Effective Laws for the Poisson Equation on Domains with Curved Oscillating Boundaries

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Abstract

In this article, we derive approximations and effective boundary laws for solutions u^ε of the Poisson equation on a domain $\Omega^\varepsilon \subset \mathbb{R}^n$ whose boundary differs from the smooth boundary of a domain $\Omega \subset \mathbb{R}^n$ by rapid oscillations of size ε. First, we construct a boundary layer correction which yields an $O(\varepsilon^{3/2})$ approximation in the energy norm, and an $O(\varepsilon^2)$ approximation in the L^2-norm if Ω is bounded. Then, we show that for $1 \le p \le 2$ an $O(\varepsilon^{1+1/p})$-approximation in the L^p-norm can already be obtained by solving an effective equation on Ω satisfying a boundary condition of Robin type.

Key Words: Homogenization, Poisson equation, oscillating boundary, curved boundary, Robin boundary condition, boundary layer, unbounded domain.

1 Introduction

We consider the Poisson equation

$$-\Delta u^\varepsilon(x) = f(x)\,,\ x \in \Omega^\varepsilon\,, \qquad u^\varepsilon(x) = 0\,,\ x \in \partial\Omega^\varepsilon \tag{1}$$

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where $\Omega^\varepsilon \subset \mathbb{R}^n$ with the additional assumption that u^ε is bounded in the case where Ω^ε is unbounded. Ω^ε is a domain with a compact boundary $\partial\Omega^\varepsilon$ having microscopic locally ε-periodic oscillations of size ε. For $\varepsilon \to 0$, the domain Ω^ε is supposed to approximate a domain Ω with smooth boundary $\Gamma = \partial\Omega$. It is clear that for small ε, this problem is difficult to solve numerically because of the intricate structure of the boundary. Therefore it is important to find a way to approximate u^ε by solving only problems on Ω. The boundary law of these problems will depend on the oscillation and can be computed by solving locally a so-called boundary layer problem.

The study of problem (1) in the case of $\partial\Omega$ being a hyperplane has a long history, see e.g. [8], [13], [4], [3] and the references therein. Studies in the case of curved boundaries $\partial\Omega$ are [1], [2], and [9]. Here, [1] treats only the two-dimensional case with uniform oscillations. However, the two-dimensional case is very special because it allows for a global isometric parameterization of the boundary, while in the multidimensional case even the correct formulation of the problem setting is not obvious, see Section 2. Also the references [2] and [9] consider a setting in the case $\Omega \subset \mathbb{R}^2$, but they allow variable oscillations. They state an $O(\varepsilon^{\frac{3}{2}})$ error estimate in the energy norm, but they do not contain proofs for this assertion. In [9], the boundary layer cell problem is posed on a domain with a fixed thickness. However, this will usually not be sufficient for obtaining the desired $O(\varepsilon^{\frac{3}{2}})$-error estimate because it introduces an $O(\varepsilon)$-error in the effective boundary condition, see Remark 4.4.

The structure of this article is the following. Since the formulation of the problems as well as the results do slightly differ, we have chosen to consider the case of bounded domains Ω and Ω^ε first, and discuss the case of unbounded domains separately in Section 9. In Section 2, we define the problem and especially the kind of ε-periodicity we are interested in. Section 3 considers the simple approximation of u^ε given by the solution u of

$$-\Delta u(x) = f(x)\,,\ x \in \Omega\,, \qquad u(x) = 0\,,\ x \in \partial\Omega\,. \tag{2}$$

The difference between u^ε and u is shown to be of size $O(\varepsilon^{\frac{1}{2}})$ in the energy norm. Next, in Sections 4, 5, 6, we construct a better approximation which involves the solution of a boundary layer problem. This approximation is shown to yield an $O(\varepsilon^{\frac{3}{2}})$-error in the energy norm and an $O(\varepsilon^2)$-error in the L^2-norm in Section 7. Finally, in Section 8, we show that interior $O(\varepsilon^2)$-approximations can also be obtained by solving the problem

$$-\Delta u^{\mathit{eff}}(x) = f(x)\,,\ x \in \Omega\,, \qquad u^{\mathit{eff}}(x) = \varepsilon c^{bl}(x)\frac{\partial u^{\mathit{eff}}}{\partial \nu}(x)\,, \qquad x \in \partial\Omega \tag{3}$$

where the function $c^{bl} : \partial\Omega \to \mathbb{R}$ can be computed from the solution to the boundary layer problem mentioned above.

2 Setting of the problem

Let $\Omega \subset \mathbb{R}^n\,,\ n \geq 2$ be a bounded domain, such that its boundary $\Gamma = \partial\Omega$ is a compact, smooth $(n-1)$-dimensional Riemannian manifold with a metric induced by the Euclidean metric in $\mathbb{R}^n$.

Let $\nu : \Gamma \to \mathbb{R}^n$ be the outer normal vector field of Γ. A standard result of differential geometry (see e.g. [5]) then implies that for a suitable choice of $\delta > 0$ the mapping

$$\mathcal{T} : \Gamma \times (-\delta, \delta) \to \mathcal{T}^\Gamma \subset \mathbb{R}^n\,, \qquad (x,t) \mapsto x + t\nu(x) \tag{4}$$

is a smooth diffeomorphism. Such a mapping $\mathcal{T}$ is called a *tubular neighborhood* of Γ.

