sum_{k=1}^N \int_{\partial P_k} \gamma(U_k-u_k)(U_k-u_k)\ dS$$

$$+\sum_{k=1}^N \int_0^t \int_{\partial P_k} \vartheta U_k^2\ dSd\xi + \sum_{k=1}^N \int_{\partial P_k} \gamma(U_k-u_k)u_k\ dS = 0, \tag{2.21}$$

where U_k denotes the trace of U at ∂P_k . Next we test the equation (1.14) with $w_k = u_k - z\Upsilon(t)/H$ and using equation (1.24) we get

$$\sum_{k=1}^N \int_{\partial P_k} \gamma(U_k-u_k)u_k\ dS = \sum_{k=1}^N \frac{R_k^2}{2}\Big\{\partial_t \int_0^H \frac{1}{2}u_k^2(z,t)\ dz +$$

$$\frac{1}{2}v(t)\Upsilon^2(t) + d\int_0^H |\partial_z u_k(z,t)|^2\ dz\Big\} + \frac{V}{4\pi}\partial_t\Upsilon^2(t) + \frac{\vartheta}{2\pi}\Upsilon^2(t) \tag{2.22}$$

After inserting (2.22) into (2.21) we get the energy equality (2.20). □

3. Uniqueness

In this section we study the uniqueness of solution to the **Problem** $(\mathcal{P})$ and to the **Problem** $(\mathcal{P}')$. For the problems **Problem** $(\mathcal{P}_\vartheta)$ and **Problem** $(\mathcal{P}'_\vartheta)$ proof is exactly the same. The proof relies on the fact that the problem has an energy functional hidden in its structure and on the monotonicity of the exchange function γ .

Let $V_2^{1,0}((K\setminus P)\times(0,T)) = C([0,T];L^2(K\setminus P))\cap L^2(0,T;H^1(K\setminus P))$ We have

Theorem 3.1. *Assume **(A1)**, **(B)** and **(C)**. Then **Problem** $(\mathcal{P})$ has a unique bounded non-negative solution* $\{U,V,\underline{u}\}$ $\in V_2^{1,0}((K\setminus P)\times(0,T))^2 \times V_2^{1,0}((0,H)\times(0,T))^N$.

Proof. Let us suppose that there exist two solutions for the **Problem** $(\mathcal{P})$. Then the difference of the solutions, denoted by $\{U,V,\underline{u}\}$, is once more in $V_2^{1,0}((K\setminus P)\times(0,T))^2 \times V_2^{1,0}((0,H)\times(0,T))^N$. We note that there are N capillary pipes P_i of the length H and consequently function $\underline{u}$ is vector valued with N components.

We proceed in several steps.

1. STEP Function U satisfies the equation

$$\partial_t U - D\Delta U + \underline{q}\cdot\nabla U + \lambda U = S(V_1 - \Phi(U_1)) - S(V_2 - \Phi(U_2)) \tag{3.1}$$

Consequently, after testing (3.1) with U, we get

$$\frac{1}{2}\int_{K\setminus P} U^2(\underline{x},t)\; dx + D\int_0^t\int_{K\setminus P} |\nabla_x U(\underline{x},\xi)|^2\; d\underline{x}d\xi + \int_0^t\int_{K\setminus P} \underline{q}\cdot\nabla UU\; d\underline{x}d\xi +$$
$$\int_0^t\int_{K\setminus P}\lambda U^2\; dxd\xi + D\sum_{i=1}^N\int_0^t\int_{\partial P_i}\nabla_x U\cdot\underline{n}_i U\; dSd\xi - \int_0^t\int_{\Sigma^-} U^2\underline{q}\cdot\underline{n}_e\; dSd\xi =$$
$$\int_0^t\int_{K\setminus P}\big(S(V_1-\Phi(U_1)) - S(V_2-\Phi(U_2))\big)U(\underline{x},\xi)\; d\underline{x}d\xi \tag{3.2}$$

Since

$$|\int_0^t\int_{K\setminus P}\big(S(V_1-\Phi(U_1)) - S(V_2-\Phi(U_2))U(\underline{x},\xi)\; d\underline{x}d\xi| \le$$
$$\|S'\|_\infty\|\Phi'\|_\infty\int_0^t\int_{K\setminus P}|U(\underline{x},\eta)|^2\; d\underline{x}d\eta + \|S'\|_\infty\int_0^t\int_{K\setminus P}|U(\underline{x},\eta)||V(\underline{x},\eta)|\; d\underline{x}d\eta, \tag{3.3}$$

$$|\int_0^t\int_{K\setminus P}\underline{q}\cdot\nabla UU\; d\underline{x}d\xi| \le \int_0^t\int_{K\setminus P}\big(\frac{\|\underline{q}\|_\infty^2}{2D}U^2 + \frac{D}{2}|\nabla U|^2\big)\; d\underline{x}d\xi \tag{3.4}$$

and

$$D\int_{\partial P_i}\nabla_x U\cdot\underline{n}_i U\; dS = \int_{\partial P_i}\big(\gamma(U_1|_{r=R_i} - (u_1)_i) - \gamma(U_2|_{r=R_i} - (u_2)_i)\big)U|_{r=R_i}\; dS \tag{3.5}$$

we get

$$\frac{1}{2}\int_{K\setminus P} U^2(\underline{x},\xi)\; d\underline{x}d\xi + \frac{D}{2}\int_0^t\int_{K\setminus P}|\nabla U|^2\; d\underline{x}d\xi + \int_0^t\int_{K\setminus P}(\lambda - \frac{\|\underline{q}\|_\infty^2}{2D})U^2\; d\underline{x}d\xi$$
$$+\sum_{i=1}^N\int_0^t\int_{\partial P_i}\big(\gamma(U_1|_{r=R_i} - (u_1)_i) - \gamma(U_2|_{r=R_i} - (u_2)_i)\big)U|_{r=R_i}\; dSd\xi \le$$
$$\|S'\|_\infty\|\Phi'\|_\infty\int_0^t\int_{K\setminus P}|U(\underline{x},\eta)|^2\; d\underline{x}d\eta + \|S'\|_\infty\int_0^t\int_{K\setminus P}|U(\underline{x},\eta)||V(\underline{x},\eta)|\; d\underline{x}d\eta \tag{3.6}$$

2. STEP Next we study the equation for V. After testing the difference of the equations (1.7) by V and integrating over $(K\setminus P)\times(0,t)$, we obtain

$$\frac{1}{2}\int_{K\setminus P} V^2(\underline{x},t)\; d\underline{x} + \int_0^t\int_{K\setminus P}\mu V^2\; d\underline{x}d\xi \le \|S'\|_\infty\int_0^t\int_{K\setminus P} V^2(\underline{x},\xi)\; d\underline{x}d\xi +$$
$$\|S'\|_\infty\|\Phi'\|_\infty\int_0^t\int_{K\setminus P}|V(\underline{x},\xi)||U(\underline{x},\xi)|\; d\underline{x}d\xi \tag{3.7}$$

3. STEP Now we study the equation for u_k :

$$\frac{\partial u_k}{\partial t} + v(t)\frac{\partial u_k}{\partial z} - d\frac{\partial^2 u_k}{\partial z^2} = \frac{2}{R_k}\int_0^{2\pi} \big\{\gamma(U_1|_{r=R_k} - (u_1)_k) - \gamma(U_2|_{r=R_k} - (u_2)_k)\big\}\, d\vartheta \quad \text{in } (0,H)\times(0,T) \tag{3.8}$$

We test (3.8) by u_k and integrate with respect to z and ξ . Then we have

$$\pi R_k^2\Big(\frac{1}{2}\int_0^H u_k^2(z,t)\, dz + \int_0^t \frac{v(\xi)}{2}u_k^2(H,\xi)\, d\xi + d\int_0^t\int_0^H |\frac{\partial u_k}{\partial z}(z,\xi)|^2\, dzd\xi\Big) = 2\pi R_k\int_0^t\int_0^H\int_0^{2\pi} u_k(z,\xi)\big\{\gamma(U_1|_{r=R_k} - (u_1)_k) - \gamma(U_2|_{r=R_k} - (u_2)_k)\big\}\, d\vartheta dz d\xi$$

After summation over k , we get

$$\frac{1}{2\pi}\sum_{k=1}^N \frac{\pi R_k^2}{2}\int_0^H u_k^2(z,t)\, dz + \frac{d}{2\pi}\sum_{k=1}^N \pi R_k^2\int_0^t\int_0^H |\frac{\partial u_k}{\partial z}(z,\xi)|^2\, dzd\eta - \sum_{k=1}^N\int_0^t\int_{\partial P_k} u_k(z,\xi)\big\{\gamma(U_1|_{r=R_k} - (u_1)_k) - \gamma(U_2|_{r=R_k} - (u_2)_k)\big\}\, dSd\xi \le 0 \tag{3.9}$$

4. STEP Now we add the estimates (3.6), (3.7) and (3.9) and obtain

$$\frac{1}{2}\int_{K\setminus P} U^2(\underline{x},t)\, d\underline{x} + \frac{1}{2}\int_{K\setminus P} V^2(\underline{x},t)\, d\underline{x} + \frac{1}{2\pi}\sum_{k=1}^N \frac{\pi R_k^2}{2}\int_0^H u_k^2(z,t)\, dz + \frac{D}{2}\int_0^t\int_{K\setminus P} |\nabla U|^2\, d\underline{x}d\xi + \frac{d}{2\pi}\sum_{k=1}^N \pi R_k^2\int_0^t\int_0^H |\frac{\partial u_k}{\partial z}(z,\xi)|^2\, dzd\eta + \sum_{k=1}^N\int_0^t\int_{\partial P_k} (U|_{r=R_k} - u_k)(z,\xi)\big\{\gamma(U_1|_{r=R_k} - (u_1)_k) - \gamma(U_2|_{r=R_k} - (u_2)_k)\big\}\, dSd\xi \le \frac{3}{2}C\int_0^t\int_{K\setminus P} (U^2(\underline{x},\xi) + V^2(\underline{x},\xi))\, d\underline{x}d\xi \tag{3.10}$$

Using monotonicity of γ and Gronwall's inequality , we easily conclude that $U(\underline{x},t) = 0 = V(\underline{x},t)$ and $u = 0$. □

Next we have

Theorem 3.2. *Assume **(A1)**, **(B)** and **(C)**. Then **Problem** $(\mathcal{P}')$ has a unique bounded non-negative solution* $\{U, V, \underline{u}, \Upsilon\} \in V_2^{1,0}((K\setminus P)\times(0,T))^2 \times V_2^{1,0}((0,H)\times(0,T))^N \times H^1(0,T)$.

