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EFFECTIVE PRESSURE INTERFACE LAW FOR TRANSPORT PHENOMENA BETWEEN AN UNCONFINED FLUID AND A POROUS MEDIUM USING HOMOGENIZATION*

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Abstract. We present modeling of the incompressible viscous flows in the domain containing unconfined fluid and a porous medium in the case when the flow in the unconfined domain dominates. For such a setting a rigorous derivation of the Beavers–Joseph–Saffman interface condition was undertaken by Jäger and Mikelić [*SIAM J. Appl. Math.*, 60 (2000), pp. 1111–1127] using the homogenization method. So far the interface law for the pressure was conceived and confirmed only numerically. In this article we derive the Beavers and Joseph law for a general body force by estimating the pressure field approximation. Different from the Poiseuille flow case, the velocity approximation is not divergence-free and the precise pressure estimation is essential. This new estimate allows us to rigorously justify the pressure jump condition using the Navier boundary layer, already used to calculate the constant in the law by Beavers and Joseph. Finally, our results confirm that the position of the interface influences the solution only at the order of physical permeability and therefore the choice of this position does not pose problems.

Key words. interface pressure jump law, Beavers and Joseph law, homogenization, boundary layer, porous media

AMS subject classifications. 35B27, 35Q30, 35Q35, 74Q15, 76D07, 76M50, 76S

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1. Introduction.

Slow viscous and incompressible simultaneous flow through an unconfined region and a porous medium occurs in a wide range of industrial processes and natural phenomena. One of the classical problems is finding effective boundary conditions at a naturally permeable wall, i.e., at the surface which separates a channel flow and a porous medium.

The effective laminar incompressible and viscous flow through a porous medium can be described using Darcy's law. The unconfined fluid flow in the channel is governed by the Stokes system, or by the Navier–Stokes system if the inertia effects in the free fluid are important. To model the coupling of both processes, it is necessary to put together two systems of partial differential equations: the second order system for the velocity and the first order equation for the pressure,

$$-\mu\Delta\mathbf{u} + \nabla p = f, \tag{1.1}$$

$$\operatorname{div}\,\mathbf{u} = 0, \tag{1.2}$$

in the unconfined fluid region, and the scalar second order equation for the pressure

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and the first order system for the seepage velocity,

$$-\mu \mathbf{v}^F = K(f - \nabla p^F), \tag{1.3}$$

$$\text{div } \mathbf{v}^F = 0, \tag{1.4}$$

in the porous medium.

The orders of the corresponding differential operators are different, and it is not clear what conditions are necessary to impose at the interface between the free fluid and the porous part of the domain. One coupling condition is based on the continuity of the normal mass flux. However, it is not enough for determination of the effective flow and it is necessary to specify more conditions.

Several laws of fluid dynamics in porous media were derived using homogenization. The most notable example is Darcy's law, being the effective equation for one phase flow through a rigid porous medium. Its formal derivation using the 2-scale expansion goes back to the classical paper by Ene and Sanchez-Palencia [9]. This derivation was made mathematically rigorous by Tartar in [27]. For the detailed proof in the case of a periodic porous medium we refer the reader to the review papers by Allaire [1] and by Mikelić [20] and for a random statistically homogeneous porous medium to the paper of Beliaev and Kozlov [3].

As in the derivation of Darcy's law, we would like to apply the homogenization technique to find the effective interface laws. However, the assumption of statistical homogeneity of the domain, which is necessary for the homogenization approach, is not valid close to the interface. Consequently, deviations from Darcy's law are expected in the thin layers near the interfaces. Furthermore, the presence of such interfaces can significantly change the structure of the model coefficients and lead to different effective constitutive laws for the flow.

It was experimentally found by Beavers and Joseph in [2] that the jump of the tangential component of the effective velocity at the interface is proportional to the shear stress originating from the free fluid. This law was justified at a physics level of rigor by Saffman in [24], where it was observed that the seepage velocity contribution could be neglected leading to the law in the form

$$\sqrt{k}\frac{\partial v_\tau}{\partial \nu} = \alpha v_\tau + O(k), \tag{1.5}$$

where α is a dimensionless parameter depending on the geometrical structure of the porous medium, ε is the characteristic pore size, and k is the scalar permeability. ν denotes the unit normal vector at the interface, and v_τ is the slip velocity of the free fluid in the channel. Saffman's modification of the law by Beavers and Joseph has been widely accepted.

As an alternative to (1.5), the continuity of the effective pressure was suggested by Ene and Sanchez-Palencia in [9]. While this interface law is acceptable from the modeling point of view, it should be noted that the well-posedness of the averaged problem is not clear.

The law (1.5) was rigorously justified by Jäger and Mikelić in [13]. Numerical calculations of the boundary layers for the experimental conditions of Beavers and Joseph are presented in [14]. They indicate the appearance of a *pressure jump* at the interface. These issues were heuristically discussed in [15].

In the experiment by Beavers and Joseph only the flows tangential to a naturally permeable wall (a porous bed) were considered. In general, the situation is much more complicated and many types of interfacial conditions have been proposed, such as continuous tangential velocity with discontinuous tangential shear stress introduced in [22], [23] by Ochoa-Tapia and Whitaker, or continuous tangential velocity and tangential shear stress in [21] by Neale and Nader, or discontinuous tangential velocity and tangential shear stress in [5] by Cieszko and Kubik. In particular, in [22], [23] the continuity of the velocity and the continuity of the "modified" normal stress were obtained at the interface using volume averaging. In order to perform the averaging it was necessary to assume Brinkman's flow in the porous part and a transition layer between the two domains. Numerical study of the hydrodynamic boundary condition at the interface between a porous and a plain medium was performed by Sahraoui and Kaviany [25]. Numerical implementation of the effective interface couplings was presented in [8], [11]. Nevertheless, determination of the practical and relevant first order interface conditions between the pure fluid and the porous matrix remains an open question that could be treated using the technique developed in [12].

This paper is a continuation of works [13], [14] and constitutes a step forward in the development of the rigorous approach to model effective interface laws for the transport phenomena between an unconfined fluid and a porous medium. We depart beyond justification of the law (1.5) developed in [13] and *undertake a rigorous derivation* of the interface laws for the viscous flow in a long channel in contact with a porous bed. The macroscopic model derived links pressure jump with the shear stress of the unconfined fluid at the interface, an effect which was predicted based on numerical simulations in [14]. Derivation of the law of Beavers and Joseph is based on the procedures proposed in [13] and discussed in [15]; however, it is nontrivially adjusted to the new setting involving a general body force. We consider a general situation when the flow in the unconfined region dominates. Nevertheless, even if the flow is much less important in the porous part, the pressures are of the same order of magnitude. Hence finding and justifying the interface law for the pressure is of fundamental interest.

The review paper [15] was concluded with the sentence "Proving the error estimate for the pressure approximation in the porous bed Ω_2^ε remains an open problem." We solve this problem and present a mathematically rigorous derivation of the pressure jump interface law, which is the next order correction of the Beavers–Joseph law. We obtain the effective equations heuristically and then rigorously justify them. A combination of homogenization and boundary layer approaches is used to achieve this end. The study of such complex flows leads to artificial compressibility effects in the upscaling process. In this paper we develop the required estimate of the pressure. Our main results are the following:

1. Confirmation of Saffman's form of the law by Beavers and Joseph in the more general setting

$$u_1^{\mathit{eff}} = -\varepsilon C_1^{bl} \frac{\partial u_1^{\mathit{eff}}}{\partial x_2} + O(\varepsilon^2), \tag{1.6}$$

   where u^{eff} is the average over the characteristic pore opening at the naturally permeable wall. Physical permeability is given by $k = k^\varepsilon = \varepsilon^2 K$, and the constant in (1.6) is proportional to $\sqrt{k^\varepsilon}$. The error is of order k^ε, as remarked by Saffman in [24]. It is important to point out that the parameter α from expression (1.5) is determined taking into account the auxiliary problems, which we formulate later in (5.1)–(5.4) and (5.7), and that it is given by $\alpha = -\frac{1}{\varepsilon C_1^{bl}} > 0$.
2. Interface between the unconfined flow and the porous bed is an artificial

mathematical boundary, and it can be chosen in a layer having the pore size thickness. We show that a perturbation of the interface position of the order $O(\varepsilon)$ implies a perturbation in the solution of $O(\varepsilon^2)$. Consequently, it influences the result only at the next order of the asymptotic expansion.

3. We obtain a uniform bound on the pressure approximation. Furthermore, we prove that there is a jump of the effective pressure on the interface and that it is proportional to the free fluid flow shear at the boundary. The proportionality constant is calculated from the boundary layer problem (5.1)–(5.4). Homogenization leads to the discontinuity of the effective pressure field at the interface, which differs from the pressure interface continuity law proposed in [9]. If the boundary layer pressure is neglected, the pressure in the neighborhood of the interface is poorly approximated.

Here, we remark that some classes of problems, like infiltration into the porous medium, are characterized by the velocity field of the same order in both domains. Such a situation requests much larger body force in the porous part than in the unconfined one. Some situations of this kind were considered in [12]. In this paper, the body force is of order $O(1)$ in both domains.

The paper is organized as follows. In section 2 we formulate the problem and main results. Section 3 is devoted to the proof of the results. We conclude the paper with two short appendices recalling the notion of very weak solutions and the definition and properties of the Navier boundary layer.

2. Statement of the problem and of the results.

2.1. Definition of the geometry. Let L, h, and H be positive real numbers. We consider a two-dimensional (2D) periodic porous medium $\Omega_2 = (0, L) \times (-H, 0)$ with a periodic arrangement of the pores. The formal description is along the following lines.

First, we define the geometrical structure inside the unit cell $Y = (0, 1)^2$. Let Y_s (the solid part) be a closed strictly included subset of $\bar{Y}$, and let $Y_F = Y \backslash Y_s$ (the fluid part). Now we make a periodic repetition of Y_s all over $\mathbb{R}^2$ and set $Y_s^k = Y_s + k$, $k \in \mathbb{Z}^2$. Obviously, the resulting set $E_s = \bigcup_{k\in\mathbb{Z}^2} Y_s^k$ is a closed subset of $\mathbb{R}^2$ and $E_F = \mathbb{R}^2 \backslash E_s$ is an open set in $\mathbb{R}^2$. We suppose that Y_s has a boundary of class $C^{0,1}$, which is located locally on one side of their boundary. Obviously, E_F is connected and E_s is not.

Now we notice that Ω_2 is covered with a regular mesh of size ε, each cell being a cube Y_i^ε, with $1 \le i \le N(\varepsilon) = |\Omega_2|\varepsilon^{-2}[1 + o(1)]$. Each cube Y_i^ε is homeomorphic to Y, by linear homeomorphism Π_i^ε, being composed of translation and a homothety of ratio $1/\varepsilon$.