For $\varepsilon > 0$, let

$$\gamma^\varepsilon :\, \Gamma \to \mathbb{R} \tag{5}$$

be a function which satisfies

$$|\gamma^\varepsilon(x)| \leq \varepsilon M < \frac{\delta}{2}\,,\ x \in \Gamma\,, \tag{6}$$

and which is locally ε-periodic in the following sense: there is an atlas $\mathcal{A}_\Gamma = \{\varphi_i\}_{i=1,\dots,N}$ of Γ consisting of charts

$$U_i \ni \mathbf{x}' \mapsto x = \varphi_i(\mathbf{x}') \in V_i \subset \Gamma \tag{7}$$

mapping open sets $U_i \subset \mathbb{R}^{n-1}$ to open sets $V_i \subset \Gamma$ such that

$$\gamma^\varepsilon(x) = \varepsilon\gamma_i(\varphi_i^{-1}(x), \frac{\varphi_i^{-1}(x)}{\varepsilon}) \tag{8}$$

with smooth functions $\gamma_i : U_i \times \mathbb{R}^{n-1} \to \mathbb{R}$ which are 1-periodic in the second variable.

The inequalities (6) ensure that γ^ε defines an oscillating surface Γ^ε by

$$\Gamma^\varepsilon = \{\mathcal{T}(x, \gamma^\varepsilon(x)) : x \in \Gamma\} \tag{9}$$

which is a submanifold of $\mathbb{R}^n$ bounding the domain

$$\Omega^\varepsilon = (\Omega \setminus \mathcal{T}_\Gamma) \cup \{\mathcal{T}(x,t) :\, x \in \Gamma,\, -\delta < t < \gamma^\varepsilon(x)\}\,. \tag{10}$$

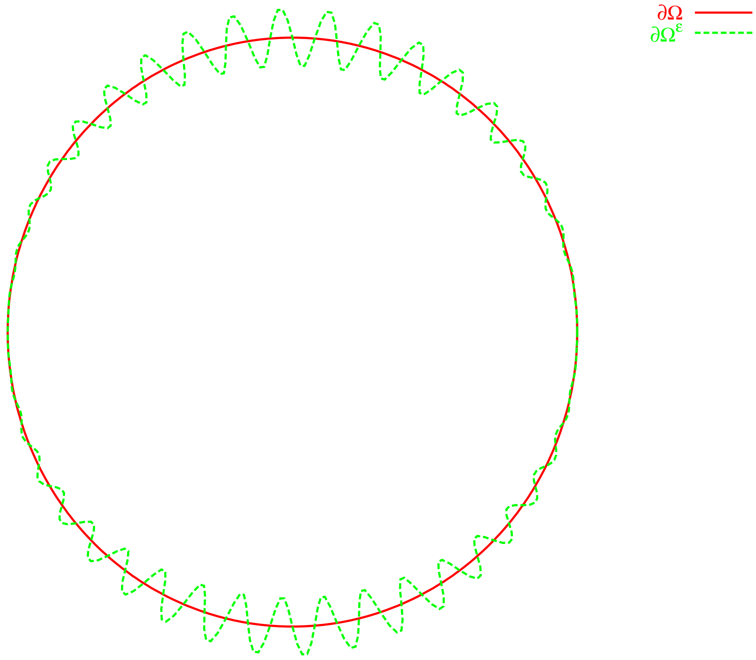

Figure 1: Ω and Ω^ε

For later use, we also define local charts for the tubular neighborhood by

$$\mathcal{T}_i : U_i \times (-\delta, \delta) \to \mathcal{T}_i^\Gamma\,, \qquad (\mathbf{x}', t) \mapsto \mathcal{T}(\varphi_i(\mathbf{x}'), t)\,. \tag{11}$$

Example 2.1 *A simple example of a domain with a smoothly varying oscillating boundary is shown in Fig. 1. Here,* $\Gamma = \partial\Omega$ *is the unit circle* S^1*. The tubular neighborhood (4) is given as*

$$\mathcal{T} : S^1 \times (-\delta, \delta) \to \mathbb{R}^2\,, \qquad (x, t) \mapsto (1 + t)x \tag{12}$$

for some $\delta \in (0, 1)$*. The oscillating boundary is given in this special case as*

$$\Gamma^\varepsilon = \{\mathcal{T}(x, \gamma^\varepsilon(x)) : x \in S^1\}$$

with

$$\gamma^\varepsilon(x) = \frac{1}{20}\,\sin^2(\arg x)\,\sin\bigl(\frac{\arg x}{\varepsilon}\bigr)\,, \qquad \varepsilon = \frac{1}{40}\,.$$

Here, $\arg : S^1 \to (-\pi, \pi]$ *is defined such that it maps* $x = \left(\begin{smallmatrix}\cos\varphi\\ \sin\varphi\end{smallmatrix}\right) \in S^1$ *to* $\varphi \in (-\pi, \pi]$*.*

In this case, we can choose for example the atlas consisting of the local charts

$$\varphi_1 : (-1, 1) \to S^1 - \left(\begin{smallmatrix}-1\\ 0\end{smallmatrix}\right)\,, \qquad \mathbf{x}' \mapsto \begin{pmatrix}\cos(2\pi\mathbf{x}')\\ \sin(2\pi\mathbf{x}')\end{pmatrix} \tag{13}$$

and

$$\varphi_2 : (0, 2) \to S^1 - \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)\,, \qquad \mathbf{x} \mapsto \begin{pmatrix}\cos(2\pi\mathbf{x}')\\ \sin(2\pi\mathbf{x}')\end{pmatrix}\,. \tag{14}$$