Proof. Let us suppose that there exist two solutions for the **Problem** $(\mathcal{P}')$. Then the difference of the solutions, denoted by $\{U, V, \underline{u}, \Upsilon\}$, is once more in $V_2^{1,0}((K\setminus P)\times(0,T))^2\times V_2^{1,0}((0,H)\times(0,T))^N\times H^1(0,T)$. We note that there are N capillary pipes P_i of the length H and consequently function $\underline{u}$ is vector valued with N components.

We proceed in several steps.

1. STEP It is exactly the same as the Step 1 from Theorem 3.1 .

2. STEP It is again exactly the same as the Step 2 from Theorem 3.1 .

3. STEP Let u_k takes value $\overline{u}$ at $z = H$. Then we test equation (3.8) by u_k and integrate with respect to z and ξ . Then we have

$$\pi R_k^2 \Big(\frac{1}{2} \int_0^H u_k^2(z,t)\ dz + \int_0^t \frac{v(\xi)}{2} \overline{u}^2(\xi)\ d\xi + d \int_0^t \int_0^H |\frac{\partial u_k}{\partial z}(z,\xi)|^2\ dzd\xi - d \int_0^t \frac{\partial u_k}{\partial z}(H,\xi)\overline{u}(\xi)\ d\xi \Big) = 2\pi \int_0^t \int_{\partial P_k} u_k(z,\xi) \big\{ \gamma(U_1|_{r=R_k} - (u_1)_k) - \gamma(U_2|_{r=R_k} - (u_2)_k) \big\}\ dSd\xi$$

After summation over k , we get

$$\frac{1}{2\pi} \sum_{k=1}^N \frac{\pi R_k^2}{2} \int_0^H u_k^2(z,t)\ dz + \frac{d}{2\pi} \sum_{k=1}^N \pi R_k^2 \int_0^t \int_0^H |\frac{\partial u_k}{\partial z}(z,\xi)|^2\ dzd\eta + \frac{V}{4}\overline{u}^2(t) + \frac{1}{2}\int_0^t (V\vartheta + v(\xi)(\sum_{k=1}^N R_k^2))\overline{u}^2(\xi)\ d\xi - \sum_{k=1}^N \int_0^t \int_{\partial P_k} u_k(z,\xi)\big\{\gamma(U_1|_{r=R_k} - (u_1)_k) - \gamma(U_2|_{r=R_k} - (u_2)_k)\big\}\ dSd\xi = 0 \tag{3.11}$$

and proceeding as in the Step 4 from the proof of Theorem 3.1, we conclude the uniqueness. □

4. Existence

Next, we prove the existence of a solution to problems $(\mathcal{P})$, $(\mathcal{P}')$, $(\mathcal{P}_\vartheta)$ and $(\mathcal{P}'_\vartheta)$. Because of maximum principle, proved in theorem 2.1, we start by considering the existence of the strong solution for *bounded* and *globally Lipschitz* continuous nonlinearities γ , S and Φ . A possible approach would be to use the sectorial operators, standard in the geometric theory of semilinear parabolic operators, and establish a local existence and uniqueness. Then one should search for the maximal time interval of the existence. This is the classical approach and we refer to the classical book of D. Henry [12] for details. Nevertheless, we have complicated interface conditions and manipulating the fractional powers of corresponding operators seems to be quite technical. From this reason we prefer to give a simpler proof by discretization in the space variables. The existence will follow from the energy estimate and appropriate time estimates.

We start by considering the **Problem** $(\mathcal{P})$ and the **Problem** $(\mathcal{P}_\vartheta)$.

Theorem 4.1. *Assume **(A1)**, **(B1)**, **(C)** and **(D)**. Then **Problem** $(\mathcal{P})$ and the **Problem** $(\mathcal{P}_\vartheta)$ admit at least one solution* $\{U, V, \underline{u}\} \in \Big(L^2(0,T; H^1(K \setminus$

$P)) \cap L^\infty(0,T;L^2(K\setminus P))\Big) \times \Big(H^1((K\setminus P)\times(0,T)) \cap W^{1,\infty}(0,T;L^2(K\setminus P))\Big) \times \Big(L^2(0,T;H^1(0,H)) \cap L^\infty(0,T;L^2(0,H))\Big)^N$, *such that* $\partial_t\{U,V,\underline{u}\} \in L^2((K\setminus P)\times(0,T)) \times H^1(0,T;L^2(K\setminus P)) \times (L^2((0,H)\times(0,T)))^N$.

Proof. It is enough to consider **Problem** $(\mathcal{P}_\vartheta)$ with $\vartheta \geq 0$.

1. STEP Let $\{\zeta_j\}_{j\in\mathbb{N}}$ be a smooth basis for $H^1(K\setminus P)$ and $\{\beta_j\}_{j\in\mathbb{N}}$ a smooth basis for $W = \{\varphi \in H^1(0,H) \mid \varphi(0) = 0\}$. Then we start by looking for an approximate solution. More precisely, we look for

$$U_m = \sum_{j=1}^m \alpha_j(t)\zeta_j, \quad V_m = \sum_{j=1}^m \delta_j(t)\zeta_j \text{ and } u_{m,k} = \sum_{j=1}^m \omega_{j,k}(t)\beta_j \tag{4.1}$$

satisfying the system

$$\int_{K\setminus P} \partial_t U_m \zeta_j \, d\underline{x} + D\int_{K\setminus P} \nabla U_m \nabla\zeta_j \, d\underline{x} + \sum_{k=1}^N \int_{\partial P_k} \Big(\gamma(U_{m,k} - u_{m,k}) + \vartheta U_{m,k}\Big)\zeta_j \, dS + \int_{K\setminus P} (\lambda+\vartheta)U_m\zeta_j \, d\underline{x} + \int_{\Sigma^-} (\vartheta - \underline{q}\cdot\underline{n}_e)U_m\zeta_j \, dS + \int_{K\setminus P} \underline{q}\nabla U_m\zeta_j \, d\underline{x} = \int_{K\setminus P} S(V_m - \Phi(U_m))\zeta_j \, d\underline{x}, \quad \forall j\in\{1,\dots,m\} \tag{4.2}$$

$$\int_{K\setminus P} \partial_t V_m\zeta_j \, d\underline{x} + \int_{K\setminus P} S(V_m - \Phi(U_m))\zeta_j \, d\underline{x} + \int_{K\setminus P} (\mu+\vartheta)V_m\zeta_j \, d\underline{x} = 0, \ \forall j\in\{1,\dots,m\} \tag{4.3}$$

$$\int_0^H \partial_t u_{m,k}\beta_l \, dz + \int_0^H v(t)\partial_z u_{m,k}\beta_l \, dz + d\int_0^H \partial_z u_{m,k}\partial_z\beta_l \, dz + \vartheta u_{m,k}(H,t)\beta_l(H) = \frac{2}{R_k^2}\int_{\partial P_k} \gamma(U_{m,k} - u_{m,k})\beta_l \, dS, \quad \forall l\in\{1,\dots,m\} \tag{4.4}$$

$$U_m(x,0) = U_{m,0}(x), \ V_m(x,0) = V_{m,0}(x), \ u_{m,k}(z,0) = u_{m,k,0}, \tag{4.5}$$

where the initial values are projected to the corresponding functional spaces.

It is obvious that the Cauchy problem (4.2) -(4.5) has a unique continuously differentiable solution on $[0,T_m]$.