We define $Y_{S_i}^\varepsilon = (\Pi_i^\varepsilon)^{-1}(Y_s)$ and $Y_{F_i}^\varepsilon = (\Pi_i^\varepsilon)^{-1}(Y_F)$. For sufficiently small $\varepsilon > 0$, we consider the set $T_\varepsilon = \{k \in \mathbb{Z}^2 | Y_{S_k}^\varepsilon \subset \Omega_2\}$ and define

$$O_\varepsilon = \bigcup_{k\in T_\varepsilon} Y_{S_k}^\varepsilon, \quad S^\varepsilon = \partial O_\varepsilon, \quad \Omega_2^\varepsilon = \Omega_2 \backslash O_\varepsilon = \Omega_2 \cap \varepsilon E_F.$$

Obviously, $\partial\Omega_2^\varepsilon = \partial\Omega_2 \cup S^\varepsilon$. The domains O_ε and Ω_2^ε represent, respectively, the solid and fluid parts of the porous medium Ω. For simplicity, we suppose $L/\varepsilon, H/\varepsilon, h/\varepsilon \in \mathbb{N}$.

We set $\Sigma = (0, L) \times \{0\}$, $\Omega_1 = (0, L) \times (0, h)$, and $\Omega = (0, L) \times (-H, h)$. Furthermore, we let $\Omega^\varepsilon = \Omega_2^\varepsilon \cup \Sigma \cup \Omega_1$. The geometry is shown in Figure 2.1.

A very important property of the porous media is the following variant of Poincaré's inequality.

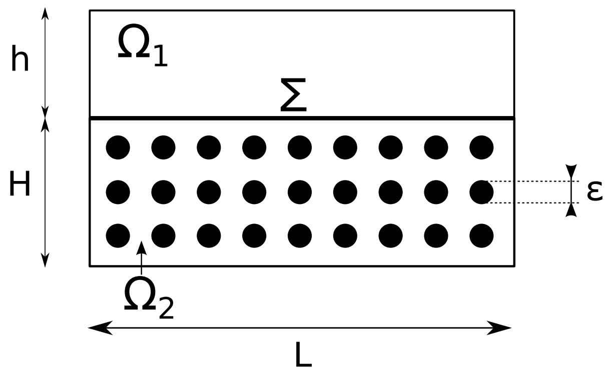

FIG. 2.1. *The geometry.*

LEMMA 2.1 (see, e.g., [26]). *Let $\varphi \in V(\Omega_2^\varepsilon) = \{\varphi \in H^1(\Omega_2^\varepsilon) \; | \varphi = 0 \text{ on } S^\varepsilon\}$. Then, it holds that*

$$\|\varphi\|_{L^2(\Sigma)} \le C\varepsilon^{1/2}\|\nabla_x \varphi\|_{L^2(\Omega_2^\varepsilon)^2}, \tag{2.1}$$

$$\|\varphi\|_{L^2(\Omega_2^\varepsilon)} \le C\varepsilon\|\nabla_x \varphi\|_{L^2(\Omega_2^\varepsilon)^2}. \tag{2.2}$$

2.2. The microscopic equations. Having defined the geometrical structure of the porous medium, we specify the flow equations. Here we consider the slow viscous incompressible flow of a single fluid through a porous medium. We suppose the no-slip condition at the boundaries of the pores (i.e., a rigid porous medium). Then, we describe it by the following nondimensional steady Stokes system in Ω^ε (the fluid part of the porous medium Ω):

$$-\Delta \mathbf{v}^\varepsilon + \nabla p^\varepsilon = \mathbf{f} \qquad \text{in} \quad \Omega^\varepsilon, \tag{2.3}$$

$$\operatorname{div} \mathbf{v}^\varepsilon = 0 \qquad \text{in} \quad \Omega^\varepsilon, \qquad \int_{\Omega_1} p^\varepsilon \, dx = 0, \tag{2.4}$$

$$\mathbf{v}^\varepsilon = 0 \quad \text{on} \quad \partial\Omega^\varepsilon \setminus \Big(\{x_1 = 0\} \cup \{x_1 = L\}\Big), \qquad \{\mathbf{v}^\varepsilon, p^\varepsilon\} \quad \text{is } L\text{-periodic in } \; x_1. \tag{2.5}$$

Here the nondimensional $\mathbf{f}$ stands for the effects of external forces or an injection at the boundary or a given pressure drop, and it corresponds to the physical forcing term multiplied by the ratio between Reynolds's number and Froude's number squared. $\mathbf{v}^\varepsilon$ denotes the nondimensional velocity, and p^ε is the nondimensional pressure. The nonconstant force f corresponds, e.g., to a nonconstant pressure drop or to injection profiles which are not parabolic.

Let

$$W^\varepsilon = \Big\{\mathbf{z} \in H^1(\Omega^\varepsilon)^2, \; \mathbf{z} = 0 \;\text{ on }\; \partial\Omega^\varepsilon \setminus \Big(\{x_1 = 0\} \cup \{x_1 = L\}\Big), \quad \text{and } \; \mathbf{z} \; \text{ is } \; L\text{-periodic in } x_1\Big\}. \tag{2.6}$$

The variational form of the problem (2.3)–(2.5) reads as follows:

Find $\mathbf{v}^\varepsilon \in W^\varepsilon$, div $\mathbf{v}^\varepsilon = 0$ in Ω^ε, and $p^\varepsilon \in L^2(\Omega^\varepsilon)$ such that

$$\int_{\Omega^\varepsilon} \nabla \mathbf{v}^\varepsilon \nabla \varphi \, dx - \int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div} \varphi \, dx = \int_{\Omega^\varepsilon} \mathbf{f}\varphi \, dx \qquad \forall \varphi \in W^\varepsilon. \tag{2.7}$$

Then for $\mathbf{f} \in C^\infty(\overline{\Omega})^2$, the elementary elliptic variational theory gives the existence of the unique velocity field $\mathbf{v}^\varepsilon \in W^\varepsilon$, div $\mathbf{v}^\varepsilon = 0$ in Ω^ε, which solves (2.7) for every $\varphi \in W^\varepsilon$, div $\varphi = 0$ in Ω^ε. The construction of the pressure field goes through De Rham's theorem (see, e.g., [28]).

2.3. Main result. We start by introducing the effective problems in Ω_1 (the unconfined fluid part) and Ω_2.

Find a velocity field u^0 and a pressure field p^{eff} such that

$$-\triangle \mathbf{u}^{eff} + \nabla p^{eff} = \mathbf{f} \qquad \text{in } \Omega_1, \tag{2.8}$$

$$\text{div } \mathbf{u}^{eff} = 0 \qquad \text{in } \Omega_1, \qquad \int_{\Omega_1} p^{eff}\, dx = 0, \tag{2.9}$$

$$\mathbf{u}^{eff} = 0 \qquad \text{on } (0,L) \times \{h\}; \quad \mathbf{u}^{eff} \text{ and } p^{eff} \quad \text{are } L\text{-periodic in} \quad x_1, \tag{2.10}$$

$$u_2^{eff} = 0 \quad \text{and} \quad u_1^{eff} + \varepsilon C_1^{bl} \frac{\partial u_1^{eff}}{\partial x_2} = 0 \quad \text{on} \quad \Sigma. \tag{2.11}$$

We note that the second boundary condition in (2.11) is the *law by Beavers and Joseph* from [2]. The constant C_1^{bl} is strictly negative and calculated through (5.7) from the viscous boundary layer described in Appendix 2.

Problem (2.8)–(2.11) has a unique solution, which in the case of Poiseuille flows (i.e., when $\mathbf{f} = -\frac{p_b - p_0}{L}\mathbf{e}^1$) reads as

$$\begin{cases} \mathbf{u}^{eff}_{pois} = \left(\dfrac{p_b - p_0}{2L} \left(x_2 - \dfrac{\varepsilon C_1^{bl} h}{h - \varepsilon C_1^{bl}} \right)(x_2 - h), 0 \right) \text{ for } 0 \le x_2 \le h; \\ p^{eff} = 0 \text{ for } 0 \le x_1 \le L. \end{cases} \tag{2.12}$$

The *effective mass flow rate* through the channel is then

$$M^{eff} = \int_{\Omega_1} u_1^{eff}\, dx, \tag{2.13}$$

which for the Poiseuille flow reads as

$$M^{eff}_{pois} = -\frac{p_b - p_0}{12} h^3 \frac{h - 4\varepsilon C_1^{bl}}{h - \varepsilon C_1^{bl}}. \tag{2.14}$$

THEOREM 2.2. *Let us suppose $\mathbf{f} \in C^\infty(\overline{\Omega})^2$ and is L-periodic with respect to x_1. For $\{\mathbf{v}^\varepsilon, p^\varepsilon\}$ given by* (2.3)–(2.5) *and $\{\mathbf{u}^{eff}, p^{eff}\}$ by* (2.8)–(2.11), *it holds that*

$$\|\mathbf{v}^\varepsilon - \mathbf{u}^{eff}\|_{L^2(\Omega_1)^2} + |M^\varepsilon - M^{eff}| \le C\varepsilon^{3/2}, \tag{2.15}$$

$$\begin{aligned} &\|\mathbf{v}^\varepsilon - \mathbf{u}^{eff}\|_{H^{1/2}(\Omega_1)^2} + \|p^\varepsilon - p^{eff}\|_{L^1(\Omega_1)} + \|\nabla(\mathbf{v}^\varepsilon - \mathbf{u}^{eff})\|_{L^1(\Omega_1)^4} \\ &\quad + \||x_2|^{1/2}\nabla(\mathbf{v}^\varepsilon - \mathbf{u}^{eff})\|_{L^2(\Omega_1)^4} + \||x_2|^{1/2}(p^\varepsilon - p^{eff})\|_{L^2(\Omega_1)^2} \le C\varepsilon, \end{aligned} \tag{2.16}$$

with M^{eff} defined in (2.13).

Next, we study the situation in the porous medium Ω_2.