With respect to these charts and for $\varepsilon = \frac{1}{40}$ *we have*

$$\gamma^\varepsilon(\varphi_i(\mathbf{x}')) = \varepsilon\gamma_i(\mathbf{x}', \frac{\mathbf{x}'}{\varepsilon})\,, \qquad i = 1, 2$$

where

$$\gamma_i(\mathbf{x}', \mathbf{y}') = 2\,\sin^2(2\pi\mathbf{x}')\,\sin\bigl(2\pi\mathbf{y}'\bigr)\,, \qquad i = 1, 2\,.$$

Example 2.2 *For* $\Omega, \Omega^\varepsilon \subset \mathbb{R}^n$ *with* $n \geq 3$*, the oscillation usually occurs only on some part of the boundary. A simple example for such a situation would be that* $\gamma_j \equiv 0$ *for all* $j = 1, \ldots, N$ *with* $j \neq i$*, while* $\gamma_i(x, y)$ *has to vanish only for* $x \in \bigcup_{j\neq i} \varphi_i^{-1}(V_i \cap V_j)$*.*

Now, let $f \in L^\infty(\mathbb{R}^n)$. We consider the problem

$$\begin{aligned} -\Delta u^\varepsilon &= f\,, \qquad x \in \Omega^\varepsilon\,, \\ u(x) &= 0\,, \qquad x \in \Gamma^\varepsilon\,. \end{aligned} \tag{15}$$

The variational formulation of this problem is: find $u^\varepsilon \in H_0^1(\Omega^\varepsilon)$ such that

$$\int_{\Omega^\varepsilon} \nabla u^\varepsilon \nabla \varphi^\varepsilon \, dx = \int_{\Omega^\varepsilon} f \varphi^\varepsilon \, dx\,, \qquad \forall \varphi^\varepsilon \in H_0^1(\Omega^\varepsilon)\,. \tag{16}$$

By the Lax-Milgram lemma, this problem has a unique solution.

3 First approximation

The solution u^ε from (15) can be approximated by the solution u of the problem

$$\begin{aligned} -\Delta u(x) &= f(x)\,, \qquad x \in \Omega\,, \\ u(x) &= 0\,, \qquad x \in \Gamma\,. \end{aligned} \tag{17}$$

We extend u to Ω^ε by 0 and denote this extension by $\widetilde{u}$. We are interested how well $\widetilde{u}$ approximates u^ε in $H^1(\Omega^\varepsilon)$ resp. $L^2(\Omega^\varepsilon)$. For proving error estimates, we introduce the corrector $\theta^\varepsilon : \Omega^\varepsilon \to \mathbb{R}$ as a solution to the problem

$$\begin{aligned} -\Delta \theta^\varepsilon(x) &= 0\,, \qquad x \in \Omega^\varepsilon \setminus \Gamma\,, \\ \theta^\varepsilon(x) &= 0\,, \qquad x \in \Gamma^\varepsilon \setminus \Omega\,, \\ \theta^\varepsilon(x) &= -u(x)\,, \qquad x \in \Gamma^\varepsilon \cap \Omega\,, \\ \left[\tfrac{\partial}{\partial \nu}\theta^\varepsilon\right](x) &= -\left[\tfrac{\partial}{\partial \nu}\widetilde{u}\right](x) = \tfrac{\partial}{\partial \nu}u(x)\,, \qquad x \in \Gamma \cap \Omega^\varepsilon\,, \end{aligned} \tag{18}$$

where the brackets denote the jump across the interface Γ:

$$[\varphi](x) = \lim_{s \downarrow 0} \left(\varphi(x + s\nu(x)) - \varphi(x - s\nu(x))\right)\,. \tag{19}$$

The variational form of this problem is: find $\theta^\varepsilon \in H^1(\Omega^\varepsilon)$ satisfying the boundary conditions from (18) on Γ^ε such that

$$\int_{\Omega^\varepsilon} \nabla \theta^\varepsilon \nabla \varphi \, dx = -\int_{\Gamma \cap \Omega^\varepsilon} \left[\tfrac{\partial}{\partial \nu}\theta^\varepsilon\right] \varphi \, ds = -\int_{\Gamma \cap \Omega^\varepsilon} \tfrac{\partial}{\partial \nu} u \varphi \, ds\,, \qquad \forall \varphi \in H_0^1(\Omega^\varepsilon)\,. \tag{20}$$

Then we can state

Theorem 3.1 *Let u^ε, u, $\widetilde{u}$, θ^ε be as defined above. Then we have*

$$\|\nabla(u^\varepsilon - \widetilde{u} - \theta^\varepsilon)\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^{\frac{3}{2}} \|f\|_{L^\infty(\Omega^\varepsilon\setminus\Omega)} \tag{21}$$

and

$$\|u^\varepsilon - \widetilde{u} - \theta^\varepsilon\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^2 \|f\|_{L^\infty(\Omega^\varepsilon\setminus\Omega)}\,. \tag{22}$$

Proof: For all $\varphi \in H^1_0(\Omega^\varepsilon)$, we have

$$\int_{\Omega^\varepsilon} \nabla(u^\varepsilon - \widetilde{u} - \theta^\varepsilon)\nabla\varphi\, dx = \int_{\Omega^\varepsilon\setminus\Omega} f\varphi\, dx\,. \tag{23}$$

Because of Hölder- and Poincaré inequality, we have

$$\begin{aligned}\int_{\Omega^\varepsilon\setminus\Omega} f\varphi\, dx \leq \|f\|_{L^\infty(\Omega^\varepsilon\setminus\Omega)}\|\varphi\|_{L^1(\Omega^\varepsilon\setminus\Omega)} &\lesssim \varepsilon^{\frac{1}{2}}\|f\|_{L^\infty(\Omega^\varepsilon\setminus\Omega)}\|\varphi\|_{L^2(\Omega^\varepsilon\setminus\Omega)} \\ &\lesssim \varepsilon^{\frac{3}{2}}\|f\|_{L^\infty(\Omega^\varepsilon\setminus\Omega)}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon\setminus\Omega)} \end{aligned}\tag{24}$$

Setting $\varphi = u^\varepsilon - \widetilde{u} - \theta^\varepsilon \in H^1_0(\Omega^\varepsilon)$ yields (21).