2. STEP In this step we prove that $T_m = T$ by obtaining the *a priori* estimates.

First, as in Proposition 2.1, we prove the energy equality (2.17) for $\{U_m, V_m, \underline{u}_m\}$. The equality (2.17), monotonicity of the non-linearities and Gronwall's inequality imply the following energy estimates :

$$\|U_m\|_{L^\infty(0,T;L^2(K\setminus P))} + \|\nabla U_m\|_{L^2(0,T;L^2(K\setminus P))} \le C \tag{4.6}$$

$$\|V_m\|_{H^1((0,T)\times(K\setminus P))} \le C \tag{4.7}$$

$$\sum_{k=1}^N \pi R_k^2 \Big\{ \sup_{0\le t\le T} \int_0^H \frac{1}{2} u_{m,k}^2(z,t)\ dz + d\int_0^T\int_0^H |\partial_z u_{m,k}(z,\xi)|^2\ dzd\xi \Big\} \le C \tag{4.8}$$

We need better estimates in time. In order to get them we test the equation (4.2) with $\partial_t U_m$. Then we get

$$\int_{K\setminus P} |\partial_t U_m|^2(\underline{x},t)\ d\underline{x} + \frac{D}{2}\partial_t \int_{K\setminus P} |\nabla U_m|^2(\underline{x},t)\ d\underline{x} + \int_{K\setminus P} S(V_m - \Phi(U_m))\partial_t U_m\ dx$$
$$+ \int_{K\setminus P} (\lambda+\vartheta) U_m \partial_t U_m\ d\underline{x} + \int_{K\setminus P} \underline{q}\nabla U_m \partial_t U_m\ d\underline{x} + \int_{\Sigma^-} (\vartheta - \underline{q}\cdot\underline{n}_e) U_m\partial_t U_m\ dS +$$
$$\sum_{k=1}^N \int_{\partial P_k} \gamma(U_{m,k}-u_{m,k})\partial_t(U_{m,k}-u_{m,k})\ dS + \sum_{k=1}^N \int_{\partial P_k} \gamma(U_{m,k}-u_{m,k})\partial_t u_{m,k}\ dS$$
$$+ \sum_{k=1}^N \int_{\partial P_k} \vartheta U_{m,k}\partial_t U_{m,k} = 0. \tag{4.9}$$

After using the equation (4.4) for transforming the term $\sum_{k=1}^N \int_{\partial P_k} \gamma(U_{m,k} - u_{m,k})\partial_t u_{m,k}\ dS$, we obtain the following equality

$$\int_0^t \int_{K\setminus P} |\partial_t U_m|^2(\underline{x},\xi)\ d\underline{x}d\xi + \frac{D}{2}\int_{K\setminus P} |\nabla U_m|^2(\underline{x},t)\ d\underline{x} + \sum_{k=1}^N \int_{\partial P_k} \frac{\vartheta U_m^2}{2}(\cdot,t)\ dS +$$
$$\frac{1}{2}\int_{K\setminus P} (\lambda+\vartheta)|U_m(\underline{x},t)|^2\ d\underline{x} + \frac{1}{2}\int_{\Sigma^-} (\vartheta - \underline{q}\cdot\underline{n}_e)|U_m(\cdot,t)|^2\ dS - \sum_{k=1}^N \int_{\partial P_k} \frac{\vartheta U_{m,0}^2}{2}(\cdot)\ dS$$
$$+ \sum_{k=1}^N \frac{R_k^2}{2}\Big\{ \int_0^t\int_0^H |\partial_t u_{m,k}|^2(z,\xi)\ dzd\xi + \frac{\vartheta}{2}|u_{m,k}(H,t)|^2 + \frac{d}{2}\int_0^H |\partial_z u_{m,k}(z,t)|^2\ dz \Big\} +$$
$$\sum_{k=1}^N \int_{\partial P_k} \int_{U_{m,0}-u_{m,k,0}}^{(U_{m,k}-u_{m,k})(t)} \gamma(\eta)d\eta\ dS = -\int_0^t \int_{K\setminus P} S(V_m - \Phi(U_m))\partial_t U_m(\underline{x},\xi)\ d\underline{x}d\xi$$
$$+ \int_{K\setminus P} \frac{D}{2}|\nabla U_{m,0}|^2(\underline{x})\ d\underline{x} + \frac{1}{2}\int_{K\setminus P} (\lambda+\vartheta)|U_{m,0}(\underline{x})|^2\ d\underline{x} + \frac{1}{2}\int_{\Sigma^-} (\vartheta - \underline{q}\cdot\underline{n}_e)|U_{m,0}(\cdot)|^2\ dS +$$
$$\sum_{k=1}^N \frac{R_k^2}{2}\Big\{ \frac{\vartheta}{2}|u_{m,k,0}(H)|^2 + \frac{d}{2}\int_0^H |\partial_z u_{m,k,0}(z)|^2\ dz - \int_0^t\int_0^H v(\xi)\partial_z u_{m,k}\partial_t u_{m,k}\ dzd\xi \Big\} -$$
$$\int_0^t \int_{K\setminus P} \underline{q}\nabla U_m \partial_t U_m\ d\underline{x}d\xi - \frac{1}{2}\int_0^t\int_{\Sigma^-} \partial_t \underline{q}\cdot\underline{n}_e |U_m|^2(\cdot,\xi)\ dSd\xi \tag{4.10}$$

Using the a priori estimates (4.6)-(4.8) and the equality (4.10) we have

$$\|\nabla U_m\|_{L^\infty(0,T;L^2(K\setminus P))} + \|\partial_t U_m\|_{L^2(0,T;L^2(K\setminus P))} \le C \tag{4.11}$$

$$\sum_{k=1}^{N} \pi R_k^2 \Big\{ \sup_{0\le t\le T} \int_0^H \frac{d}{2} |\partial_z u_{m,k}|^2(z,t)\ dz + \int_0^T \int_0^H |\partial_t u_{m,k}(z,t)|^2\ dzdt \Big\} \le C \tag{4.12}$$

3. STEP We note that the strong L^2- convergence of $\{U_m\}_{m\in\mathbb{N}}$ implies the same convergence of the sequence $\{V_m\}_{m\in\mathbb{N}}$. Then the a priori estimates (4.6)-(4.8), (4.11)-(4.12) allow us to choose strongly and weakly convergent subsequences. The obtained convergences allow an easy passing to the limit in the approximate problem. Thus all clusters are strong solutions for the **Problem** $(\mathcal{P}_\vartheta)$. As the estimates do not depend on $\vartheta \ge 0$, we have simultaneously existence for the **Problem** $(\mathcal{P})$. □

Now we consider the problems $(\mathcal{P}')$ and $(\mathcal{P}'_\vartheta)$. Here the calculations are bit more involved. We have

Theorem 4.2. *Assume* ***(A1)****,* ***(B1)****,* ***(C)*** *and* ***(D)****. Then the* ***Problem*** $(\mathcal{P}')$ *and the* ***Problem*** $(\mathcal{P}'_\vartheta)$ *admit at least one solution*

$$\begin{gathered}\{U, V, \underline{u}, \Upsilon\} \in \big(L^2(0,T;H^1(K\setminus P)) \cap L^\infty(0,T;L^2(K\setminus P))\big)\times \\ \big(H^1((K\setminus P)\times(0,T)) \cap W^{1,\infty}(0,T;L^2(K\setminus P))\big) \\ \times\Big(L^2(0,T;H^1(0,H)) \cap L^\infty(0,T;L^2(0,H))\Big)^N \times H^1(0,T),\ \textit{such that} \\ \partial_t\{U, V, \underline{u}, \Upsilon\} \in L^2((K\setminus P)\times(0,T)) \times H^1(0,T;L^2(K\setminus P))\times \\ (L^2((0,H)\times(0,T)))^N \times L^2(0,T)\ \textit{and}\ \underline{u}(H,t) = \Upsilon(t)\underline{1}.\end{gathered}$$

Proof. As before, it is enough to consider **Problem** $(\mathcal{P}'_\vartheta)$ with $\vartheta \ge 0$.

1. STEP Let $\{\zeta_j\}_{j\in\mathbb{N}}$ be a smooth basis for $H^1(K\setminus P)$ and $\{\xi_j\}_{j\in\mathbb{N}}$ a smooth basis for $H_0^1(0,H)$. Then we start by looking for an approximate solution. More precisely, we look for

$$\begin{cases} U_m = \sum_{j=1}^m \alpha_j(t)\zeta_j, \quad V_m = \sum_{j=1}^m \delta_j(t)\zeta_j, \\ w_{m,k} = \sum_{j=1}^m \omega_{j,k}(t)\xi_j \ \text{ and } \ u_m(t) \end{cases} \tag{4.13}$$

satisfying the system

$$\int_{K\setminus P}\partial_t U_m\zeta_j\ d\underline{x}+D\int_{K\setminus P}\nabla U_m\nabla\zeta_j\ d\underline{x}+\sum_{k=1}^N\int_{\partial P_k}\gamma(U_{m,k}-w_{m,k}-\frac{z}{H}u_m(t))\zeta_j\ dS+$$
$$\sum_{k=1}^N\int_{\partial P_k}\vartheta U_{m,k}\zeta_j\ dS+\int_{K\setminus P}(\lambda+\vartheta)U_m\zeta_j\ d\underline{x}+\int_{\Sigma^-}(\vartheta-\underline{q}\cdot\underline{n}_e)U_m\zeta_j\ dS$$
$$+\int_{K\setminus P}\underline{q}\nabla U_m\zeta_j\ d\underline{x}=\int_{K\setminus P}S(V_m-\Phi(U_m))\zeta_j\ d\underline{x},\quad\forall j\in\{1,\dots,m\}\tag{4.14}$$

$$\int_{K\setminus P}\partial_t V_m\zeta_j\ d\underline{x}+\int_{K\setminus P}S(V_m-\Phi(U_m))\zeta_j\ d\underline{x}+$$
$$\int_{K\setminus P}(\mu+\vartheta)V_m\zeta_j\ d\underline{x}=0,\ \forall j\in\{1,\dots,m\}\tag{4.15}$$

$$\int_0^H\partial_t w_{m,k}\xi_l\ dz+\partial_t u_m(t)\int_0^H\frac{z}{H}\xi_l\ dz+\int_0^H v(t)\partial_z w_{m,k}\xi_l\ dz+v(t)u_m(t)\int_0^H\frac{1}{H}\xi_l\ dz+$$
$$d\int_0^H\partial_z u_{m,k}\partial_z\xi_l\ dz=\frac{2}{R_k^2}\int_{\partial P_k}\gamma(U_{m,k}-w_{m,k}-\frac{z}{H}u_m(t))\xi_l\ dS,\ \forall l\in\{1,\dots,m\}\tag{4.16}$$

$$\frac{du_m}{dt}+\vartheta u_m=-\frac{\pi d}{HV}u_m(t)\sum_{k=1}^N R_k^2+\frac{2\pi}{V}\sum_{k=1}^N\int_{\partial P_k}\gamma(U_{m,k}-w_{m,k}-\frac{z}{H}u_m(t))\frac{z}{H}\ dS$$
$$-\frac{\pi}{V}\sum_{k=1}^N R_k^2\int_0^H v(t)\partial_z w_{m,k}\frac{z}{H}\ dz-\frac{\pi}{V}\sum_{k=1}^N R_k^2\int_0^H\partial_t w_{m,k}\frac{z}{H}\ dz-$$
$$\frac{\pi H}{3V}\frac{du_m(t)}{dt}\sum_{k=1}^N R_k^2-\frac{\pi}{V}\frac{v(t)u_m(t)}{2}\sum_{k=1}^N R_k^2\tag{4.17}$$

$$U_m(x,0)=U_{m,0}(x),\ V_m(x,0)=V_{m,0}(x),\ w_{m,k}(z,0)=P_m(u_{k,0}-\frac{z}{H}u_0),\ u_m(0)=u_0,\tag{4.18}$$

where the initial values are projected to the corresponding functional spaces.