THEOREM 2.3. *Let the permeability tensor K be given by* (3.54). *The effective porous media pressure $\tilde{p}^0$ is the L-periodic in x_1 function satisfying*

$$\text{div}\left(K(\mathbf{f}(x) - \nabla \tilde{p}^0) \right) = 0 \quad in \quad \Omega_2, \tag{2.17}$$

$$\tilde{p}^0 = p^{eff} + C^{bl}_\omega \frac{\partial u_1^{eff}}{\partial x_2}(x_1, 0) \quad on \quad \Sigma; \quad K(\mathbf{f}(x) - \nabla \tilde{p}^0)|_{\{x_2 = -H\}} \cdot \mathbf{e}^2 = 0, \tag{2.18}$$

with $\{\mathbf{u}^{\mathit{eff}}, p^{\mathit{eff}}\}$ being the solution to the problem (2.8)–(2.11) and C^{bl}_ω being the pressure stabilization constant defined by (5.9). In addition we have for all $\delta > 0$,

$$\frac{1}{\varepsilon^2}\mathbf{v}^\varepsilon - K(\mathbf{f} - \nabla\tilde{p}^0) \rightharpoonup 0 \quad \text{weakly in} \quad L^2((0,L)\times(-H,-\delta))^2 \quad \text{as} \quad \varepsilon \to 0; \tag{2.19}$$

$$p^\varepsilon - \tilde{p}^0 \to 0 \quad \text{strongly in} \quad L^2(\Omega_2) \quad \text{as} \quad \varepsilon \to 0; \tag{2.20}$$

$$||p^\varepsilon - p^{\mathit{eff}}||_{H^{-1/2}(\Sigma)} \le C\sqrt{\varepsilon}. \tag{2.21}$$

REMARK 2.4. *If we include the vicinity of Σ, the velocity $\mathbf{v}^\varepsilon$ has to be corrected by a boundary layer term $\beta^{bl,\varepsilon}(x) = \varepsilon\beta^{bl}(x/\varepsilon)$, defined through (5.1)–(5.4), and the convergence result (2.19) reads as*

$$\frac{1}{\varepsilon^2}\left(\mathbf{v}^\varepsilon + \beta^{bl,\varepsilon}\frac{\partial u_1^{\mathit{eff}}}{\partial x_2}(x_1,0)\right) - K(\mathbf{f}(x) - \nabla\tilde{p}^0) \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega_2)^2 \quad \text{as} \quad \varepsilon \to 0. \tag{2.22}$$

REMARK 2.5. *Let $\Omega_{a\varepsilon} = (0,L)\times(a\varepsilon,h)$ for $a<0$, and let $\{\mathbf{u}^{a,\mathit{eff}}, p^{a,\mathit{eff}}\}$ be a solution for (2.8)–(2.11) in $\Omega_{a\varepsilon}$, with (2.11) replaced by*

$$u_2^{a,\mathit{eff}} = 0 \quad \text{and} \quad u_1^{a,\mathit{eff}} + \varepsilon C_1^{a,bl}\frac{\partial u_1^{a,\mathit{eff}}}{\partial x_2} = 0 \quad \text{on} \quad \Sigma_a = (0,b)\times a\varepsilon. \tag{2.23}$$

Problem (2.8)–(2.10), (2.23) has a unique smooth solution $\{\mathbf{u}^{a,\mathit{eff}}, p^{a,\mathit{eff}}\}$, its derivatives are bounded independently of ε, and, by (5.13), $C_1^{a,bl} = C_1^{bl} - a$. Then a simple calculation gives

$$\begin{aligned} 0 &= u_1^{a,\mathit{eff}}(x_1,\varepsilon a) + \varepsilon C_1^{a,bl}\frac{\partial u_1^{a,\mathit{eff}}}{\partial x_2}(x_1,\varepsilon a) = u_1^{a,\mathit{eff}}(x_1,0) + \varepsilon C_1^{bl}\frac{\partial u_1^{a,\mathit{eff}}}{\partial x_2}(x_1,0) \\ &\quad + \frac{(\varepsilon a)^2}{2}\left(\frac{\partial^2 u_1^{a,\mathit{eff}}}{\partial x_2^2}(x_1,\xi_1) + \frac{\partial^2 u_1^{a,\mathit{eff}}}{\partial x_2^2}(x_1,\xi_2)\right) \quad \text{for} \quad \xi_1,\xi_2 \in (0,\varepsilon a). \end{aligned}$$

Therefore, a perturbation of the interface position for an $O(\varepsilon)$ implies a perturbation in the solution of $O(\varepsilon^2)$ in $H^k(\Omega_1)$. Consequently, there is a freedom in fixing the position of Σ. It influences the result only at the next order of the asymptotic expansion.

The physical permeability K_{phys} is proportional to ε^2. Our result on the influence of the interface position on the effective slip is in agreement with the observation of Kaviany in [16, pages 79–83]. In fact, it has been noticed by Larson and Higdon in [17], [18] that changes of $O(1)$ in the slip coefficients are possible after the change of order $O(\sqrt{K_{phys}})$ of the interface position. Therefore, the exact position of Σ does not pose problems, since it influences the solution only at order $O(K_{phys})$.

3. Law by Beavers and Joseph. In this section we extend the justification of the law (1.5) from [13] to the case with a general body force. Our boundary conditions are simpler than those of the experiment from [2], and we consider the 2D Stokes system. The Beavers and Joseph setting could be reduced to our setting if Ω is sufficiently long in the x_1 direction. Then we may assume the periodic boundary conditions at the inlet/outlet boundary, and the flow is governed by a force coming from the pressure drop and is equal to $\frac{p_b - p_0}{b}\mathbf{e}^1$. We assume a nonconstant force, which can describe a larger class of the problems.

3.1. The impermeable interface approximation. Intuitively, the main flow is in the unconfined domain Ω_1. Following the approach from [13] we study the problem

$$-\triangle \mathbf{v}^0 + \nabla p^0 = \mathbf{f} \qquad \text{in } \Omega_1, \tag{3.1}$$

$$\text{div } \mathbf{v}^0 = 0 \qquad \text{in } \Omega_1, \tag{3.2}$$

$$\mathbf{v}^0 = 0 \qquad \text{on } \partial\Omega_1 \setminus \Big(\{x_1 = 0\} \cup \{x_1 = L\} \Big), \tag{3.3}$$

$$\{\mathbf{v}^0, p^0\} \qquad \text{is } L\text{-periodic in } x_1. \tag{3.4}$$

Problem (3.1)–(3.4) has a unique solution $\{\mathbf{v}^0, p^0\} \in H^1(\Omega_1)^2 \times L^2_0(\Omega_1)$ (see, e.g., [28]). In fact this solution is C^∞ for $\mathbf{f} \in C^\infty$. Therefore, for the lowest order approximation $\{\mathbf{v}^0, p^0\}$ we impose on the interface the no-slip condition

$$\mathbf{v}^0 = 0 \qquad \text{on} \quad \Sigma. \tag{3.5}$$

We observe that in the Beavers and Joseph setting $\mathbf{f} = -\frac{p_b - p_0}{L}\mathbf{e}^1$ and the unique solution for this problem in $H^1(\Omega_1)^2 \times L^2_0(\Omega_1)$ is the classic Poiseuille flow in Ω_1, satisfying the no-slip conditions at Σ. It is given by

$$\mathbf{v}^0 = \left(\frac{p_b - p_0}{2L} x_2(x_2 - h), 0 \right) \text{ for } \quad 0 \le x_2 \le h; \qquad p^0 = 0 \text{ for } \quad 0 \le x_1 \le L \tag{3.6}$$

(see [13], [15] for further details).

We extend $\mathbf{v}^0$ to Ω_2 by setting $v^0 = 0$ for $-H \le x_2 < 0$. For p^0 we use a smooth extension to Ω_2, $\tilde{p}^0$, which we shall specify. The question is in which sense this solution approximates the solution $\{\mathbf{v}^\varepsilon, p^\varepsilon\}$ of the original problem (2.3)–(2.5).

A direct consequence of the weak formulation (2.7) is that the difference $\mathbf{v}^\varepsilon - \mathbf{v}^0$ satisfies the following variational equation:

$$\begin{aligned} \int_{\Omega^\varepsilon} \nabla(\mathbf{v}^\varepsilon - \mathbf{v}^0)\nabla\varphi \, dx - \int_{\Omega^\varepsilon} (p^\varepsilon - \tilde{p}^0) \text{ div } \varphi = \int_\Sigma \frac{\partial v_1^0}{\partial x_2}\varphi_1 \, dS \\ - \int_\Sigma [\tilde{p}^0]\varphi_2 \, dS + \int_{\Omega_2^\varepsilon} (\mathbf{f} - \nabla\tilde{p}^0)\varphi \, dx \qquad \forall \varphi \in W^\varepsilon. \end{aligned} \tag{3.7}$$

Taking $\varphi = \mathbf{v}^\varepsilon - \mathbf{v}^0$ in (3.7) and applying Lemma 2.1 leads to the following result, proved in [13].

PROPOSITION 3.1. *Let* $\{\mathbf{v}^\varepsilon, p^\varepsilon\}$ *be the solution for* (2.3)–(2.5) *and* $\{\mathbf{v}^0, p^0\}$ *be defined by* (3.1)–(3.4). *Then, it holds that*

$$\sqrt{\varepsilon}\|\nabla(\mathbf{v}^\varepsilon - v^0)\|_{L^2(\Omega^\varepsilon)^4} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{v}^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\mathbf{v}^\varepsilon\|_{L^2(\Sigma)} \le C\varepsilon. \tag{3.8}$$

Furthermore, using estimate (3.8) and the notion of very weak solutions for the Stokes system in Ω_1, introduced in [6], [7] (see also Appendix 1), we conclude the following additional estimates.

COROLLARY 3.2 (see [13]). *Let* $\{\mathbf{v}^\varepsilon, p^\varepsilon\}$ *be the solution for* (2.3)–(2.5) *and* $\{\mathbf{v}^0, p^0\}$ *be defined by* (3.1)–(3.4). *Then, it holds that*

$$\sqrt{\varepsilon}\|p^\varepsilon - p^0\|_{L^2(\Omega_1)} + \|\mathbf{v}^\varepsilon - v^0\|_{L^2(\Omega_1)^2} \le C\varepsilon. \tag{3.9}$$

This provides the uniform a priori estimates for $\{\mathbf{v}^\varepsilon, p^\varepsilon\}$. Moreover, we have found that the viscous flow in Ω_1 corresponding to an impermeable wall is an $O(\varepsilon)$ L^2-approximation for $\mathbf{v}^\varepsilon$. The Beavers and Joseph law should correspond to the next order velocity correction. Since the Darcy velocity is of order $O(\varepsilon^2)$, we justify Saffman's version of the law.

3.2. Justification of the law by Beavers and Joseph. At the interface Σ the approximation from subsection 3.1 leads to the shear stress jump equal to $-\frac{\partial v_1^0}{\partial x_2}|_\Sigma$. Contrary to the pressure difference, which could be easily set to zero by the appropriate choice of $\tilde{p}^0$, the shear stress jump requires construction of the corresponding boundary layer. For the intuitive argument of how to obtain the shear stress jump correction using the natural stretching variable $y = \frac{x}{\varepsilon}$, we refer the reader to [15, page 503]. In the present paper we present the rigorous construction, based on the Navier boundary layer and following the scheme originally used in [13].

Let $\{\beta^{bl}, \omega^{bl}\}$ be the boundary layer given by (5.1)–(5.4).