For proving (22), we set

$$\Omega' = \{x \in \Omega : d(x,\Gamma) > M\varepsilon\}\,. \tag{25}$$

We then note that $v = u^\varepsilon - \widetilde{u} - \theta^\varepsilon$ satisfies

$$\|v\|_{L^2(\Omega^\varepsilon\setminus\Omega')} \lesssim \varepsilon\|\nabla v\|_{L^2(\Omega^\varepsilon\setminus\Omega')} \lesssim \varepsilon^{\frac{5}{2}} \tag{26}$$

due to Poincaré's inequality. Furthermore, a standard trace estimate yields

$$\|v\|_{L^2(\partial\Omega')} \lesssim \varepsilon^{\frac{1}{2}}\|\nabla v\|_{L^2(\Omega^\varepsilon\setminus\Omega')} \lesssim \varepsilon^2\,.$$

Now we can apply Theorem 3.2 (in the special case $\Delta v = 0$) to obtain

$$\|v\|_{L^2(\Omega')} \lesssim \|v\|_{L^2(\partial\Omega')} \lesssim \varepsilon^2\,. \tag{27}$$

(22) now follows by combining (26) and (27). □

Theorem 3.2 *Let Ω' be as defined in (25). Let $v \in C^2(\Omega') \cap C^0(\bar{\Omega}')$ satisfy* $\mathrm{supp}(\Delta v) \subset \mathcal{T}_\Gamma \cap \Omega'$ *together with*

$$\|v\|_{L^2(\partial\Omega')} \lesssim \varepsilon^2 \tag{28}$$

and

$$|\Delta v(x)| \lesssim e^{-\lambda\frac{d(x,\partial\Omega')}{\varepsilon}}\,,\ x \in \Omega'\,. \tag{29}$$

Then

$$\|v\|_{L^2(\Omega')} \lesssim \varepsilon^2\,. \tag{30}$$

Proof: In the case $\Delta v \equiv 0$, (30) is a special case of the very weak estimates for the Laplace equation, see e.g. [12], Chap. 5, (1.21). For proving (30) under the assumption (29), let $w \in H_0^1(\Omega')$ be the solution of the problem

$$\begin{aligned} -\Delta w(x) = v(x)\,, \quad & x \in \Omega'\,, \\ w(x) = 0\,, \quad & x \in \partial\Omega'\,. \end{aligned} \tag{31}$$

First, we note that by H^2-regularity of the problem (31), we have $\|w\|_{H^2(\Omega')} \lesssim \|v\|_{L^2(\Omega')}$. Multiplying now (31) by v and integrating over Ω' we obtain

$$\int_{\Omega'} v^2\,dx = -\int_{\Omega'} v\Delta w\,dx = \int_{\Omega'} \nabla w \nabla v\,dx - \int_{\partial\Omega'} \frac{\partial w}{\partial \nu} v\,ds\,.$$

Here, the second term on the right-hand side can be estimated as

$$\begin{aligned} \Big|\int_{\partial\Omega'} \frac{\partial w}{\partial \nu} v\,ds\Big| &\lesssim (\|\nabla w\|_{L^2(\Omega')} + \|\nabla^2 w\|_{L^2(\Omega')})\|v\|_{L^2(\partial\Omega')} \\ &\lesssim \|v\|_{L^2(\Omega')}\|v\|_{L^2(\partial\Omega')}\,. \end{aligned}$$

Thus, together with (28) this term can be estimated as desired. For estimating the first term $\int_{\Omega'} \nabla w \nabla v\,dx$, we set, for $t < 0$,

$$S(t) = \mathcal{T}(\Gamma \times \{t\}) = \{x \in \Omega' :\ d(x, \Gamma) = -t\}$$

and obtain by partial integration and Fubini's theorem

$$\Big|\int_{\Omega'} \nabla w \nabla v\,dx\Big| = \Big|\int_{\Omega'} w\Delta v\,dx\Big| \le \int_{-\delta}^{-M\varepsilon} \|w\Delta v\|_{L^1(S(t))}\,dt \lesssim \int_{-\delta}^{-M\varepsilon} \|w\|_{L^2(S(t))} e^{\lambda\frac{t}{\varepsilon}}\,dt$$