Showing that the Cauchy problem (4.14) -(4.18) has a unique continuously differentiable solution on $[0,T_m]$ is equivalent to show that the matrix containing the coefficients in front of the time derivatives of $\frac{d\omega_{j,k}}{dt}$, $j\in\{1,\dots,m\},\ k\in\{1,\dots,N\}$ and u_m , is non-degenerate. Without loosing generality, we can suppose that $\{\xi_j\}$ is an orthonormal basis for $L^2(0,H)$ and an orthogonal basis for $H_0^1(0,H)$. Then

$$\frac{d\omega_{j,k}}{dt}=-\frac{du_m}{dt}\int_0^H\frac{z}{H}\xi_j\ dz+\mathcal{F}_{jk}(\vec{\omega}_1,\dots,\vec{\omega}_N,\vec{\alpha},\vec{\delta},u_m),\tag{4.19}$$

where $\mathcal{F}_{jk}$ are determined by (4.16).

Next we plug the expressions for $\frac{d\omega_{j,k}}{dt}$ into (4.17). It turns out that (4.17) can be written in the form

$$\{1+\frac{H\pi}{3V}\sum_{k=1}^{N}R_k^2-\frac{\pi}{V}(\sum_{k=1}^{N}R_k^2)\sum_{j=1}^{m}(\int_0^H \xi_j\frac{z}{H}\ dz)^2\}\frac{du_m}{dt}=\mathcal{F}(\vec{\omega}_1,\dots,\vec{\omega}_N,\vec{\alpha},\vec{\delta},u_m) \tag{4.20}$$

Since

$$\sum_{j=1}^{m}(\int_0^H \xi_j\frac{z}{H}\ dz)^2<\sum_{j=1}^{\infty}(\int_0^H \xi_j\frac{z}{H}\ dz)^2=\frac{H}{3}$$

we see that (4.20) gives an expression for $\frac{du_m(t)}{dt}$. Hence the coefficient matrix of the system (4.14)-(4.18) is non-degenerate and this Cauchy problem has a unique C^1 solution on $[0,T_m]$, for some $T_m>0$.

2.STEP In this step we prove that $T_m=T$ by obtaining the *a priori* estimates.

First, as in Proposition 2.2, we prove the energy equality (2.20) for $\{U_m,V_m,\underline{u}_m,u_m\}$. The equality equality (2.20), monotonicity of the non-linearities and Gronwall's inequality imply the following energy estimates :

$$\|U_m\|_{L^\infty(0,T;L^2(K\setminus P))}+\|\nabla U_m\|_{L^2(0,T;L^2(K\setminus P))}+\|V_m\|_{H^1((0,T)\times(K\setminus P))}\le C \tag{4.21}$$

$$\sum_{k=1}^{N}\pi R_k^2\Big\{\sup_{0\le t\le T}\int_0^H\frac{1}{2}u_{m,k}^2(z,t)\ dz+d\int_0^T\int_0^H|\partial_z u_{m,k}(z,\xi)|^2\ dzd\xi\Big\}\le C \tag{4.22}$$

We need better estimates in time. In order to get them we test the equation (4.14) with $\partial_t U_m$. Then we get

$$\int_{K\setminus P}|\partial_t U_m|^2(\underline{x},t)\ d\underline{x}+\frac{D}{2}\partial_t\int_{K\setminus P}|\nabla U_m|^2(\underline{x},t)\ d\underline{x}+\int_{K\setminus P}S(V_m-\Phi(U_m))\partial_t U_m\ dx$$
$$+\int_{K\setminus P}(\lambda+\vartheta)U_m\partial_t U_m\ d\underline{x}+\int_{K\setminus P}\underline{q}\nabla U_m\partial_t U_m\ d\underline{x}+\int_{\Sigma^-}(\vartheta-\underline{q}\cdot\underline{n}_e)U_m\partial_t U_m\ dS+$$
$$\sum_{k=1}^{N}\int_{\partial P_k}\gamma(U_{m,k}-u_{m,k})\partial_t(U_{m,k}-u_{m,k})\ dS+\sum_{k=1}^{N}\int_{\partial P_k}\gamma(U_{m,k}-u_{m,k})\partial_t u_{m,k}\ dS+$$
$$\sum_{k=1}^{N}\int_{\partial P_k}\vartheta\partial U_{m,k}U_{m,k}\ dS=0. \tag{4.23}$$

After using the equation (4.16) for transforming the term $\sum_{k=1}^{N}\int_{\partial P_k}\gamma(U_{m,k}-$

$u_{m,k})\partial_t u_{m,k}\ dS,$ we obtain the following equality

$$\int_0^t \int_{K\setminus P} |\partial_t U_m|^2(\underline{x},\xi)\ d\underline{x}d\xi + \frac{D}{2}\int_{K\setminus P} |\nabla U_m|^2(\underline{x},t)\ d\underline{x} + \sum_{k=1}^N \int_{\partial P_k} \frac{\vartheta U_m^2}{2}(\cdot,t)\ dS+$$

$$\frac{1}{2}\int_{K\setminus P}(\lambda+\vartheta)|U_m(\underline{x},t)|^2\ d\underline{x} + \frac{1}{2}\int_{\Sigma^-}(\vartheta - \underline{q}\cdot\underline{n}_e)|U_m(\cdot,t)|^2\ dS - \sum_{k=1}^N \int_{\partial P_k}\frac{\vartheta U_{m,0}^2}{2}(\cdot)\ dS+$$

$$\sum_{k=1}^N \frac{R_k^2}{2}\left\{\int_0^t\int_0^H |\partial_t u_{m,k}|^2(z,\xi)\ dzd\xi + \frac{d}{2}\int_0^H |\partial_z u_{m,k}(z,t)|^2\ dz\right\} + \frac{V}{2\pi}\int_0^t |\partial_t u_m|^2(\tau)\ d\tau+$$

$$\frac{V\vartheta}{4\pi}u_m^2(t) + \sum_{k=1}^N \int_{\partial P_k}\int_{U_{m,0}-u_{m,k,0}}^{(U_{m,k}-u_{m,k})(t)} \gamma(\eta)d\eta\ dS = -\int_0^t\int_{K\setminus P} S(V_m - \Phi(U_m))\partial_t U_m(\underline{x},\xi)\ d\underline{x}d\xi$$

$$+\int_{K\setminus P}\frac{D}{2}|\nabla U_{m,0}|^2(\underline{x})\ d\underline{x} + \frac{1}{2}\int_{K\setminus P}(\lambda+\vartheta)|U_{m,0}(\underline{x})|^2\ d\underline{x} + \frac{1}{2}\int_{\Sigma^-}(\vartheta - \underline{q}\cdot\underline{n}_e)|U_{m,0}(\cdot)|^2\ dS+$$

$$\sum_{k=1}^N \frac{R_k^2}{2}\left\{\frac{d}{2}\int_0^H |\partial_z u_{m,k,0}(z)|^2\ dz - \int_0^t\int_0^H v(\xi)\partial_z u_{m,k}\partial_t u_{m,k}\ dzd\xi\right\} + \frac{V\vartheta}{4\pi}u_0^2-$$

$$\int_0^t\int_{K\setminus P}\underline{q}\nabla U_m \partial_t U_m\ d\underline{x}d\xi - \frac{1}{2}\int_0^t\int_{\Sigma^-}\partial_t\underline{q}\cdot\underline{n}_e|U_m|^2(\cdot,\xi)\ dSd\xi \tag{4.24}$$

Using the a priori estimates (4.21)-(4.22) and the equality (4.24) we have

$$\|\nabla U_m\|_{L^\infty(0,T;L^2(K\setminus P))} + \|\partial_t U_m\|_{L^2(0,T;L^2(K\setminus P))} \le C \tag{4.25}$$

$$\sum_{k=1}^N \pi R_k^2\left\{\sup_{0\le t\le T}\int_0^H \frac{d}{2}|\partial_z u_{m,k}|^2(z,t)\ dz + \int_0^T\int_0^H |\partial_t u_{m,k}(z,t)|^2\ dzdt\right\} \le C \tag{4.26}$$

$$\|u_m\|_{H^1(0,T)} \le C \tag{4.27}$$

3. STEP We note that the strong L^2- convergence of $\{U_m\}_{m\in I\!N}$ implies the same convergence of the sequence $\{V_m\}_{m\in I\!N}$. Then the a priori estimates (4.21)-(4.22), (4.25)-(4.27) allow us to choose strongly and weakly convergent subsequences. The obtained convergences allow an easy passing to the limit in the approximate problem. Thus all clusters are strong solutions for **Problem** $(\mathcal{P}'_\vartheta)$. As the estimates do not depend on $\vartheta \ge 0$, we have simultaneously existence for the **Problem** $(\mathcal{P}')$. □

Remark 4.1. The strong solutions obtained in the previous theorems are unique.