Now we set

$$\beta^{bl,\varepsilon}(x) = \varepsilon\beta^{bl}\left(\frac{x}{\varepsilon}\right) \qquad \text{and} \qquad \omega^{bl,\varepsilon}(x) = \omega^{bl}\left(\frac{x}{\varepsilon}\right), \quad x \in \Omega^\varepsilon, \tag{3.10}$$

where $\beta^{bl,\varepsilon}$ is extended by zero to $\Omega \setminus \Omega^\varepsilon$. Let H be Heaviside's function. Then for every $q \geq 1$ we obtain

$$\begin{aligned} \frac{1}{\varepsilon}\|\beta^{bl,\varepsilon} - \varepsilon(C_1^{bl}, 0)H(x_2)\|_{L^q(\Omega)^2} + \|\omega^{bl,\varepsilon} - C_\omega^{bl} H(x_2)\|_{L^q(\Omega^\varepsilon)} \\ + \|\nabla\beta^{bl,\varepsilon}\|_{L^q(\Omega_1\cup\Sigma\cup\Omega_2)^4} = C\varepsilon^{1/q}. \end{aligned} \tag{3.11}$$

Hence, our boundary layer is not concentrated around the interface and there are some stabilization constants. We will see that these constants are closely linked to our effective interface law.

As in [12], stabilization of $\beta^{bl,\varepsilon}$ towards a nonzero constant velocity $\varepsilon\big(C_1^{bl}, 0\big)$, at the upper boundary, generates a counterflow. It is given by the following Stokes system in Ω_1:

$$-\triangle \mathbf{z}^\sigma + \nabla p^\sigma = 0 \qquad \text{in } \Omega_1, \tag{3.12}$$

$$\operatorname{div} \mathbf{z}^\sigma = 0 \qquad \text{in } \Omega_1, \tag{3.13}$$

$$\mathbf{z}^\sigma = 0 \quad \text{on } \{x_2 = h\} \quad \text{and } \mathbf{z}^\sigma = \frac{\partial v_1^0}{\partial x_2}|_\Sigma \mathbf{e}^1 \quad \text{on } \{x_2 = 0\}, \tag{3.14}$$

$$\{\mathbf{z}^\sigma, p^\sigma\} \qquad \text{is } L\text{-periodic in } x_1. \tag{3.15}$$

In the setting of the experiment by Beavers and Joseph, $\mathbf{z}^\sigma$ was proportional to the 2D Couette flow $\mathbf{d} = (1 - \frac{x_2}{h})\mathbf{e}^1$.

Now, after [12], we expected that the approximation for the velocity would read as

$$\mathbf{v}^\varepsilon = \mathbf{v}^0 - (\beta^{bl,\varepsilon} - \varepsilon(C_1^{bl}, 0))\frac{\partial v_1^0}{\partial x_2}|_\Sigma - \varepsilon C_1^{bl}\mathbf{z}^\sigma + O(\varepsilon^2). \tag{3.16}$$

Concerning the pressure, there are additional complications due to the stabilization of the boundary layer pressure to C_ω^{bl} when $y_2 \to +\infty$. Consequently, $\omega^{bl,\varepsilon} - H(x_2)C_\omega^{bl}\frac{\partial v_1^0}{\partial x_2}|_\Sigma$ is small in Ω_1 and we should take into account the pressure stabilization effect.

At the flat interface Σ, the normal component of the normal stress reduces to the pressure field. Subtraction of the stabilization pressure constant at infinity leads to the pressure jump on Σ:

$$[p^\varepsilon]_\Sigma = p^0(x_1,+0) - \tilde p^0(x_1,-0) = -C_\omega^{bl}\frac{\partial v_1^0}{\partial x_2}|_\Sigma + O(\varepsilon) \quad \text{for} \quad x_1\in(0,L). \tag{3.17}$$

Therefore, the pressure approximation is

$$\begin{aligned} p^\varepsilon(x) = p^0 H(x_2) + \tilde p^0 H(-x_2) - \big(\omega^{bl,\varepsilon}(x) - H(x_2)C_\omega^{bl}\big)\frac{\partial v_1^0}{\partial x_2}|_\Sigma \\ - \varepsilon C_1^{bl} p^\sigma H(x_2) + O(\varepsilon). \end{aligned} \tag{3.18}$$

Following the ideas from [12], these heuristic calculations could be made rigorous. Let us define the errors in velocity and in the pressure:

$$\mathcal{U}^\varepsilon(x) = \mathbf{v}^\varepsilon - \mathbf{v}^0 + (\beta^{bl,\varepsilon} - \varepsilon C_1^{bl}\mathbf{e}^1 H(x_2))\frac{\partial v_1^0}{\partial x_2}|_\Sigma + \varepsilon C_1^{bl}\mathbf{z}^\sigma, \tag{3.19}$$

$$\begin{aligned} \mathcal{P}^\varepsilon(x) = p^\varepsilon - p^0 H(x_2) - \tilde p^0 H(-x_2) + \big(\omega^{bl,\varepsilon}(x) \\ - H(x_2)C_\omega^{bl}\big)\frac{\partial v_1^0}{\partial x_2}|_\Sigma + \varepsilon C_1^{bl} p^\sigma H(x_2). \end{aligned} \tag{3.20}$$

REMARK 3.3. *A rigorous argument, showing that $\mathcal{U}^\varepsilon$ is of order $O(\varepsilon^2)$, allows justifying Saffman's modification of the Beavers and Joseph law* (1.5)*: On the interface Σ we obtain*

$$\frac{\partial v_1^\varepsilon}{\partial x_2}|_\Sigma = \frac{\partial v_1^0}{\partial x_2}|_\Sigma - \frac{\partial \beta_1^{bl}}{\partial y_2}|_{\Sigma, y=x/\varepsilon} + O(\varepsilon) \quad \text{and} \quad \frac{v_1^\varepsilon}{\varepsilon} = -\beta_1^{bl}(x_1/\varepsilon,0)\frac{\partial v_1^0}{\partial x_2}|_\Sigma + O(\varepsilon).$$

After averaging over Σ with respect to y_1, we obtain the Saffman version of the law by Beavers and Joseph

$$u_1^{eff} = -\varepsilon C_1^{bl}\frac{\partial u_1^{eff}}{\partial x_2} \quad \text{on} \quad \Sigma, \tag{3.21}$$

where u_1^{eff} is the average of v_1^ε over the characteristic pore opening at the naturally permeable wall. The higher order terms are neglected.

For simplicity we denote

$$\sigma_{12}^0(x_1) = \frac{\partial v_1^0}{\partial x_2}|_\Sigma.$$

Then, the variational equation for $(\beta^{bl,\varepsilon} - \varepsilon C_1^{bl}\mathbf{e}^1 H(x_2))\frac{\partial v_1^0}{\partial x_2}|_\Sigma$ reads as

(3.22)

$$\begin{aligned} &\int_{\Omega^\varepsilon} \nabla\Big((\beta^{bl,\varepsilon} - \varepsilon C_1^{bl}\mathbf{e}^1 H(x_2))\sigma_{12}^0\Big) : \nabla\varphi \, dx - \int_{\Omega^\varepsilon} \sigma_{12}^0\big(\omega^{bl,\varepsilon}(x) - H(x_2)C_\omega^{bl}\big) \operatorname{div}\varphi \, dx \\ &\quad = -\int_\Sigma \varphi_1\sigma_{12}^0 \, dS - \int_\Sigma C_\omega^{bl}\varphi_2\sigma_{12}^0 \, dS \\ &\qquad - \int_{\Omega^\varepsilon}\sum_i \Big(\Delta\sigma_{12}^0(\beta_i^{bl,\varepsilon} - \varepsilon C_1^{bl}\delta_{1i}H(x_2))\varphi_i \\ &\qquad\qquad - \partial_{x_i}\sigma_{12}^0(\omega^{bl,\varepsilon} - \varepsilon C_\omega^{bl})\varphi_i \\ &\qquad\qquad - 2(\beta_i^{bl,\varepsilon} - \varepsilon C_1^{bl}\delta_{1i}H(x_2)) \operatorname{div}(\varphi_i\nabla\sigma_{12}^0)\Big)\, dx \qquad \forall\varphi\in W^\varepsilon. \end{aligned}$$

Next, the variational form of (3.12)–(3.15) reads as

$$\int_{\Omega^\varepsilon} \nabla \mathbf{z}^\sigma : \nabla\varphi \, dx - \int_{\Omega^\varepsilon} p^\sigma \operatorname{div} \varphi \, dx = -\int_\Sigma \left(-\varphi_2 p^\sigma + \varphi \cdot \frac{\partial \mathbf{z}^\sigma}{\partial x_2} \right) dS \qquad \forall \varphi \in W^\varepsilon. \tag{3.23}$$

Now we are ready to write the variational equation for $\{\mathcal{U}^\varepsilon, \mathcal{P}^\varepsilon\}$ and obtain the higher order error estimates as in [13]. Nevertheless, contrary to [13], $\mathcal{U}^\varepsilon$ is no longer divergence-free and we need more effort to control $\mathcal{P}^\varepsilon$.

THEOREM 3.4. *Let $\mathcal{U}^\varepsilon$ be defined by* (3.19) *and $\mathcal{P}^\varepsilon$ by* (3.20)*. Let $\tilde p^0$ be a smooth function satisfying the interface condition* (3.17)*. Then, the following estimates hold:*

$$\varepsilon \|\nabla \mathcal{P}^\varepsilon\|_{H^{-1}(\Omega^\varepsilon)} + \varepsilon \|\nabla \mathcal{U}^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|\mathcal{U}^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \varepsilon^{1/2} \|\mathcal{U}^\varepsilon\|_{L^2(\Sigma)^2} \le C\varepsilon^2. \tag{3.24}$$

Proof. First, we remark that for $y_2 > 0$ the mean with respect to y_1 of $\omega^{bl}(y) - C^{bl}_\omega$ is zero. Consequently, the problem

$$\frac{\partial \pi^{bl}_\omega}{\partial y_1} = \omega^{bl}(y) - C^{bl}_\omega \qquad \forall y_1 \in (0,1);\ \pi^{bl}_\omega \text{ is 1-periodic; } \int_0^1 \pi^{bl}_\omega(y_1, y_2)\, dy_1 = 0 \tag{3.25}$$

has a unique smooth solution.