The function

$$w \circ \mathcal{T} \in H^2(\Gamma \times (-\delta, -M\varepsilon)) \subset H^2((-\delta, -M\varepsilon), L^2(\Gamma)) \subset C^1((-\delta, -M\varepsilon), L^2(\Gamma))$$

satisfies $(w \circ \mathcal{T})(\cdot, -M\varepsilon) \equiv 0$, so that we obtain the estimate

$$\|(w \circ \mathcal{T})(\cdot, t)\|_{L^2(\Gamma)} \lesssim \|w \circ \mathcal{T}\|_{H^2(\Gamma \times (-\delta, -M\varepsilon))} |t + M\varepsilon|$$

and using a standard transformation formula also

$$\|w\|_{L^2(S(t))} \lesssim \|w\|_{H^2(\Omega')} |t + M\varepsilon|\,.$$

Therefore, we get

$$\int_{-\delta}^{-M\varepsilon} \|w\|_{L^2(S(t))} e^{\lambda \frac{t}{\varepsilon}}\, dt \lesssim \|w\|_{H^2(\Omega')} \int_{-\delta}^{-M\varepsilon} |t + M\varepsilon| e^{\lambda \frac{t}{\varepsilon}}\, dt \lesssim \varepsilon^2 \|v\|_{L^2(\Omega')} .$$

Thus, (30) is proved. □

The following theorem then gives an estimate for the size of θ^ε which in turn yields an estimate for the difference $u^\varepsilon - \widetilde{u}$.

Theorem 3.3 *Let u^ε, u, $\widetilde{u}$, θ^ε be as defined above. Then*

$$\|\nabla \theta^\varepsilon\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^{\frac{1}{2}} \left(\|u\|_{W^{1,\infty}(\Omega)} + \|f\|_{L^\infty(\Omega)} \right) \tag{32}$$

which implies also

$$\|\nabla (u^\varepsilon - \widetilde{u})\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^{\frac{1}{2}} \left(\|u\|_{W^{1,\infty}(\Omega)} + \|f\|_{L^\infty(\Omega)} \right) . \tag{33}$$

Proof: Let $\psi : \mathbb{R}^n \to [0,1]$ be a smooth cut-off function satisfying $\psi(x) = 1$ for $d(x,\Gamma) \leq M\varepsilon$, $\psi(x) = 0$ on $\Omega^\varepsilon \setminus S$ where

$$S = \{ x \in \Omega : d(x,\Gamma) < 2M\varepsilon \} , \tag{34}$$

and $\|\nabla\psi\|_\infty \lesssim \varepsilon^{-1}$, $\|\Delta\psi\|_\infty \lesssim \varepsilon^{-2}$. Suitable application of Poincaré's inequality on S yields for all $\varphi \in H_0^1(\Omega^\varepsilon)$

$$\begin{aligned}
&\int_{\Omega^\varepsilon} \nabla(\theta^\varepsilon + \psi\widetilde{u}) \nabla\varphi\, dx \\
&= -\int_{\Omega^\varepsilon \setminus \Gamma} \Delta(\psi\widetilde{u})\varphi\, dx \\
&= -\int_{\Omega^\varepsilon \setminus \Gamma} (\Delta\psi)\widetilde{u}\varphi\, dx - 2\int_{\Omega^\varepsilon \setminus \Gamma} \nabla\psi\nabla\widetilde{u}\varphi\, dx - \int_{\Omega^\varepsilon \setminus \Gamma} \psi f \varphi\, dx \\
&\lesssim \varepsilon^{\frac{1}{2}} \left(\|\nabla\widetilde{u}\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} \right) \|\nabla\varphi\|_{L^2(\Omega^\varepsilon)} .
\end{aligned}$$

Setting $\varphi = \theta^\varepsilon + \psi\widetilde{u}$ we obtain $\|\nabla(\theta^\varepsilon + \psi\widetilde{u})\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^{\frac{1}{2}}$, and because of

$$\|\nabla(\psi\widetilde{u})\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^{\frac{1}{2}} \|\nabla\widetilde{u}\|_{L^\infty(\Omega)}$$

the assertion follows by applying the triangle inequality. □

Remark 3.4

1. *θ^ε is the solution of a problem posed on Ω^ε, and is therefore as difficult to compute as the solution u^ε itself. The following sections construct an approximation which is easier to calculate.*

2. *In general, the estimate (32) is optimal. This is also a side-result of the more explicit approximation constructed below.*

4 The boundary layer

Boundary correctors for domains with smooth curved boundaries in the case of elliptic problems with rapidly oscillating coefficients have been defined in [11] by using boundary layers parametrized by the boundary parameters. Using similar ideas we define boundary layers β_i in the coordinate charts of the atlas $\mathcal{A}$ as follows.

Let $\varphi_i : U_i \to V_i \subset \Gamma$ be a chart in the atlas $\mathcal{A}$ of Γ and let $E_i = U_i \times \mathbb{R}^n$. We set

$$A_i(\mathbf{x}') = \begin{pmatrix} ((D\varphi_i(\mathbf{x}'))^t(D\varphi_i(\mathbf{x}')))^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{x}' \in U_i \tag{35}$$

which is the $n \times n$-matrix corresponding to the Laplace operator in the chart $\mathcal{T}_i$ of the tubular neighborhood.

The following definition unifies the settings of [7], [10] and [3], see Remark 4.4 below.