Let us now prove the **regularity** for **Problem** $(\mathcal{P}_\vartheta)$ and **Problem** $(\mathcal{P})$. The extension of the results to **Problem** $(\mathcal{P}'_\vartheta)$ and **Problem** $(\mathcal{P}')$ are straightforward.

Theorem 4.3. *(regularity theorem) Let us suppose **(A1)**, **(B1)** , **(C)** and **(D1)**. Then the strong solutions for **Problems** $(\mathcal{P}_\vartheta)$, $(\mathcal{P})$, $(\mathcal{P}'_\vartheta)$, $(\mathcal{P}')$ belong* *to* $\big(C^{2,1}((K\setminus P)\times(0,T))^2\times C^{2,1}((0,H)\times(0,T))^N\big)\cap\big(C(\overline{K\setminus P}\times[0,T])^2\times H^{1,1/2}([0,H]\times[0,T])^N\big)$.

Proof. We apply the regularity theory from [17]. We proceed in several steps.

First, direct application of Th. 9.1, page 341 from [17] gives $u_k\in W_2^{2,1}((0,H)\times(0,T))$.

Next, we use u_k as data in the equation for U. Using once more Th. 9. 1 from [17], we get $U\in W_2^{2,1}((K\setminus P)\times(0,T))$ and the same is true for V. Consequently, using the embedding lemma 3.3., page 80, from [17], we conclude that $U|_{P_k}\in L^{10/3}((0,T)\times\partial P_k)$.

Now, we go back to the equation for u_k and find out that the right hand side belongs to $L^{10/3}((0,H\times(0,T))$. Thus $u_k\in W^{2,1}_{10/3}((0,H)\times(0,T))\subseteq H^{1,1/2}([0,H]\times[0,T])$.

Finally, we need the internal regularity for solution U of the parabolic problem with the nonlinear Neumann conditions (involving γ) and semilinear nonlinearities S and Φ. The classical theory from [17], chapter 5.7, and [9], chapter 7.5, implies that $\{U,V\}\in C^{2,1}((K\setminus P)\times(0,T))^2\cap\big(C(\overline{K\setminus P}\times[0,T])^2$. □

Remark 4.2. Now, for $\vartheta>0$, we can apply the maximum principle, proved in theorem 2.1, to conclude that solution satisfies the bounds (2.4)-(2.6). This justifies the assumption that non-linearities are bounded and globally Lipschitz.

Remark 4.3. If $\vartheta=0$, the classical maximum principle from theorem 2.1 doesn't apply directly. Nevertheless, for sequence $\{U^\vartheta,V^\vartheta,\underline{u}^\vartheta\}$, both the energy estimates (4.6)-(4.8), (4.11)-(4.12) and the L^∞-bounds (2.4)-(2.6) apply independently of ϑ. Then using the weak compactness, we conclude there are clusters $\{U,V,\underline{u}\}$, which satisfy the bounds (2.4)-(2.6), the energy estimates (4.6)-(4.8), (4.11)-(4.12) and the equations. The uniqueness theorem applies and, consequently, there is a unique limit. This proves that for $\vartheta=0$ the solution satisfies the bounds (2.4)-(2.6).

5. Homogenization of a periodic network of parallel pipes

In this section we consider the model with many pipes obtained by periodic repetition of an elementary section of size ε in the smooth domain $Q\subset\mathbb{R}^2$. An elementary section is a fixed open circle $\mathcal{Y}_C=\{(x,y)\in Y:\ x^2+y^2<\rho_C^2<1/4\}$ inside the unit cell $Y=(0,1)^2$. Other possibility is to have a finite number of circles at positive distances from each other and from ∂Y. Then $\mathcal{Y}_C$ would be their union. For simplicity we suppose here only one circle.

Let $\varepsilon\mathbb{Z}^2$ be a set of lattice points with edge of length ε, i.e. $\varepsilon\mathbb{Z}^2=\{p^i_\varepsilon:\ i\in\mathbf{Z}^2\}$. We make the periodic repetition of $\mathcal{Y}_C$ and set $\mathcal{P}^i_\varepsilon=p^i_\varepsilon+\varepsilon\mathcal{Y}_C$, $Y^i_\varepsilon=p^i_\varepsilon+\varepsilon Y$. The set of capillary pipes is given by $P_\varepsilon=\bigcup_i\{\mathcal{P}^i_\varepsilon:\ Y^i_\varepsilon\subset Q\}$. The porous medium part is

$$M^\varepsilon=\big(Q\setminus\overline{P_\varepsilon}\big)\times(0,H) \tag{5.1}$$

After covering Q with this mesh of size ε, we see that there are $N_\varepsilon = (\varepsilon^{-2})C(1+O(1))$ capillary pipes .

After [5] and [16] there exists an extension operator $\tilde{\Pi} \in \mathcal{L}(H^1(Y \setminus \bar{\mathcal{Y}}_C), H^1(Y))$ such that

$$\|\nabla(\tilde{\Pi}\phi)\|_{L^2(Y)^2} \leq \|\nabla\phi\|_{L^2(Y\setminus\bar{\mathcal{Y}}_C)^2}, \quad \forall \phi \in H^1(Y \setminus \bar{\mathcal{Y}}_C).$$

Then for every $\varepsilon > 0$ there exists an extension operator $\Pi^\varepsilon \in \mathcal{L}(H^1(Q\setminus\overline{P_\varepsilon}), H^1(Q))$ such that

$$\|\nabla(\Pi^\varepsilon\phi)\|_{L^2(Q)^2} \leq \|\nabla\phi\|_{L^2(Q\setminus\overline{P_\varepsilon})^2}, \quad \forall \phi \in H^1(Q \setminus \overline{P_\varepsilon}). \tag{5.2}$$

We note that this approach generalizes to a huge class of arbitrary elementary sections strictly included in the unit cell. Then the extension operator is constructed as in [4] .

Now we define auxiliary problems corresponding to various values of a given constant vector $\lambda \in I\!R^2$.

$$\begin{cases} -\Delta w_\lambda = 0 \ \text{ in } \ \mathcal{Y}_C; \quad \dfrac{\partial w_\lambda}{\partial n}|_{\partial\mathcal{Y}_C} = 0 \\ w_\lambda - \lambda\cdot(y_1, y_2) \ \text{ is } \ Y-\text{periodic.} \end{cases} \tag{5.3}$$

If $w^k = w_{e_k}$, then the effective diffusion matrix is given by $A_{ij} = \int_{\mathcal{Y}_C} \nabla w^i \cdot \nabla w^j \, dy_1 dy_2$. It is well-known that A is positive definite and symmetric matrix. Furthermore

$$\begin{cases} \tilde{\eta}^\varepsilon_\lambda = \nabla w_\lambda(\dfrac{x}{\varepsilon}, \dfrac{y}{\varepsilon})\chi_{Q\setminus\overline{P_\varepsilon}} \rightharpoonup A\lambda \text{ weakly in } L^\alpha_{loc}(I\!R^2), \\ \\ \chi_{Q\setminus\overline{P_\varepsilon}} \rightharpoonup \theta = |Y\setminus\bar{\mathcal{Y}}_C| = 1 - \rho_C^2\pi \text{ weakly in } L^\beta_{loc}(I\!R^2), \ \forall \beta \in [1, +\infty). \end{cases} \tag{5.4}$$

Remark 5.1. Let us suppose that $\mathcal{Y}_C$ is a circle of small radius ρ . Then, following [16] , we find

$$A = (1 - 2\rho^2\pi)I + o(\rho^2) \tag{5.5}$$

Next we need an auxiliary result for the interfaces. Homogenization of the non-homogeneous Neumann problem for the Laplace's operator in perforated domains was studied in [6] and the following result was proved on pages 120-122 :

Lemma 5.1. *Let* $\phi \in H^1(Q)$. *Then we have*

$$\varepsilon^2\rho_C \sum_i \int_0^{2\pi} \phi|_{\partial\mathcal{P}^i_\varepsilon} \, d\vartheta \to |\partial\mathcal{Y}_C| \int_Q \phi \, dxdy \quad \text{as} \quad \varepsilon \to 0. \tag{5.6}$$

Furthermore,

$$|\varepsilon^2\rho_C \sum_i \int_0^{2\pi} \phi|_{\partial\mathcal{P}^i_\varepsilon} \, d\vartheta - \frac{|\partial\mathcal{Y}_C|}{|Y\setminus\bar{\mathcal{Y}}_C|} \int_{Q\setminus\overline{\mathcal{P}_\varepsilon}} \phi \, dxdy| \leq C\varepsilon\|\phi\|_{H^1(Q)} \tag{5.7}$$

Next we suppose that the non-linearity $\gamma(\cdot)$ has the form $\varepsilon\gamma(\cdot)$. This assumptions guarantees the balance between the volume and surface terms in the limit $\varepsilon \to 0$.

Since **Problem** $(\mathcal{P}')$ is the most interesting case, we concentrate only on it. For other case, the result is analogous and slightly simpler. We leave the details to the reader.