Next by subtracting (3.22) and (3.23) from (3.7) we obtain

$$\begin{aligned} &\int_{\Omega^\varepsilon} \nabla \mathcal{U}^\varepsilon : \nabla\varphi \, dx - \int_{\Omega^\varepsilon} \mathcal{P}^\varepsilon \operatorname{div} \varphi \, dx = \varepsilon \int_\Sigma \Bigg(-\varphi_2 p^\sigma \\ &+ \varphi \cdot \frac{\partial \mathbf{z}^\sigma}{\partial x_2} \Bigg) dS + \int_{\Omega_2^\varepsilon} (\mathbf{f} - \nabla \tilde p^0) \varphi \, dx - \int_{\Omega^\varepsilon} \sum_i \Delta \sigma^0_{12} (\beta_i^{bl,\varepsilon} - \varepsilon C_1^{bl} \delta_{1i} H(x_2)) \varphi_i \, dx \\ &- \int_{\Omega_2^\varepsilon} \partial_{x_1} \sigma^0_{12} \omega^{bl,\varepsilon} \varphi_1 \, dx - \int_{\Omega_1} \varepsilon \pi^{bl}_\omega \left(\frac{x}{\varepsilon} \right) (\varphi_1 \partial^2_{x_1} \sigma^0_{12} + \partial_{x_1} \varphi_1 \partial_{x_1} \sigma^0_{12}) \, dx \\ &+ 2 \int_{\Omega^\varepsilon} (\beta_1^{bl,\varepsilon} - \varepsilon C_1^{bl} H(x_2)) (\varphi_1 \partial^2_{x_1} \sigma^0_{12} + \partial_{x_1} \varphi_1 \partial_{x_1} \sigma^0_{12}) \, dx \qquad \forall \varphi \in W^\varepsilon, \end{aligned} \tag{3.26}$$

$$\operatorname{div} \mathcal{U}^\varepsilon = (\beta_1^{bl,\varepsilon} - \varepsilon C_1^{bl} H(x_2)) \frac{d}{dx_1} \sigma^0_{12} \quad \text{ in } \Omega^\varepsilon. \tag{3.27}$$

From (3.26) we find out that

$$\left| \int_{\Omega^\varepsilon} \nabla \mathcal{U}^\varepsilon : \nabla\varphi \, dx - \int_{\Omega^\varepsilon} \mathcal{P}^\varepsilon \operatorname{div} \varphi \, dx \right| \le C\varepsilon^{3/2} ||\nabla\varphi||_{L^2(\Omega^\varepsilon)^4} + C\varepsilon ||\mathbf{f} - \nabla \tilde p^0||_{L^2(\Omega_2)^2} ||\nabla \varphi||_{L^2(\Omega_2^\varepsilon)^4} \tag{3.28}$$

and

$$||\operatorname{div} \mathcal{U}^\varepsilon||_{L^2(\Omega^\varepsilon)^2} \le C\varepsilon^{3/2}. \tag{3.29}$$

The size of $\operatorname{div} \mathcal{U}^\varepsilon$ does not allow us to obtain the appropriate estimate, and we should diminish it further.

Let $\mathbf{Q}^{bl}$ be given by (5.14)–(5.16). Furthermore, let $\mathbf{Q}^{bl,\varepsilon}(x) = \varepsilon^2 \mathbf{Q}^{bl}(x/\varepsilon)$, and let $\mathbf{w}^Q$ be defined by

$$\begin{cases} \Delta \mathbf{w}^Q - \nabla p^Q = 0 \quad \text{in } \Omega_1; \\ \text{div } \mathbf{w}^Q = \dfrac{1}{|\Omega_1|} \displaystyle\int_\Sigma \frac{d}{dx_1} \sigma_{12}^0 \, dS = 0 \quad \text{in } \Omega_1; \\ \mathbf{w}^Q = -\dfrac{d}{dx_1} \sigma_{12}^0 \mathbf{e}^2 \text{ on } \Sigma, \quad \mathbf{w}^Q = 0 \text{ on } \{x_2 = h\}; \\ \{\mathbf{w}^Q, p^Q\} \quad \text{is } L\text{-periodic in } x_1. \end{cases} \tag{3.30}$$

We introduce the following error functions, where the compressibility effects are reduced to the next order:

$$\mathcal{U}^\varepsilon(x) = \mathcal{U}_0^\varepsilon(x) + \mathbf{Q}^{bl,\varepsilon}(x) \frac{d}{dx_1} \sigma_{12}^0 + \varepsilon^2 H(x_2) \left(\int_{Z_{BL}} (C_1^{bl} H(y_2) - \beta_1^{bl}(y)) \, dy \right) \mathbf{w}^Q, \tag{3.31}$$

$$\mathcal{P}^\varepsilon(x) = \mathcal{P}_0^\varepsilon(x,t) + \varepsilon^2 H(x_2) \left(\int_{Z_{BL}} (C_1^{bl} H(y_2) - \beta_1^{bl}(y)) \, dy \right) p^Q, \tag{3.32}$$

$$\text{div } \mathcal{U}_0^\varepsilon = -Q_1^{bl,\varepsilon}(x) \, \frac{d^2}{dx_1^2} \sigma_{12}^0 \quad \text{in } \Omega^\varepsilon. \tag{3.33}$$

Then $\mathcal{U}_0^\varepsilon \in W^\varepsilon$ and $||\text{div } \mathcal{U}_0^\varepsilon||_{L^2(\Omega^\varepsilon)^4} \le C\varepsilon^{5/2}$. Next we construct a function $\Phi^{1,\varepsilon} \in H^1(\Omega_1)^2$ such that

$$\begin{cases} \text{div } \Phi^{1,\varepsilon} = -Q_1^{bl,\varepsilon}(x) \, \dfrac{d^2}{dx_1^2} \sigma_{12}^0 \quad \text{in } \Omega_1; \\ \Phi^{1,\varepsilon} = \dfrac{\mathbf{e}^2}{|\Sigma|} \displaystyle\int_{\Omega_1} Q_1^{bl,\varepsilon}(x) \frac{d^2}{dx_1^2} \sigma_{12}^0 \, dx \text{ on } \Sigma, \quad \Phi^{1,\varepsilon} = 0 \text{ on } \{x_2 = h\}; \\ \Phi^{1,\varepsilon} \quad \text{is } L\text{-periodic in } x_1. \end{cases} \tag{3.34}$$

We note that $||\Phi^{1,\varepsilon}||_{H^1(\Omega_1)^2} \le C\varepsilon^2$. Next we extend $Q^{bl,\varepsilon}$ by zero to the rigid part of the porous medium and choose a function $\Phi^{2,\varepsilon} \in H^1(\Omega_2)^2$ such that

$$\begin{cases} \text{div } \Phi^{2,\varepsilon} = -Q_1^{bl,\varepsilon}(x) \, \dfrac{d^2}{dx_1^2} \sigma_{12}^0 \quad \text{in } \Omega_2; \\ \Phi^{2,\varepsilon} = -\dfrac{\mathbf{e}^2}{|\Sigma|} \displaystyle\int_{\Omega_2} Q_1^{bl,\varepsilon}(x) \frac{d^2}{dx_1^2} \sigma_{12}^0 \, dx \text{ on } \Sigma, \quad \Phi^{2,\varepsilon} = 0 \text{ on } \{x_2 = -H\}; \\ \Phi^{2,\varepsilon} \quad \text{is } L\text{-periodic in } x_1. \end{cases} \tag{3.35}$$

We note that $\Phi^{1,\varepsilon} = \Phi^{2,\varepsilon}$ on Σ and $||\Phi^{2,\varepsilon}||_{H^1(\Omega_1)^2} \le C\varepsilon^2$. Let $X_2 = \{\mathbf{z} \in H^1(\Omega_2)^2$, $\mathbf{z} = 0$ on $\{x_1 = L\}$ and $\mathbf{z}$ is L-periodic in $x_1\}$ and $X_2^\varepsilon = \{\mathbf{z} \in X_2, \mathbf{z} = 0 \text{ on } \partial\Omega_2^\varepsilon \setminus \partial\Omega_2\}$. In the seminal paper [27], Tartar constructed a continuous linear restriction operator

operator $R_\varepsilon \in \mathcal{L}\big(X_2, X_2^\varepsilon\big)$ such that

$$\text{div }(R_\varepsilon\varphi) = \text{ div }\varphi + \sum_{k\in T_\varepsilon} \frac{1}{|Y^\varepsilon_{F_k}|}\chi_{Y^\varepsilon_{F_k}} \int_{Y^\varepsilon_{S_k}} \text{div }\varphi\, dx \qquad \forall\varphi \in X_2, \tag{3.36}$$

$$\|R_\varepsilon\varphi\|_{L^2(\Omega_2^\varepsilon)^2} \le C\big\{\varepsilon\|\nabla\varphi\|_{L^2(\Omega_2)^4} + \|\varphi\|_{L^2(\Omega_2)^2}\big\} \qquad \forall\varphi \in X_2, \tag{3.37}$$

$$\|\nabla(R_\varepsilon\varphi)\|_{L^2(\Omega^\varepsilon)^4} \le \frac{C}{\varepsilon}\big\{\varepsilon\|\nabla\varphi\|_{L^2(\Omega_2)^4} + \|\varphi\|_{L^2(\Omega_2)^2}\big\} \qquad \forall\varphi \in X_2. \tag{3.38}$$

Furthermore, $\varphi = R_\varepsilon\varphi$ on Σ. For more details we also refer the reader to [1], [20]. This construction allows us to work with the divergence-free velocity error function $\overline{\mathcal{U}}^\varepsilon$ given by

$$\overline{\mathcal{U}}^\varepsilon = \mathcal{U}_0^\varepsilon - H(x_2)\Phi^{1,\varepsilon} - H(-x_2)R_\varepsilon\Phi^{2,\varepsilon}. \tag{3.39}$$

Now we write the analogue of the variational equation (3.26) for $\{\overline{\mathcal{U}}^\varepsilon, \mathcal{P}_0^\varepsilon\}$ and, since $||\nabla R_\varepsilon\Phi^{2,\varepsilon}||_{L^2(\Omega_2)^4} \le C\varepsilon$, we find out that the leading order force term is of the same order as in the estimate (3.28). Now we test the analogue of variational equation (3.26) for $\{\overline{\mathcal{U}}^\varepsilon, \mathcal{P}_0^\varepsilon\}$ with $\varphi = \overline{\mathcal{U}}^\varepsilon$ to obtain

$$||\nabla\overline{\mathcal{U}}^\varepsilon||_{L^2(\Omega^\varepsilon)^4} \le C\varepsilon. \tag{3.40}$$

We remark that $\overline{\mathcal{U}}^\varepsilon$ differs from $\mathcal{U}^\varepsilon$ for $O(\varepsilon^2)$ in L^2-norm and for $O(\varepsilon)$ in H^1-norm. Therefore (3.40) gives us the middle part of the estimate (3.24). In what concerns the $L^2(\Sigma)$-norm of $\mathcal{U}^\varepsilon$, it follows by using (2.1). The remaining pressure estimate follows easily from the weak formulation and the estimates on $\mathcal{U}^\varepsilon$. ☐

Next we use Theorem 3.4 and the results on the Stokes system with L^2-boundary values from [10], [4] to conclude the following result.

COROLLARY 3.5. *Let $\mathcal{U}^\varepsilon$ be defined by* (3.19) *and $\mathcal{P}^\varepsilon$ by* (3.20). *Let $\tilde{p}^0$ be a smooth function satisfying the interface condition* (3.17). *Then, the following estimate holds:*

$$\sqrt{\varepsilon}\|\mathcal{P}^\varepsilon\|_{L^2(\Omega_1)} + \|\mathcal{U}^\varepsilon\|_{H^{1/2}(\Omega_1)^2} \le C\varepsilon^{3/2}. \tag{3.41}$$

Now we introduce the effective flow equations in Ω_1 through the boundary value problem (2.8)–(2.11), containing the slip condition of Beavers and Joseph. Since our expansion is performed using the solution $\{\mathbf{v}^0, p^0\}$ of the problem (3.1)–(3.4), we need to know the relationship between the solutions to these two boundary value problems.