Definition 4.1 *The boundary layers $\beta_i : E_i \to \mathbb{R}$ are defined such that $\beta_i(\mathbf{x}', \mathbf{y}) = \beta_i(\mathbf{x}', (\mathbf{y}', \mathbf{y}_n))$ solves on each fiber $E_{\mathbf{x}'} = \{\mathbf{x}'\} \times \mathbb{R}^n$ the equation*

$$\begin{aligned} -\operatorname{div}_{\mathbf{y}}\big(A_i(\mathbf{x}')\nabla_{\mathbf{y}}\beta_i(\mathbf{x}', \mathbf{y})\big) &= 0\,, \quad 0 \neq \mathbf{y}_n < \gamma_i(\mathbf{x}', \mathbf{y}')\,, \\ \beta_i(\mathbf{x}', \mathbf{y}) &\text{ is 1-periodic in } \mathbf{y}_1, \ldots, \mathbf{y}_{n-1}. \\ \left[\frac{\partial \beta_i}{\partial \mathbf{y}_n}(\mathbf{x}', \mathbf{y})\right] &= 1\,, \quad 0 = \mathbf{y}_n < \gamma_i(\mathbf{x}', \mathbf{y}')\,, \\ \beta_i(\mathbf{x}', \mathbf{y}) &= 0\,, \quad \mathbf{y}_n \geq \max(\gamma_i(\mathbf{x}', \mathbf{y}'), 0)\,, \\ \beta_i(\mathbf{x}', \mathbf{y}) &= -\mathbf{y}_n\,, \quad \gamma_i(\mathbf{x}', \mathbf{y}') \leq \mathbf{y}_n \leq 0\,, \\ |\nabla_{\mathbf{y}}\beta_i(\mathbf{x}', \mathbf{y})| &\to 0\,, \quad \mathbf{y}_n \to -\infty\,. \end{aligned} \tag{36}$$

Remark 4.2 *We note that problem (36) is member of a family of problems parametrized by the position $s \in \mathbb{R}$ where the interface condition holds,*

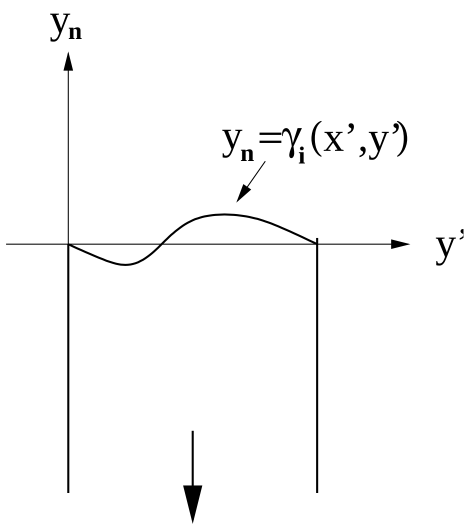

Figure 2: Boundary layer cell.

namely

$$
\begin{aligned}
-\operatorname{div}_{\mathbf{y}}\big(A_i(\mathbf{x}')\nabla_{\mathbf{y}}\beta_i^{(s)}(\mathbf{x}',\mathbf{y})\big) &= 0\,, \qquad s \neq \mathbf{y}_n < \gamma_i(\mathbf{x}',\mathbf{y}')\,,\\
\beta_i^{(s)}(\mathbf{x}',\mathbf{y}) &\text{ is 1-periodic in } \mathbf{y}_1,\dots,\mathbf{y}_{n-1}.\\
\left[\frac{\partial \beta_i^{(s)}}{\partial \mathbf{y}_n}(\mathbf{x}',\mathbf{y})\right] &= 1\,, \qquad s = \mathbf{y}_n < \gamma_i(\mathbf{x}',\mathbf{y}')\,,\\
\beta_i^{(s)}(\mathbf{x}',\mathbf{y}) &= 0\,, \qquad \mathbf{y}_n \geq \max(\gamma_i(\mathbf{x}',\mathbf{y}'),s)\,,\\
\beta_i^{(s)}(\mathbf{x}',\mathbf{y}) &= s-\mathbf{y}_n\,, \qquad s \leq \mathbf{y}_n \leq \gamma_i(\mathbf{x}',\mathbf{y}')\,,\\
|\nabla_{\mathbf{y}}\beta_i^{(s)}(\mathbf{x}',\mathbf{y})| &\to 0\,, \qquad \mathbf{y}_n \to -\infty\,.
\end{aligned}
\tag{37}
$$

For arbitrary $s_1 < s_2 \in \mathbb{R}$, the unique solvability of (37) shows that the corresponding solutions $\beta_i^{(s_1)}, \beta_i^{(s_2)}$ are related as

$$
(\beta_i^{(s_2)} - \beta_i^{(s_1)})(\mathbf{x}',\mathbf{y}) = \begin{cases} s_2 - s_1 & \mathbf{y}_n \leq s_1 \\ s_2 - \mathbf{y}_n & s_1 \leq \mathbf{y}_n \leq s_2 \\ 0 & \mathbf{y}_n \geq s_2 \end{cases}. \tag{38}
$$

Theorem 4.3 *We have*

1. *$\beta_i \in C^0(E_i) \cap C^\infty(\{(\mathbf{x}',\mathbf{y}) \in E_i :\ 0 \neq \mathbf{y}_n < \gamma_i(\mathbf{x}',\mathbf{y}')\})$.*

2. *There is some* $\lambda_i = \lambda_i(\gamma_i, A_i) > 0$ *and a function* $c_i^{bl} \in C^\infty(U_i, \mathbb{R})$ *such that*

$$|\beta_i(\mathbf{x}', (\mathbf{y}', \mathbf{y}_n)) - c_i^{bl}(\mathbf{x}')| \lesssim e^{\lambda_i \mathbf{y}_n}\,, \qquad (\mathbf{x}', (\mathbf{y}', \mathbf{y}_n)) \in E_i\,. \tag{39}$$