After these auxiliary results we write the Problem $(\mathcal{P}')$ in the weak form :

$$\text{Find } U^\varepsilon \in L^2(0,T;H^1(M^\varepsilon)) \times L^\infty(M^\varepsilon \times (0,T)),\ \Upsilon^\varepsilon \in H^1(0,T),$$
$$\underline{u}^\varepsilon - \frac{z}{H}\Upsilon^\varepsilon \underline{1} \in L^2(0,T;H_0^1(0,H))^{N_\varepsilon} \cap L^\infty((0,H)\times(0,T))^{N_\varepsilon} \text{ and}$$
$$V^\varepsilon \in H^1(M^\varepsilon \times (0,T)) \cap L^\infty(M^\varepsilon \times (0,T)), \quad \text{such that } \partial_t U^\varepsilon \in L^2(M^\varepsilon \times (0,T)),$$
$$\partial_t \underline{u}^\varepsilon \in L^2((0,H)\times(0,T))^{N_\varepsilon}, \text{ with non-negative initial values}$$
$$\underline{u}^\varepsilon(\cdot,0) = \underline{u}_0(\cdot),\ \|\underline{u}^\varepsilon\|_{L^\infty(0,H)} \le M,\ \underline{u}_0(0) = \underline{0},\ \underline{u}_0(H) = u_0\underline{1},\ u_0 \in (0,M), \tag{5.8}$$
$$U^\varepsilon(\cdot,0) = U_0(\cdot) \in (0,M), \text{ and } V^\varepsilon(\cdot,0) = V_0(\cdot) \in (0,\Phi(M)), \tag{5.9}$$

which satisfy the following variational equations

$$\frac{d}{dt}\int_{M^\varepsilon} U^\varepsilon \phi \, d\underline{x} + \int_{M^\varepsilon} \Big\{ D\nabla U^\varepsilon \cdot \nabla\phi - S\big(V^\varepsilon - \Phi(U^\varepsilon)\big)\phi \Big\}\, d\underline{x} +$$
$$\int_{M^\varepsilon} \underline{q}\nabla U^\varepsilon \varphi \, d\underline{x} + \int_{M^\varepsilon} \lambda U^\varepsilon \varphi \, d\underline{x} - \int_{\Sigma^-} \underline{q}\cdot\underline{n}_e U^\varepsilon \varphi \, dS +$$
$$\varepsilon \sum_{i=1}^{N_\varepsilon} \int_0^H \int_{\partial\mathcal{P}_\varepsilon^i} \gamma\Big(U^\varepsilon - u_i^\varepsilon\Big)\phi \, dSdz = 0, \quad \forall \phi \in H^1(M^\varepsilon),\ t > 0, \tag{5.10}$$
$$2\pi\varepsilon \int_0^H \int_{\partial\mathcal{P}_\varepsilon^i} g(z)\gamma\Big(U^\varepsilon|_{\partial\mathcal{P}_\varepsilon^i} - u_i^\varepsilon\Big)\, dSdz = \frac{d}{dt}\int_0^H \int_{\mathcal{P}_\varepsilon^i} u_i^\varepsilon g \, dz +$$
$$v(t)\int_0^H \int_{\mathcal{P}_\varepsilon^i} \frac{\partial u_i^\varepsilon}{\partial z} g \, dz + d\int_0^H \int_{\mathcal{P}_\varepsilon^i} \frac{\partial u_i^\varepsilon}{\partial z}\frac{dg}{dz}\, dz, \quad \forall g \in H_0^1(0,H) \tag{5.11}$$
$$\frac{\partial V^\varepsilon}{\partial t} + \mu V^\varepsilon + S\big(V^\varepsilon(x,t) - \Phi(U^\varepsilon(x,t))\big) = 0, \quad x \in M^\varepsilon,\ t > 0, \tag{5.12}$$
$$\frac{d\Upsilon^\varepsilon}{dt} = \frac{2\pi}{V}\sum_{i=1}^{N_\varepsilon}\int_0^H \int_{\partial\mathcal{P}_\varepsilon^i} \gamma\Big(U^\varepsilon - u_i^\varepsilon\Big)\frac{z}{H}\, dSdz - \frac{\pi}{V}\sum_{i=1}^{N_\varepsilon} \varepsilon^2 \rho_C^2 \Big\{ \partial_t \int_0^H u_i^\varepsilon \frac{z}{H}\, dz$$
$$+ v(t)\int_0^H \partial_z u_i^\varepsilon \frac{z}{H}\, dz + \frac{d}{H}\Upsilon^\varepsilon \Big\},\ \Upsilon^\varepsilon(0) = u_0, \quad u_i^\varepsilon|_{z=H} = \Upsilon^\varepsilon(t),\ \forall i, \tag{5.13}$$

where $u_i^\varepsilon = u^\varepsilon|_{\partial\mathcal{P}_\varepsilon^i}$ on $\mathcal{P}_\varepsilon^i$, $\forall i$. The existence of a smooth solution for the equations (5.10)-(5.13), satisfying initial conditions (5.8)-(5.9) was established in preceding sections. In order to study the limit $\varepsilon \to 0$ we need a priori estimates uniform with respect to ε.

Proposition 5.1. *Let the extension of* V^ε *be defined by*

$$\partial_t(\hat{\Pi}^\varepsilon \overline{V}^\varepsilon) + \mu\hat{\Pi}^\varepsilon V^\varepsilon = -S(\hat{\Pi}^\varepsilon \overline{V}^\varepsilon - \Phi(\Pi^\varepsilon U^\varepsilon)), \quad \hat{\Pi}^\varepsilon \overline{V}^\varepsilon(x,0) = V_0(x). \tag{5.14}$$

Then the functions $\{U^\varepsilon, V^\varepsilon, \underline{u}^\varepsilon, \Upsilon^\varepsilon\}$ *, defined by Problem* $(\mathcal{P}')$ *, are non-negative and satisfy the following a priori estimate*

$$\|\Pi^\varepsilon U^\varepsilon\|_{L^\infty(0,T;H^1(K))} + \|\partial_t \Pi^\varepsilon U^\varepsilon\|_{L^2(0,T;L^2(K))} \le C \tag{5.15}$$

$$\|\partial_t \hat{\Pi}^\varepsilon V^\varepsilon\|_{L^2(0,T;L^2(K))} + \sup_{0\le t\le T} \|\hat{\Pi}^\varepsilon V^\varepsilon(\cdot + h) - \hat{\Pi}^\varepsilon V^\varepsilon(\cdot)\|_{L^2(K)} \le C\sqrt{h}, \quad \forall h > 0, \tag{5.16}$$

$$\|\Pi^\varepsilon U^\varepsilon\|_{L^\infty(K\times(0,T))} \le M;\ \|\hat{\Pi}^\varepsilon V^\varepsilon\|_{L^\infty(K\times(0,T))} \le \Phi(M) \tag{5.17}$$

$$\sup_{1\le i\le N_\varepsilon} \|u_i^\varepsilon\|^2_{L^\infty(\mathcal{P}_\varepsilon^i\times(0,T))} + \sum_{i=1}^{N_\varepsilon} \left(\int_0^T \int_0^H \int_{\mathcal{P}_\varepsilon^i} |\partial_t u_i^\varepsilon|^2 \, d\underline{x} dt + \sup_{0\le t\le T} \int_0^H \int_{\mathcal{P}_\varepsilon^i} |\partial_z u_i^\varepsilon|^2 \, d\underline{x} \right) \le C. \tag{5.18}$$

Proof. First we note that (5.17) follows from the maximum principle. Next, in order to get the energy estimate we test (5.11) by $g = u_i^\varepsilon - \Upsilon^\varepsilon z/H$, sum with respect to i and add (5.13) tested with $V\Upsilon^\varepsilon$. Then we test (5.10) with $\varphi = U^\varepsilon$ and (5.12) by $h = \Phi^{-1}(V^\varepsilon)$. Finally, we combine all three integral equalities. Then, as in derivation of the a priori estimates (4.21)-(4.22) in the existence proof, it follows that

$$\sup_{0\le t\le T} \Big\{ \int_{M^\varepsilon} \left(|U^\varepsilon(t)|^2 + \int_0^{V^\varepsilon} \Phi^{-1}(\eta)\, d\eta \right) dxdydz + \sum_{i=1}^{N_\varepsilon} \int_0^H \int_{\mathcal{P}_\varepsilon^i} |u_i^\varepsilon(t)|^2 \, dx + V\cdot \Upsilon^\varepsilon(t)^2 \Big\}$$
$$+ D\int_0^T \int_{M^\varepsilon} |\nabla U^\varepsilon|^2 \, dxdydz + d\sum_{i=1}^{N_\varepsilon} \int_0^T \int_0^H \int_{\mathcal{P}_\varepsilon^i} |\frac{\partial u_i^\varepsilon}{\partial z}|^2 \, dxdydz \le$$
$$C\varepsilon^2 \sum_{i=1}^{N_\varepsilon} \|u_{i0}\|^2_{L^2(0,H)} + C + C\int_K \left(|U_0|^2 + \int_0^{V_0} \Phi^{-1}(\eta)\, d\eta \right) \tag{5.19}$$

where C depends on the boundary data and nonlinearities, but not on ε .