PROPOSITION 3.6. *Let $\mathbf{f} \in C^\infty(\overline{\Omega}_1)^2$ and be L-periodic in x_1. Let $\{\mathbf{u}^{\mathit{eff}}, p^{\mathit{eff}}\}$ be the solution of the problem* (2.8)–(2.11), *$\{\mathbf{v}^0, p^0\}$ of the problem* (3.1)–(3.4), *and $\{\mathbf{z}^\sigma, p^\sigma\}$ of the problem* (3.12)–(3.15). *Then we have*

$$||\mathbf{u}^{\mathit{eff}} - \mathbf{v}^0||_{H^k(\Omega_1)^2} + ||p^{\mathit{eff}} - p^0||_{H^{k-1}(\Omega_1)} \le C\varepsilon \qquad \forall k \in \mathbb{N}; \tag{3.42}$$

$$\begin{aligned} &||\mathbf{u}^{\mathit{eff}} - \mathbf{v}^0 + \varepsilon C_1^{bl}\mathbf{z}^\sigma||_{H^k(\Omega_1)^2} + ||p^{\mathit{eff}} - p^0 + \varepsilon C_1^{bl}p^\sigma||_{H^{k-1}(\Omega_1)} \\ &\quad \le C\varepsilon^2 \qquad \forall k \in \mathbb{N}. \end{aligned} \tag{3.43}$$

Proof. The elliptic regularity for the Stokes operator (see, e.g., [28]) gives C^∞ regularity for the functions $\{\mathbf{u}^{\mathit{eff}}, p^{\mathit{eff}}\}$, $\{\mathbf{v}^0, p^0\}$, and $\{\mathbf{z}^\sigma, p^\sigma\}$. It is easy to see that $\{\mathbf{u}^{\mathit{eff}}, p^{\mathit{eff}}\}$ is bounded in $H^k(\Omega_1)^4$, independently of ε, for every integer k.

Let $\mathbf{U} = \mathbf{u}^{\mathit{eff}} - \mathbf{v}^0$ and $P = p^{\mathit{eff}} - p^0$. Then for every $\varphi \in \mathcal{V} = \{\varphi \in H^1(\Omega_1)^2 |\ \varphi$ is L-periodic in x_1, $\varphi = 0$ on $\{x_2 = h\}$, $\varphi_2 = 0$ on $\Sigma\}$ we obtain

$$\int_{\Omega_1} \nabla\mathbf{U} : \nabla\varphi\ dx - \int_{\Omega_1} P \text{ div }\varphi\ dx - \frac{1}{\varepsilon C_1^{bl}}\int_\Sigma U_1\varphi_1\ dS = -\int_\Sigma \sigma_{12}^0\varphi_1\ dS. \tag{3.44}$$

Using $\varphi = \mathbf{U}$ as a test function yields

$$\left\{ \begin{array}{l} ||\mathbf{U}||_{H^1(\Omega_1)^2} + \frac{1}{\sqrt{\varepsilon}}||U_1||_{L^2(\Sigma)} \leq C\sqrt{\varepsilon}, \\ ||P||_{L^2(\Omega_1)} \leq C\sqrt{\varepsilon}. \end{array} \right. \tag{3.45}$$

Differentiating the equations with respect to x_1 leads to the estimate

$$\left\{ \begin{array}{l} \left|\left|\frac{\partial \mathbf{U}}{\partial x_1}\right|\right|_{H^1(\Omega_1)^2} + \frac{1}{\sqrt{\varepsilon}}\left|\left|\frac{\partial U_1}{\partial x_1}\right|\right|_{L^2(\Sigma)} \leq C\sqrt{\varepsilon}, \\ \left|\left|\frac{\partial P}{\partial x_1}\right|\right|_{L^2(\Omega_1)} \leq C\sqrt{\varepsilon}. \end{array} \right. \tag{3.46}$$

Since $\frac{\partial U_2}{\partial x_1} = 0$ on Σ, we have for the velocity trace that $\mathbf{U} \in H^1(\Sigma)^2$ and its norm is smaller than $C\varepsilon$. Using [10], [4] we obtain that

$$||\mathbf{U}||_{H^{3/2}(\Omega_1)^2} + ||P||_{H^{1/2}(\Omega_1)} \leq C\varepsilon. \tag{3.47}$$

After bootstrapping, we conclude that the estimate (3.42) holds true.

Using corrections $\mathbf{U}^1 = \mathbf{u}^{\mathit{eff}} - \mathbf{v}^0 + \varepsilon C_1^{bl}\mathbf{z}^\sigma$ and $P^1 = p^{\mathit{eff}} - p^0 + \varepsilon C_1^{bl} p^\sigma$, for every $\varphi \in \mathcal{V} = \{\varphi \in H^1(\Omega_1)^2 |\ \varphi$ is L-periodic in x_1, $\varphi = 0$ on $\{x_2 = h\}$, $\varphi_2 = 0$ on $\Sigma\}$ we obtain

$$\int_{\Omega_1} \nabla \mathbf{U}^1 : \nabla\varphi \, dx - \int_{\Omega_1} P^1 \operatorname{div} \varphi \, dx - \frac{1}{\varepsilon C_1^{bl}} \int_\Sigma U_1^1 \varphi_1 \, dS = \varepsilon \int_\Sigma g\varphi_1 \, dS, \tag{3.48}$$

where $g = -C_1^{bl} \frac{\partial z_1^\sigma}{\partial x_2}|_\Sigma \in C^\infty(\overline{\Sigma})$ is uniformly bounded with respect to ε. Repeating the argument used in the first part of the proof to $\{\mathbf{U}^1, P^1\}$ yields the estimate (3.43). ∎

Proof of Theorem 2.2. We remark that on Σ

$$\mathbf{v}^\varepsilon - \mathbf{u}^{\mathit{eff}} = \mathcal{U}^\varepsilon - (\beta^{bl,\varepsilon} - \varepsilon(C_1^{bl}, 0))\frac{\partial v_1^0}{\partial x_2}(x_1, 0). \tag{3.49}$$

Now Theorem 3.4, Corollary 3.5, and Propositions 4.2 and 4.3 from Appendix 1 imply the desired result. ∎

3.3. Justification of the interface pressure jump law and the effective equations in the porous medium. We have already seen that, after extension by zero to the rigid part, the velocity $\mathcal{U}^\varepsilon$ satisfies the a priori estimates (3.24), (3.41), with Ω^ε replaced by Ω. Furthermore, it would be more comfortable to work with the pressure field $\mathcal{P}^\varepsilon$ defined on Ω. Following the approach from [19], we define the pressure extension $\tilde{\mathcal{P}}^\varepsilon$ by

$$\tilde{\mathcal{P}}^\varepsilon = \left\{ \begin{array}{ll} \mathcal{P}^\varepsilon & \text{in } \Omega^\varepsilon, \\ \frac{1}{|Y_{F_i}^\varepsilon|} \int_{Y_{F_i}^\varepsilon} \mathcal{P}^\varepsilon & \text{in the } Y_{S_i}^\varepsilon \text{ for each } i, \end{array} \right. \tag{3.50}$$

where $Y_{F_i}^\varepsilon$ is the fluid part of the cell Y_i^ε. Note that the solid part of the porous medium is a union of all $Y_{S_i}^\varepsilon$. Then, following Tartar's results from [27] we have

$$\langle \nabla\tilde{\mathcal{P}}^\varepsilon, \varphi\rangle_\Omega = \langle \nabla\mathcal{P}^\varepsilon, \tilde{R}_\varepsilon\varphi\rangle_{\Omega^\varepsilon} \qquad \forall\varphi \in H^1(\Omega)^2,$$

where

$$\tilde{R}_\varepsilon \varphi(x) = \begin{cases} \varphi(x) & \text{for } x \in \Omega_1 \cup \Sigma; \\ R_\varepsilon \varphi(x) & \text{for } x \in \Omega_2^\varepsilon. \end{cases} \tag{3.51}$$

Using the estimate (3.24) and properties (3.36)–(3.38) of the restriction operator R_ε, we arrive at the following result.

COROLLARY 3.7 (a priori estimate for the pressure field in Ω_2). *Let $\tilde{\mathcal{P}}^\varepsilon$ be defined by* (3.50). *Then it satisfies the estimates*

$$\|\nabla \tilde{\mathcal{P}}^\varepsilon\|_{W'} \le C \quad \text{and} \quad \|\tilde{\mathcal{P}}^\varepsilon\|_{L^2(\Omega_2)} \le C, \tag{3.52}$$

where $W = \{\mathbf{z} \in H^1(\Omega_2)^2 : \mathbf{z} = 0$ on $\{x_2 = -H\} \cup \{x_2 = 0\}$, $\mathbf{z}$ is L-periodic$\}$.

We remark that in Ω_2 we have strong L^2-compactness of the family $\{\tilde{\mathcal{P}}^\varepsilon\}$. From the properties of Tartar's restriction operator (see [27] or [1]), we arrive at the following result.

LEMMA 3.8. *The sequence $\{\tilde{\mathcal{P}}^\varepsilon\}$ is strongly relatively compact in $L^2(\Omega_2)$.*

Following the homogenization derivation of the Darcy law from [9], [27], [1], or [20], we consider the following auxiliary problems in Y_F.

For $1 \le i \le 2$, find $\{\mathbf{w}^i, \pi^i\} \in H^1_{per}(Y_F)^2 \times L^2(Y_F)$, $\int_{Y_F} \pi^i(y)\, dy = 0$, such that

$$\begin{cases} -\Delta_y \mathbf{w}^i(y) + \nabla_y \pi^i(y) = \mathbf{e}^i & \text{in } Y_F, \\ \mathrm{div}_y \mathbf{w}^i(y) = 0 & \text{in } Y_F, \\ \mathbf{w}^i(y) = 0 & \text{on } (\partial Y_F \setminus \partial Y). \end{cases} \tag{3.53}$$

Obviously, these problems always admit unique solutions. Let us introduce the *permeability matrix* K by

$$K_{ij} = \int_{Y_F} \nabla_y \mathbf{w}^i : \nabla_y \mathbf{w}^j \; dy = \int_{Y_F} w^i_j \; dy, \quad 1 \le i, j \le 2. \tag{3.54}$$

Then, after [26], permeability tensor K is symmetric and positive definite. Consequently, the *drag tensor* K^{-1} is also positive definite.