3. *For every* $(\mathbf{x}', \mathbf{y}) \in E_i$ *and* $\vec{k} \in \mathbb{N}^{n-1}$, $\vec{l} \in \mathbb{N}^n$ *with* $|\vec{l}| \geq 1$ *we have*

$$|D_{\mathbf{x}'}^{\vec{k}} D_{\mathbf{y}}^{\vec{l}} \beta_i(\mathbf{x}', (\mathbf{y}', \mathbf{y}_n))| \lesssim e^{\lambda_i \mathbf{y}_n}\,, \qquad 0 \neq \mathbf{y}_n < \gamma_i(\mathbf{x}', \mathbf{y}')\,. \tag{40}$$

Defining $\bar{\beta}_i : E \to \mathbb{R}$ *as*

$$\bar{\beta}_i(\mathbf{x}', (\mathbf{y}', \mathbf{y}_n)) := \beta_i(\mathbf{x}', (\mathbf{y}', \mathbf{y}_n)) - c_i^{bl}(\mathbf{x}') \tag{41}$$

we have for all $\vec{k} \in \mathbb{N}^{n-1}$, $\vec{l} \in \mathbb{N}^n$ *that*

$$|D_{\mathbf{x}'}^{\vec{k}} D_{\mathbf{y}}^{\vec{l}} \bar{\beta}_i(\mathbf{x}', (\mathbf{y}', \mathbf{y}_n))| \lesssim e^{\lambda_i \mathbf{y}_n}\,, \qquad 0 \neq \mathbf{y}_n < \gamma_i(\mathbf{x}', \mathbf{y}')\,. \tag{42}$$

The constants in (40) and (42) depend only on $\vec{k}$, $\vec{l}$, γ_i, *and* A_i.

Proof: Because of Remark 4.2, it is sufficient to consider the case where $\gamma_i(\mathbf{x}', \mathbf{y})$ is strictly positive, say $\gamma_i(\mathbf{x}', \mathbf{y}) > 1$ everywhere. For $Z' = (-\frac{1}{2}, \frac{1}{2})^{n-1}$ let $Z = Z' \times (-\infty, 1)$. We define an additional transformation

$$\begin{aligned} \Phi_i : U_i \times Z \quad &\to \quad E_i\,, \\ (\mathbf{x}', (\hat{\mathbf{y}}', \hat{\mathbf{y}}_n)) \quad &\mapsto \quad \begin{cases} (\mathbf{x}', (\hat{\mathbf{y}}', \hat{\mathbf{y}}_n)) & \hat{\mathbf{y}}_n \leq 0 \\ (\mathbf{x}', (\hat{\mathbf{y}}', \gamma_i(\mathbf{x}', \hat{\mathbf{y}}')\hat{\mathbf{y}}_n)) & \hat{\mathbf{y}}_n > 0 \end{cases}\,, \end{aligned}$$

and consider the transformed function

$$\widehat{\beta}_i : U_i \times Z \to \mathbb{R}\,, \qquad (\mathbf{x}', (\hat{\mathbf{y}}', \hat{\mathbf{y}}_n)) \mapsto \beta_i \circ \Phi_i(\mathbf{x}', (\hat{\mathbf{y}}', \hat{\mathbf{y}}_n))\,. \tag{43}$$

Because Φ_i is smooth, application of the chain rule transfers decay estimates for $\widehat{\beta}_i$, $\widehat{\beta}_i - c^{bl}$, and their derivatives immediately into the corresponding estimates (40) and (42) for β_i, $\bar{\beta}_i$ and their derivatives.

Now, $\widehat{\beta}_i(\mathbf{x}', \cdot)$ is the solution of the following problem: given the function space

$$\begin{aligned} V = \{v \in H^1_{\text{loc}}(Z) : \; & v(\hat{\mathbf{y}}', \hat{\mathbf{y}}_n) \text{ is 1-periodic in } \hat{\mathbf{y}}_1, \ldots, \hat{\mathbf{y}}_{n-1}, \\ & v(\hat{\mathbf{y}}', 1) = 0, \int_Z |\nabla v(\hat{\mathbf{y}})|^2 \, d\hat{\mathbf{y}} < \infty\} \end{aligned}$$

find $\widehat{\beta}_i(\mathbf{x}', \cdot) \in V$, such that

$$\int_Z (\nabla_{\hat{\mathbf{y}}}\varphi)^t a(\mathbf{x}', \hat{\mathbf{y}}) \nabla_{\hat{\mathbf{y}}} \widehat{\beta}_i(\mathbf{x}', \hat{\mathbf{y}})\, d\hat{\mathbf{y}} = -\int_{Z' \times \{0\}} \varphi(\hat{\mathbf{y}}) d\hat{\mathbf{y}}\,, \qquad \varphi \in V\,, \tag{44}$$

where

$$a(\mathbf{x}', \hat{\mathbf{y}}) = (\nabla_{\hat{\mathbf{y}}}\Phi_i)^{-t} A_i(\mathbf{x}') (\nabla_{\hat{\mathbf{y}}}\Phi_i)^{-1} \det(\nabla_{\hat{\mathbf{y}}}\Phi_i)\,. \tag{45}$$