Further time estimates for U^ε , u^ε and V^ε follow from the equality (4.24). We have

$$\frac{D}{2} \sup_{0\le t\le T} \int_{M^\varepsilon} |\nabla U^\varepsilon(t)|^2 \, dxdydz + \int_0^T \int_{M^\varepsilon} |\partial_t U^\varepsilon|^2 \, d\underline{x} dt +$$
$$\sum_{i=1}^{N_\varepsilon} \Big\{ \int_0^T \int_0^H \int_{\mathcal{P}_\varepsilon^i} |\partial_t u_i^\varepsilon(t)|^2 \, d\underline{x} + \frac{d}{2} \sup_{0\le t\le T} \int_0^H \int_{\mathcal{P}_\varepsilon^i} |\frac{\partial u_i^\varepsilon}{\partial z}|^2 \, dxdydz \Big\} \le$$
$$C\varepsilon^2 \sum_{i=1}^{N_\varepsilon} \left(\|\partial_z u_{i0}\|^2_{L^2(0,H)} + \|u_{i0}\|^2_{L^\infty(0,H)} \right) + C + C\int_K |\nabla U_0|^2 \, d\underline{x} \tag{5.20}$$

$$\|\partial_t \Upsilon^\varepsilon\|_{L^2(0,T)} + \|\partial_t V^\varepsilon\|_{L^2(M^\varepsilon\times(0,T))} \le C \tag{5.21}$$

Next we note that (5.19)-(5.20) apply to $\Pi^\varepsilon U^\varepsilon$, as well, proving (5.15) and (5.18).

For V^ε we introduce the extension by (5.14). Then

$$\int_K |\hat{\Pi}^\varepsilon V^\varepsilon(x+h,t) - \hat{\Pi}^\varepsilon V^\varepsilon(x,t)|^2 \, d\underline{x} \leq \int_K |V_0(x+h) - V_0(x)|^2 \, d\underline{x} + C\int_0^t \int_K |\Pi^\varepsilon U^\varepsilon(x+h,\xi) - \Pi^\varepsilon U^\varepsilon(x,\xi)|^2 \, d\underline{x} d\xi \leq C|h|, \quad \forall h \in I\!R^3, \ \forall t \in (0,T), \tag{5.22}$$

proving (5.16). □

Next we extend u^ε to K by

$$\tilde{u}^\varepsilon(x,y,z,t) = u_i^\varepsilon(z,t) \text{ if } (x,y) \in Y_\varepsilon^i, \tag{5.23}$$

Obviously, u^ε is a non-negative function, uniformly bounded in L^∞ with respect to ε. Furthermore

$$\|\partial_z \tilde{u}^\varepsilon\|_{L^2(K\times(0,T))} + \|\partial_t \tilde{u}^\varepsilon\|_{L^2(K\times(0,T))} \leq C \tag{5.24}$$

Nevertheless, since they are locally constant with respect to x and y , these extensions don't have derivatives with respect to x and y , in the sense of distributions, in L^2 . This means that we should estimate the translations with respect to x and y , if we wish to prove compactness of the sequence u^ε . We note the analogy with the approach from [1] .

Proposition 5.2. *Let us suppose that* $\forall k \in Z\!\!\!Z^2$ *we have*

$$\varepsilon^2 \rho_C \int_0^H \sum_{i=1}^{N_\varepsilon} |u_{i+k,0} - u_{i,0}|^2 \, dz \leq C|k|. \tag{5.25}$$

Let $\tilde{u}^\varepsilon$ *be extended by zero outside* K *. Then* $\forall h = (h_1,h_2) \in I\!R^2$ *we have*

$$\sup_{0\leq t\leq T} \int_0^H \int_Q |\tilde{u}^\varepsilon(x+h_1,y+h_2,z,t) - \tilde{u}^\varepsilon(x,y,z,t)|^2 \, dxdydz \leq C\big(\varepsilon^{3/2} + |h|\big). \tag{5.26}$$

Proof. The idea is to use the equation (5.11) and the a priori estimates (5.15)-(5.18).

Clearly, it is enough to prove the result for $h = (k_1\varepsilon, k_2\varepsilon)$, $k \in Z\!\!\!Z^2$.

Let $u_i^{\varepsilon,k} = u_i^\varepsilon(x+k_1\varepsilon, y+k_2\varepsilon, z, t)$ .We test the equation (5.11) with $g = u_i^{\varepsilon,k} - u_i^\varepsilon$ and get

$$\frac{1}{2}\int_0^H \int_{\mathcal{P}_\varepsilon^i} |u_i^{\varepsilon,k} - u_i^\varepsilon|^2(t) + d\int_0^t \int_0^H \int_{\mathcal{P}_\varepsilon^i} |\partial_z(u_i^{\varepsilon,k} - u_i^\varepsilon)|^2 = \frac{1}{2}\int_0^H \int_{\mathcal{P}_\varepsilon^i} |u_{i,0}^{\varepsilon,k} - u_{i,0}^\varepsilon|^2 + \mathcal{I}, \tag{5.27}$$

where

$$\mathcal{I} = 2\pi \int_0^t \int_0^H \int_0^{2\pi} \varepsilon^2 \rho_C \big(\gamma(U_i^{\varepsilon,k} - u_i^{\varepsilon,k}) - \gamma(U_i^\varepsilon - u_i^\varepsilon)\big)(u_i^{\varepsilon,k} - u_i^\varepsilon) \, d\vartheta dz d\eta \tag{5.28}$$

At this stage we make use of an auxiliary function, systematically used in [15], β, being the solution with zero mean to the problem

$$-\Delta\beta = -\frac{|\partial\mathcal{Y}_C|}{|\mathcal{Y}_C|} \text{ in } \mathcal{Y}_C; \qquad \frac{\partial\beta}{\partial n} = 1 \text{ on } \partial\mathcal{Y}_C \tag{5.29}$$

Then $\beta^\varepsilon(x,y) = \beta(x/\varepsilon, y/\varepsilon)$ is uniformly bounded and its derivatives behave as ε^{-1}.

Next we note that the term $\mathcal{I}$ involves U^ε and we estimate it as follows :

$$\begin{aligned}|\mathcal{I}| = |\int_0^t \int_0^H \int_0^{2\pi} \varepsilon^3 \rho_C \big(\gamma(U_i^{\varepsilon,k} - u_i^{\varepsilon,k}) - \gamma(U_i^\varepsilon - u_i^\varepsilon)\big)(u_i^{\varepsilon,k} - u_i^\varepsilon)\frac{\partial\beta^\varepsilon}{\partial n}\ d\vartheta dz d\eta| \\ \leq C\int_0^t \int_0^H \int_{\mathcal{P}_i^\varepsilon} |\Pi^\varepsilon U^\varepsilon(\cdot + \varepsilon k, z, \eta) - \Pi^\varepsilon U^\varepsilon(x,y,z,t)| \cdot |u_i^{\varepsilon,k} - u_i^\varepsilon|\ dxdydzd\eta \\ + C\|\varepsilon\nabla_{x,y}\beta^\varepsilon\|\varepsilon^3\end{aligned} \tag{5.30}$$

Finally we insert (5.30) into (5.27) and get

$$\begin{aligned}\int_0^H \int_{\mathcal{P}_\varepsilon^i} |u_i^{\varepsilon,k} - u_i^\varepsilon|^2(t) + d\int_0^t \int_0^H \int_{\mathcal{P}_\varepsilon^i} |\partial_z(u_i^{\varepsilon,k} - u_i^\varepsilon)|^2 \leq C\varepsilon^2 \int_0^H |u_{i,0}^k - u_{i,0}|^2\ dz \\ + C\int_0^t \int_0^H \int_{\mathcal{P}_\varepsilon^i} |\Pi^\varepsilon U^\varepsilon(\cdot + \varepsilon k, z, \eta) - \Pi^\varepsilon U^\varepsilon(x,y,z,t)|^2 + C\varepsilon^3\end{aligned} \tag{5.31}$$

Insertion of the assumptions on the data and (5.15) into (5.31) implies the desired result. □

These estimates lead to the following compactness result

Proposition 5.3. *There are subsequences of* $\{\Pi^\varepsilon U^\varepsilon, \hat\Pi^\varepsilon V^\varepsilon, \tilde u^\varepsilon, \Upsilon^\varepsilon\}$, *denoted by the same indices, and functions* $\{U, V, u, \Upsilon\} \in H^1(K\times(0,T))^2 \times L^\infty(K\times(0,T)) \times H^1(0,T)$, *with* $\partial_z u \in L^2(K\times(0,T))$ *and* $\partial_t u \in L^2(K\times(0,T))$ *such that*

$$\Pi^\varepsilon U^\varepsilon \to U \text{ weakly in } H^1(K\times(0,T)) \text{ and strongly in } L^2(K\times(0,T)) \tag{5.32}$$

$$\tilde u^\varepsilon \to u \text{ weak* in } L^\infty(K\times(0,T)),\ \partial_t\Upsilon^\varepsilon \to \partial_t\Upsilon \text{ weakly in } L^2(0,T) \tag{5.33}$$

$$\{\partial_z\tilde u^\varepsilon, \partial_t\tilde u^\varepsilon\} \to \{\partial_z u, \partial_t u\} \text{ weakly in } L^2(K\times(0,T))^2 \tag{5.34}$$

$$\tilde u^\varepsilon \to u \text{ strongly in } L^2(K\times(0,T)) \tag{5.35}$$

$$\hat\Pi^\varepsilon V^\varepsilon \to V \text{ weakly in } H^1(K\times(0,T)) \text{ and strongly in } L^2(K\times(0,T)) \tag{5.36}$$

$$\Pi^\varepsilon U^\varepsilon \to U \quad \text{and} \quad \hat\Pi^\varepsilon V^\varepsilon \to V \quad \text{weak* in } L^\infty(K\times(0,T)), \tag{5.37}$$

$$\Upsilon^\varepsilon = \tilde u^\varepsilon|_{z=H} \to \Upsilon = u|_{z=H} \quad \text{uniformly on } [0,T] \tag{5.38}$$