Proof of Theorem 2.3. Let the function $\hat{p}^0$ be the solution for the boundary value problem

$$\mathrm{div}\left(K(\mathbf{f}(x) - \nabla \hat{p}^0) \right) = 0 \ \text{ in } \ \Omega_2, \tag{3.55}$$

$$\hat{p}^0 = p^0 + C^{bl}_\omega \sigma^0_{12}(x_1) \ \text{ on } \ \Sigma; \quad K(\mathbf{f}(x) - \nabla \hat{p}^0)|_{\{x_2=-H\}} \cdot \mathbf{e}^2 = 0. \tag{3.56}$$

We take as test function in (3.26) $\varphi(x)\psi(y)$, with $\varphi \in C_0^\infty(\Omega_2)$ and $\psi \in H^1_{per}(Y_F)^2$, $\mathrm{div}_y \psi = 0$. Then, after passing to the subsequence,

$$\frac{\mathcal{U}^\varepsilon}{\varepsilon^2} \to \mathcal{U}^{imp}(x,y), \quad \nabla \frac{\mathcal{U}^\varepsilon}{\varepsilon} \to \nabla_y \mathcal{U}^{imp}(x,y), \quad \text{and} \quad \tilde{\mathcal{P}}^\varepsilon \to \mathcal{P}^{imp}(x)$$

and we have

$$\begin{aligned} &\int_{\Omega_2} \int_{Y_F} \nabla_y \mathcal{U}^{imp} : \nabla_y \psi \varphi \; dydx - \int_{\Omega_2} \int_{Y_F} \mathcal{P}^{imp}(x) \psi(y) \nabla_x \varphi(x) \; dydx \\ &\qquad = \int_{\Omega_2} \int_{Y_F} (\mathbf{f} - \nabla \hat{p}^0) \psi(y) \varphi(x) \; dydx, \end{aligned} \tag{3.57}$$

implying that

$$\mathcal{U}^{imp}(x,y) = \sum_{j=1}^{2} \mathbf{w}^j(y)\left(f_j(x) - \frac{\partial(\hat{p}^0 + \mathcal{P}^{imp})}{\partial x_j} \right) \text{ (a.e.) in } \Omega_2. \tag{3.58}$$

Consequently, we obtain $\hat{p}^0 + \mathcal{P}^{imp} \in H^1(\Omega_2)$.

By Corollary 3.5 it holds that $\varepsilon^{-1}\nabla \mathcal{U}^\varepsilon \rightharpoonup 0$ strongly in $L^2(\Omega_1)^2$. Next taking $\varphi \in C_0^\infty(\Omega)$ and using a priori estimates (3.24), the variational equation (3.26) yields a generalized form of (3.57) leading to

$$\mathcal{P}^{imp} = 0 \quad \text{on} \quad \Sigma. \tag{3.59}$$

Averaging div $\mathcal{U}^\varepsilon$ in Ω_2 results in

$$\text{div}\,\Big(K(\mathbf{f}(x) - \nabla(\mathcal{P}^{imp} + \hat{p}^0)) \Big) = 0 \text{ in } \Omega_2.$$

Hence the function $\mathcal{P}^{imp} + \hat{p}^0$ is L-periodic in x_1 and satisfies

$$\text{div}\,\Big(K(\mathbf{f}(x) - \nabla(P^{imp} + \hat{p}^0)) \Big) = 0 \text{ in } \Omega_2; \tag{3.60}$$

$$\mathcal{P}^{imp} + \hat{p}^0 = p^0 + C_\omega^{bl}\sigma_{12}^0(x_1) \text{ on } \Sigma; \tag{3.61}$$

$$K(\mathbf{f}(x) - \nabla(\mathcal{P}^{imp} + \hat{p}^0))|_{\{x_2=-H\}} \cdot \mathbf{e}^2 = 0, \tag{3.62}$$

and we have $\mathcal{P}^{imp} = 0$. Let $\tilde{p}^0$ be the solution to the problem (2.17)–(2.18). Using Proposition 3.6 we find out that $\tilde{p}^0$ and $\hat{p}^0$ differ for $C\varepsilon$ in any $H^k(\Omega_2)$, $k \in \mathbb{N}$. Hence we have established (2.19)–(2.20). It remains to prove the last stated result, i.e.,

$$||p^\varepsilon - p^0||_{H^{-1/2}(\Sigma)} \le C\sqrt{\varepsilon}. \tag{3.63}$$

We use the variational equation (3.26) with test function having support in Ω_1 and Corollary 3.5 to obtain

$$\begin{aligned} &\left|\left| \text{div}\, \left(\nabla \mathcal{U}_2^\varepsilon - \mathcal{P}^\varepsilon \mathbf{e}^2 - 2\beta_2^{bl,\varepsilon} \frac{d\sigma_{21}^0}{dx_1}\mathbf{e}^1 \right) \right|\right|_{L^2(\Omega_1)} \\ &+ \left|\left| \nabla \mathcal{U}_2^\varepsilon - \mathcal{P}^\varepsilon \mathbf{e}^2 - 2\beta_2^{bl,\varepsilon} \frac{d\sigma_{21}^0}{dx_1}\mathbf{e}^1 \right|\right|_{L^2(\Omega_1)^2} \le C\varepsilon. \end{aligned} \tag{3.64}$$

Estimate (3.64) implies the following estimate for the trace:

$$\left|\left| \frac{\partial \mathcal{U}_2^\varepsilon}{\partial x_2} - \mathcal{P}^\varepsilon \right|\right|_{H^{-1/2}(\Sigma)} \le C\varepsilon. \tag{3.65}$$

Next we remark that

$$\frac{\partial \mathcal{U}_2^\varepsilon}{\partial x_2} = \text{ div}\, \mathcal{U}^\varepsilon - \frac{\partial \mathcal{U}_1^\varepsilon}{\partial x_1},$$

and on Σ, using Theorem 3.4, we obtain

$$||\mathcal{P}^\varepsilon||_{H^{-1/2}(\Sigma)} \le \left|\left| \frac{\partial \mathcal{U}_2^\varepsilon}{\partial x_2} \right|\right|_{H^{-1/2}(\Sigma)} + C\varepsilon \le \left|\left| \frac{\partial \mathcal{U}_1^\varepsilon}{\partial x_1} \right|\right|_{H^{-1/2}(\Sigma)} + C\varepsilon. \tag{3.66}$$

A direct calculation shows that

$$\left|\left|\left[\frac{\partial \mathcal{U}_2^\varepsilon}{\partial x_2} - \mathcal{P}^\varepsilon\right]\right|\right|_{L^\infty(\Sigma)} \le C\varepsilon,$$

and our result is valid for the traces taken from either the unconfined side or the side corresponding to the porous medium. ∎

4. Appendix 1: Very weak solutions to the Stokes system in Ω_1. Let $\mathbf{G}_1 \in L^2(\Omega_1)^2$, $G_2 \in L^2(\Omega_1)^4$, and $\xi \in L^2(\Sigma)^2$. We consider the following Stokes system in Ω_1:

$$\begin{cases} -\Delta \mathbf{b} + \nabla P = \mathbf{G}_1 + \text{ div } G_2 \quad \text{in } \Omega_1; \\ \text{div } \mathbf{b} = 0 \quad \text{in } \Omega_1; \\ \mathbf{b} = \xi \quad \text{on } \Sigma_T = \Sigma \cup \{x_2 = h\}; \\ \{\mathbf{b}, P\} \quad \text{is } L\text{-periodic in } x_1. \end{cases} \tag{4.1}$$

Our aim is to show the existence of a very weak solution $(\mathbf{b}, P) \in L^2(\Omega_1)^2 \times H^{-1}(\Omega_1)$ to problem (4.1). To this end, we use the transposition method from [6], [7].

Thus, let us test problem (4.1) with a smooth test function $(\mathbf{\Phi}, \pi)$, satisfying $\mathbf{\Phi} = 0$ on Σ_T and being L-periodic in x_1. Furthermore, π is L-periodic in x_1. We obtain

$$\begin{aligned} \langle \mathbf{G}_1 + \text{ div } G_2, \mathbf{\Phi}\rangle = \langle -\text{ div } (\nabla \mathbf{b} - PI), \mathbf{\Phi}\rangle = -\int_{\Omega_1} P \text{ div } \mathbf{\Phi}\, dx \\ + \int_{\Sigma_T} (2D(\mathbf{\Phi}) - \pi I)\nu\xi\, dSdt + \int_{\Omega_1} \mathbf{b} \cdot \Big(-\Delta \mathbf{\Phi} + \nabla \pi\Big)\, dx. \end{aligned} \tag{4.2}$$

Let $(\mathbf{g}, s) \in W^{q-2,r}(\Omega_1)^2 \times W^{q-1,r}(\Omega_1)$, $1 < r < +\infty$, $1 \le q \le 2$, and $\mathcal{H} = \{z \in W^{q-1,r}(\Omega_1), \int_{\Omega_1} z\, dx = 0\}$, and denote by $\mathcal{H}^*$ its dual. Now let $\{\mathbf{\Phi}, \pi\}$ be given by

$$\begin{cases} -\Delta \mathbf{\Phi} + \nabla \pi = \mathbf{g} \quad \text{in } \Omega_1; \\ \text{div } \mathbf{\Phi} = s \quad \text{in } \Omega_1; \\ \Phi = 0 \quad \text{on } \Sigma_T, \quad \{\mathbf{\Phi}, \pi\} \quad \text{is } L\text{-periodic in } x_1. \end{cases} \tag{4.3}$$

After the elliptic regularity for the Stokes system in [28], for $q \ne 1 + 1/r$, we obtain $\mathbf{\Phi} \in W^{q,r}(\Omega_1)^2$, $\pi \in W^{q-1,r}(\Omega_1)$, with $\int_{\Omega_1} \pi = 0$, and the following estimates hold:

$$\|\mathbf{\Phi}\|_{W^{q,r}(\Omega_1)^2} + \|\nabla \pi\|_{W^{q-2,r}(\Omega)} dt \le C \left(\|\mathbf{g}\|_{W^{q-2,r}(\Omega_1)^2} + \|\nabla s\|_{W^{q-2,r}(\Omega_1)}\right). \tag{4.4}$$

Now, analogously to the approach in [6], [7] where the stationary Stokes system was treated, for $q > 1 + 1/r$, we consider the linear form

$$\ell(\mathbf{g}, s) = \langle \mathbf{G}_1 + \text{ div } G_2, \mathbf{\Phi}\rangle_{\Omega_1} - \langle \xi, (\nabla \mathbf{\Phi} - \pi I)\nu\rangle_{\Sigma_T}, \tag{4.5}$$

where $(\mathbf{\Phi}, \pi)$ is given by (4.3). Since $(\mathbf{\Phi}, \pi)$ satisfies (4.4), the linear form $\ell : W^{q-2,r}(\Omega_1)^2 \times \mathcal{H} \to \mathbb{R}$ is continuous, and we set the following definition.

DEFINITION 4.1 (a very weak variational formulation for the Stokes problem (4.1)). *$\{\mathbf{b}, P\}$ is a very weak solution of the problem* (4.1) *if*

$$\{\mathbf{b}, P\} \in W^{2-q,r/(r-1)}(\Omega_1)^2 \times \mathcal{H}^* \tag{4.6}$$

and satisfies

$$\langle \mathbf{g}, \mathbf{b}\rangle_{\Omega_1} - \langle P, s\rangle_{\mathcal{H}^*,\mathcal{H}} = \ell(\mathbf{g}, s) \qquad \forall \mathbf{g} \in L^r(\Omega_1)^2,\ \forall s \in \mathcal{H}. \tag{4.7}$$

Because of the linearity and continuity of ℓ, Riesz's theorem implies the following result.

PROPOSITION 4.2. *Let $1 < r < +\infty$, $1 + 1/r < q \leq 2$, and $\langle \xi_2, 1\rangle_{\Sigma_T} = 0$. Then, there exists a unique very weak solution $\{\mathbf{b}, P\}$ for* (4.1). *It satisfies the following estimates:*

$$\|\mathbf{b}\|_{W^{2-q,r/(r-1)}(\Omega_1)^2} \leq c\Big\{\|\mathbf{G}_1\|_{L^1(\Omega_1)^2} + \|G_2\|_{W^{1-q,r/(r-1)}(\Omega_1)^4} + \|\xi\|_{W^{1+1/r-q,r/(r-1)}(\Sigma_T)^2}\Big\}. \tag{4.8}$$

Another approach is to use the result from the article [10] directly, which reads as the following result.

PROPOSITION 4.3. *Let $\mathbf{G}_1 = 0$ and $G_2 = 0$. Then for $\xi \in L^2(\Sigma_T)$, $\int_{\Sigma_T} \xi_2 = 0$, there exists a unique very weak solution $\{\mathbf{b}, P\}$ of* (4.1) *satisfying the following estimates:*

$$\|\mathbf{b}\|_{H^{1/2}(\Omega_1)^2} \leq c\|\xi\|_{L^2(\Sigma_T)^2}. \tag{4.9}$$

Furthermore,

$$\||x_2|^{1/2}\nabla \mathbf{b}\|_{L^2(\Omega_1)^2} + \||x_2|^{1/2}\pi\|_{L^2(\Omega_1)^2} \leq c\|\xi\|_{L^2(\Sigma_T)^2}. \tag{4.10}$$

5. Appendix 2: Navier's boundary layer and compressibility corrections. In this appendix, for completeness of the paper, we recall the derivation of Navier's boundary layer developed in [12], [13] and also presented in [15].

As observed in hydrology, the phenomena relevant to the boundary occur in a thin layer surrounding the interface between a porous medium and a free flow. In this appendix we are going to present a sketch of the construction of the main boundary layer, used for determining the coefficient α in (1.5) and the coefficient C^{bl}_ω in the interface pressure jump law (2.18). Since the law by Beavers and Joseph is an example of the Navier slip condition, we call it *Navier's boundary layer*.

In addition to the notation from subsection 2.1, we introduce the interface $S = (0,1) \times \{0\}$, the free fluid slab $Z^+ = (0,1) \times (0,+\infty)$, and the semi-infinite porous slab $Z^- = \cup_{k=1}^{\infty}(Y_F - \{0,k\})$. The flow region is then $Z_{BL} = Z^+ \cup S \cup Z^-$.

We consider the following problem.

Find $\{\beta^{bl}, \omega^{bl}\}$ with square-integrable gradients satisfying

$$-\triangle_y \beta^{bl} + \nabla_y \omega^{bl} = 0 \qquad \text{in } Z^+ \cup Z^-, \tag{5.1}$$

$$\mathrm{div}_y \beta^{bl} = 0 \qquad \text{in } Z^+ \cup Z^-, \tag{5.2}$$

$$\big[\beta^{bl}\big]_S(\cdot,0) = 0 \quad \text{and} \quad \big[\{\nabla_y \beta^{bl} - \omega^{bl} I\}\mathbf{e}^2\big]_S(\cdot,0) = \mathbf{e}^1 \quad \text{on } S, \tag{5.3}$$

$$\beta^{bl} = 0 \quad \text{on } \bigcup_{k=1}^{\infty}(\partial Y_s - \{0,k\}), \qquad \{\beta^{bl}, \omega^{bl}\} \quad \text{is 1-periodic in } y_1. \tag{5.4}$$

By the Lax–Milgram lemma, there exists a unique $\beta^{bl} \in L^2_{loc}(Z_{BL})^2$, $\nabla_y z \in L^2(Z_{BL})^4$ satisfying (5.1)–(5.4) and $\omega^{bl} \in L^2_{loc}(Z^+ \cup Z^-)$, which is unique up to a constant and satisfying (5.1). We note that due to the incompressibility and the continuity of β^{bl} on S, considering $\nabla \beta^{bl}$ or the symmetrized gradient $(\nabla + \nabla^t)\beta^{bl}$ is equivalent.

The goal of this subsection is to show that system (5.1)–(5.4) describes a boundary layer, i.e., that β^{bl} and ω^{bl} stabilize exponentially towards constants, when $|y_2| \to \infty$. Since we are studying an incompressible flow, it is useful to prove properties of the conserved averages.

LEMMA 5.1 (see [12]). *Any solution $\{\beta^{bl}, \omega^{bl}\}$ satisfies*

$$\int_0^1 \beta_2^{bl}(y_1, b)\ dy_1 = 0 \qquad \forall b \in \mathbb{R} \tag{5.5}$$

$$\text{and} \quad \int_0^1 \omega^{bl}(y_1, b_1)\ dy_1 = \int_0^1 \omega^{bl}(y_1, b_2)\ dy_1 \qquad \forall b_1 > b_2 \geq 0,$$

(5.6)
$$\int_0^1 \beta_1^{bl}(y_1, b_1)\ dy_1 = \int_0^1 \beta_1^{bl}(y_1, b_2)\ dy_1 = -\int_{Z_{BL}} |\nabla \beta^{bl}(y)|^2\ dy \qquad \forall b_1 > b_2 \geq 0.$$

PROPOSITION 5.2 (see [12]). *Let*

$$C_1^{bl} = \int_0^1 \beta_1^{bl}(y_1, 0) dy_1. \tag{5.7}$$

Then, for every $y_2 \geq 0$ and $y_1 \in (0,1)$,

$$|\beta^{bl}(y_1, y_2) - (C_1^{bl}, 0)| \leq Ce^{-\delta y_2} \qquad \forall \delta < 2\pi. \tag{5.8}$$

COROLLARY 5.3 (see [12]). *Let*

$$C_\omega^{bl} = \int_0^1 \omega^{bl}(y_1, 0)\, dy_1. \tag{5.9}$$

Then, for every $y_2 \geq 0$ and $y_1 \in (0,1)$, we have

$$|\ \omega^{bl}(y_1, y_2) - C_\omega^{bl}\ | \leq e^{-2\pi y_2}. \tag{5.10}$$

In the last step we study the decay of β^{bl} and ω^{bl} in the semi-infinite porous slab Z^-.

PROPOSITION 5.4 (see [12, pages 411–412]). *Let β^{bl} and ω^{bl} be defined by (5.1)–(5.4). Then, there exist positive constants C and γ_0 such that*

$$|\beta^{bl}(y_1, y_2)| + |\nabla \beta^{bl}(y_1, y_2)| \leq Ce^{-\gamma_0 |y_2|} \qquad \text{for every} \quad (y_1, y_2) \in Z^-. \tag{5.11}$$

Furthermore, the limit $\kappa_\infty = \lim_{k \to -\infty} \frac{1}{|Y_F|} \int_{Z_k} \omega^{bl}(y)\, dy$ exists and it holds that

$$|\omega^{bl}(y_1, y_2) - \kappa_\infty| \leq Ce^{-\gamma_0 |y_2|} \qquad \text{for every} \quad (y_1, y_2) \in Z^-. \tag{5.12}$$

REMARK 5.5. *Without losing generality, we take $\kappa_\infty = 0$. If the geometry of Z^- is axially symmetric with respect to reflections around the axis $y_1 = 1/2$, then $C_\omega^{bl} = 0$. For the proof, we refer the reader to [14]. In [14] a detailed numerical analysis of the problem (5.1)–(5.4) is given. Through numerical experiments it is shown that for a general geometry of Z^-, $C_\omega^{bl} \neq 0$.*

It is important to be sure that the law by Beavers and Joseph does not depend on the position of the interface. We have the following result.

LEMMA 5.6. *Let $a < 0$, and let $\beta^{a,bl}$ be the solution of* (5.1)–(5.4) *with S replaced by $S_a = (0,1) \times \{a\}$, with Z^+ replaced by $Z_a^+ = (0,1) \times (a, +\infty)$, and where $Z_a^- = Z_{BL} \setminus (S_a \cup Z_a^+)$. Then, it holds that*

$$C_1^{a,bl} = C_1^{bl} - a. \tag{5.13}$$

This simple result implies the invariance of the obtained law on the position of the interface. It is in agreement with the law of Saffman for the slip coefficient formulated in [24]. The law was confirmed numerically by Sahraoui and Kaviany in [25]. For more discussion, we refer the reader to [16, formulas (2.193)–(2.195), page 74 and Figure 2.22 and formula (2.211), page 81].

The remainder of the section is devoted to auxiliary functions correcting the compressibility effects. We define $\mathbf{Q}^{bl}$ by

$$\operatorname{div}_y \mathbf{Q}^{bl}(y) = \beta_1^{bl}(y) - C_1^{bl} H(y_2) \quad \text{in } \ Z^+ \cup Z^-, \tag{5.14}$$

$$\mathbf{Q}^{bl} = 0 \text{ on } \bigcup_{k=1}^{\infty} (\partial Y_s - \{0, k\}), \quad \mathbf{Q}^{bl} \text{ is 1-periodic in } \ y_1, \tag{5.15}$$

$$[\mathbf{Q}^{bl}]_S = \mathbf{e}^2 \int_{Z_{BL}} (C_1^{bl} H(y_2) - \beta_1^{bl}(y)) \, dy = -\mathbf{e}^2 \int_{Z^-} \beta_1^{bl}(y) \, dy. \tag{5.16}$$

PROPOSITION 5.7 (see [12, page 411]). *Problem* (5.14)–(5.16) *has at least one solution $\mathbf{Q}^{bl} \in H^1(Z^+ \cup Z^-)^2 \cap C^\infty_{loc}(Z^+ \cup Z^-)^2$. Furthermore, $\mathbf{Q}^{bl} \in W^{1,q}(Z^+)^2$, $\mathbf{Q}^{bl} \in W^{1,q}(Z^-)^2$ for all $q \in [1, +\infty)$ and there exists $\gamma_0 > 0$ such that*

$$e^{\gamma_0 y_3} \mathbf{Q}^{bl} \in H^1(Z^+ \cup Z^-)^2. \tag{5.17}$$

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