In order to pass to the limit in the interface integrals containing u^ε we prove the following result

Proposition 5.4. *We have*

$$\sum_{i=1}^{N_\varepsilon}\int_0^T\int_0^H\int_0^{2\pi}\varepsilon^2\rho_C\varphi|_{\partial\mathcal{P}_\varepsilon^i}\gamma(U^\varepsilon|_{\partial\mathcal{P}_\varepsilon^i}-u_i^\varepsilon)\ d\vartheta dzdt\to$$
$$|\partial\mathcal{Y}_C|\int_0^T\int_K\gamma(U-u)\varphi\ dxdydzdt,\quad\forall\varphi\in L^2(0,T;H^1(\Omega))\tag{5.39}$$

Proof. Since we don't have good estimates for the derivatives of u^ε with respect to x and y, we can't directly use results from [6]. We proceed as in the estimate for the translations in x and y and introduce β, as the solution with zero mean to the problem (5.29). Then we have

$$\lim_{\varepsilon\to0}\sum_{i=1}^{N_\varepsilon}\int_0^T\int_0^H\int_0^{2\pi}\varepsilon^2\rho_C\varphi|_{\partial\mathcal{P}_\varepsilon^i}\gamma(U^\varepsilon|_{\partial\mathcal{P}_\varepsilon^i}-u_i^\varepsilon)\ d\vartheta dzdt=$$
$$\lim_{\varepsilon\to0}\varepsilon^2\int_0^T\int_0^H\int_{P_\varepsilon}\mathrm{div}_{x,y}(\varphi\nabla_{x,y}\beta^\varepsilon\gamma(\Pi^\varepsilon U^\varepsilon-\tilde u^\varepsilon))\ dxdydzdt=$$
$$\lim_{\varepsilon\to0}\int_0^T\int_0^H\int_{P_\varepsilon}\frac{|\partial\mathcal{Y}_C|}{|\mathcal{Y}_C|}\gamma(\Pi^\varepsilon U^\varepsilon-\tilde u^\varepsilon)\varphi\ dxdydzdt=|\partial\mathcal{Y}_C|\int_0^T\int_K\gamma(U-u)\varphi\ dxdydzdt$$

and the result is proved. □

The derivation of the homogenized problem is now immediate. We have

Theorem 5.1. *Let $\theta=|Y\setminus\bar{\mathcal{Y}}_C|$ be the porosity and let the A be the effective diffusion matrix. Then all cluster points $\{U,u,V,\Upsilon\}$ satisfy the system*

$$\theta\partial_tU-D\,div\,\big(A\nabla U\big)+|\partial\mathcal{Y}_C|\gamma(U-u)+\lambda\theta U=\theta S(V-\Phi(U))\tag{5.40}$$
$$2\pi\frac{|\partial\mathcal{Y}_C|}{|\mathcal{Y}_C|}\gamma(U-u)=\frac{\partial u}{\partial t}+v(t)\frac{\partial u}{\partial z}-d\frac{\partial^2u}{\partial z^2}\tag{5.41}$$
$$\frac{\partial V}{\partial t}+\mu V=-S(V-\Phi(U))\tag{5.42}$$
$$\frac{1}{1-\theta}\frac{\partial\Upsilon}{\partial t}+(\frac{d|Q|}{VH}+\frac{(1-\theta)|Q|}{V})\Upsilon+\frac{1}{VH}\partial_t\int_K zu\ dxdydz=\frac{v(t)|\mathcal{Y}_C|}{VH}\int_K u\ dxdydz+$$
$$\frac{2\pi|\partial\mathcal{Y}_C|}{VH(1-\theta)}\int_K\gamma(U-u)z\ dxdydz\tag{5.43}$$

in $K\times(0,T)$, together with the following initial and boundary conditions

$$\begin{cases}A\nabla U\cdot\underline{n}_e=0\quad on\quad\Sigma^+\times(0,T);\\ DA\nabla U\cdot\underline{n}_e=U\underline{q}\cdot\underline{n}_e\quad on\quad\Sigma^-\times(0,T);\end{cases}\tag{5.44}$$

$$u|_{z=H}=\Upsilon(t),\ u|_{z=0}=0\ on\ (0,T)\ \ and\ \ u|_{t=0}=\lim_{\varepsilon\to0}\varepsilon^2\sum_{i=1}^{N_\varepsilon}u_{i0}(z)\ on\ K\tag{5.45}$$

$$U|_{t=0}=U_0,\ \Upsilon(0)=u_0\quad and\ V|_{t=0}=V_0\ \ on\ K.\tag{5.46}$$

Theorem 5.2. *The problem (5.40)-(5.46) admits a unique solution in* $H^1(K\times(0,T))\times L^\infty(K\times(0,T))\times H^1(K\times(0,T))\times H^1(0,T)$, $(\partial_z u, \partial_t u)\in L^2(K\times(0,T))^2$.

Proof. The proof uses the energy estimates. We suppose 2 solutions and write the system for the difference. Then the first equation is tested by $U=U_1-U_2$, the second by $u=u_1-u_2-z(\Upsilon_1-\Upsilon_2)/H$, the 3rd by $V=V_1-V_2$ and the fourth by $\Upsilon=\Upsilon_1-\Upsilon_2$. We have

$$\theta\int_K \frac{1}{2}|U(t)|^2 + D\int_0^t\int_K A\nabla U\nabla U + |\partial\mathcal{Y}_C|\int_0^t\int_K \big(\gamma(U_1-u_1)-\gamma(U_2-u_2)\big)U$$
$$+\lambda\theta\int_0^t\int_K |U|^2 - \int_0^t\int_{\Sigma^-} \underline{q}\cdot\underline{n}_e U^2 = \theta\int_0^t\int_K \big(S(V_1-\Phi(U_1)) - S(V_2-\Phi(U_2))U \tag{5.47}$$

$$\frac{1}{2\pi}|\mathcal{Y}_C|\int_K\frac{1}{2}|u(t)|^2 + \frac{d}{2\pi}|\mathcal{Y}_C|\int_0^t\int_K|\partial_z u|^2 - |\partial\mathcal{Y}_C|\int_0^t\int_K\big(\gamma(U_1-u_1)-\gamma(U_2-u_2)\big)u$$
$$=\int_0^t\frac{1}{2\pi}\big\{\Upsilon(\tau)|\mathcal{Y}_C|\int_K\frac{z}{H}u + |\mathcal{Y}_C|v(\tau)\int_K\partial_z u\frac{z}{H}\Upsilon + \frac{d|Q|}{H}|\mathcal{Y}_C|\Upsilon^2(\tau) -$$
$$|\partial\mathcal{Y}_C|\int_K\big(\gamma(U_1-u_1)-\gamma(U_2-u_2)\big)z\Upsilon/H\big\} \tag{5.48}$$

$$\int_K\frac{1}{2}|V(t)|^2 + \int_0^t\int_K\mu V^2 + \int_0^t\int_K\big(S(V_1-\Phi(U_1)) - S(V_2-\Phi(U_2))\big)V = 0 \tag{5.49}$$

$$\frac{V}{2\pi}\frac{1}{2}\Upsilon^2(t) = -\int_0^t\frac{1}{2\pi}\big\{\Upsilon(\tau)|\mathcal{Y}_C|\int_K\frac{z}{H}u + |\mathcal{Y}_C|v(\tau)\int_K\partial_z u\frac{z}{H}\Upsilon +$$
$$\frac{d|Q|}{H}|\mathcal{Y}_C|\Upsilon^2(\tau) - |\partial\mathcal{Y}_C|\int_K\big(\gamma(U_1-u_1)-\gamma(U_2-u_2)\big)z\Upsilon/H\big\} \tag{5.50}$$

We add (5.47) to (5.50) to get

$$\frac{1}{2}\left\{\theta\int_K(U^2(t)+V^2(t)) + \frac{|\mathcal{Y}_C|}{2\pi}\int_K u^2(t) + \frac{V}{2\pi}\Upsilon^2(t)\right\} + \frac{|\mathcal{Y}_C|d}{2\pi}\int_0^t\int_K|\frac{\partial u}{\partial z}|^2 +$$
$$|\partial\mathcal{Y}_C|\int_0^t\int_K\big(\gamma(U_1-u_1)-\gamma(U_2-u_2)\big)(U-u) + D\int_0^t\int_K A\nabla U\nabla U + \lambda\int_0^t\int_K|U|^2$$
$$-\int_0^t\int_{\Sigma^-}\underline{q}\cdot\underline{n}_e U^2 + |Y\setminus\bar{\mathcal{Y}}_C|\int_0^t\int_K\mu V^2 \le C\int_0^t\int_K(|V|+|U|)^2\,dxdydzdt. \tag{5.51}$$

Now the uniqueness is trivial. □

Corollary 5.1. *The whole sequence* $\{\Pi^\varepsilon U^\varepsilon, \tilde{u}^\varepsilon, \hat{\Pi}^\varepsilon V^\varepsilon, \Upsilon^\varepsilon\}$ *converges to the unique solution* $\{U,u,V,\Upsilon\}$ *for the system (5.40)-(5.46).*

Remark 5.2. We note that our homogenized model corresponds to the models found the direct modeling of the solute transport, involving insufficiently fast surface reactions. For more details we refer to the classical paper [23] . Problems related to the system (5.40)-(5.46), with modeling borrowed from [23] , are studied in [10] .

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