First, we prove exponential decay of $\nabla_{\hat{\mathbf{y}}}\widehat{\beta}_i$ for $\hat{\mathbf{y}}_n \to -\infty$. For an arbitrary $\alpha > 0$, we set

$$W^1_\alpha = \{v \in V : e^{-\alpha \hat{\mathbf{y}}_n} \nabla v \in L^2(Z)\}\,, \ \|v\|_{W^1_\alpha} = \|e^{-\alpha \hat{\mathbf{y}}_n} \nabla v\|_{L^2(Z)}$$

and

$$W^0_\alpha = \{v \in W^1_\alpha : e^{-\alpha \hat{\mathbf{y}}_n} v \in L^2(Z)\}\,, \qquad \|v\|^2_{W^0_\alpha} = \|v\|^2_{W^1_\alpha} + \|e^{-\alpha \hat{\mathbf{y}}_n} v\|_{L^2(Z)}\,.$$

Then we observe that the right-hand side $F(\varphi) = -\int_{Z' \times \{0\}} \varphi(\hat{\mathbf{y}})\, d\hat{\mathbf{y}}$ of (44) satisfies for all $\alpha > 0$ and $\varphi \in W^0_\alpha$ the estimate

$$F(e^{2\alpha \hat{\mathbf{y}}_n} \varphi) = F(\varphi) \lesssim \|\varphi\|_{W^0_\alpha}\,. \tag{46}$$

This is precisely condition (10.37) of [8], such that Theorem 10.1 of [8] (which is based on a Lemma by Tartar) is applicable. This yields the existence of a $\lambda_i \in (0, \alpha]$ depending only on the ellipticity of the coefficient matrix a, such that (44) has a unique solution $\widehat{\beta}_i \in W^1_{\lambda_i}$. Standard results about regularity of solutions of elliptic differential equations then yield for every $\vec{l} \in \mathbb{N}\,,\ |\vec{l}| \geq 1$ the estimate

$$|D^{\vec{l}}_{\hat{\mathbf{y}}} \widehat{\beta}_i(\mathbf{x}', \hat{\mathbf{y}})| \lesssim e^{\lambda_i \hat{\mathbf{y}}_n}\,, \qquad 0 \neq \hat{\mathbf{y}}_n < 1\,, \tag{47}$$

which also implies the existence of some constant $c^{bl}_i(\mathbf{x}')$ with

$$|\widehat{\beta}_i(\mathbf{x}', \hat{\mathbf{y}}) - c^{bl}_i(\mathbf{x}')| \lesssim e^{\lambda_i \hat{\mathbf{y}}_n}\,, \qquad 0 \neq \hat{\mathbf{y}}_n < 1\,. \tag{48}$$

The negative sign of $c^{bl}_i(\mathbf{x}')$ follows from the maximum principle because the right-hand side F is negative in a generalized sense.

From (48) we obtain immediately (39), (40) and (42) for $\vec{k} = 0$.

Differentiating (44) with respect to $\mathbf{x}'$ then yields the following equation for the derivative $\zeta_i(\mathbf{x}', \hat{\mathbf{y}}) := \nabla_{\mathbf{x}'} \widehat{\beta}_i(\mathbf{x}', \hat{\mathbf{y}})$:

$$\int_Z (\nabla_{\hat{\mathbf{y}}}\varphi)^t a(\mathbf{x}', \hat{\mathbf{y}}) \nabla_{\hat{\mathbf{y}}} \zeta_i\, d\hat{\mathbf{y}} = L(\varphi)$$

with

$$L(\varphi) = \int_Z (\nabla_{\hat{\mathbf{y}}}\varphi)^t D_{\mathbf{x}'} a(\mathbf{x}',\hat{\mathbf{y}}) \nabla_{\hat{\mathbf{y}}} \widehat{\beta}_i(\mathbf{x}',\hat{\mathbf{y}}) \nabla_{\hat{\mathbf{y}}}\varphi(\hat{\mathbf{y}})\, d\hat{\mathbf{y}}$$
$$\lesssim \|\widehat{\beta}_i(\mathbf{x}',\cdot)\|_{W^1_{\lambda_i}} \|\varphi\|_{W^1_{\lambda_i}}\,, \qquad \varphi \in W^1_{\lambda_i}\,.$$

We easily see that $L(\cdot)$ again satisfies condition (46) such that the application of Theorem 10.1 from [8] yields the estimate $\|\zeta_i\|_{W^1_{\lambda_i}} \lesssim \|\widehat{\beta}_i(\mathbf{x}',\cdot)\|_{W^1_{\lambda_i}}$. Again, standard results yield pointwise estimates for $D^{\vec{l}}_{\hat{\mathbf{y}}}\zeta_i$ for $\mathbf{y}_n \neq 0$ such that (40) follows for $|\vec{k}| = 1$. Furthermore, one easily sees (e.g. by Fourier expansion in the region $\mathbf{y}_n \leq 0$), that $c_i^{bl}(\mathbf{x}')$ can be computed by

$$c_i^{bl}(\mathbf{x}') = F(\widehat{\beta}_i) = \int_{Z'} \widehat{\beta}_i(\mathbf{x}',(\mathbf{y}',0))d\mathbf{y}'\,.$$

Since F is linear and continuous in $W^1_{\lambda_i}$, we obtain the differentiability of c_i^{bl} and the estimate (42) for $|\vec{k}| = 1$.

Finally, repeated differentiation proves (40) and (42) for arbitrary $\vec{k} \in \mathbb{N}^{n-1}$. □

Remark 4.4 *For the numerical solution of (37), one will choose $s \in \mathbb{R}$ usually such that for all $\mathbf{x}' \in U_i$ and $\mathbf{y}' \in \mathbb{R}^{n-1}$ either $\gamma_i(\mathbf{x}',\mathbf{y}') \geq s$ or $\gamma_i(\mathbf{x}',\mathbf{y}') \leq s$. If this is the case already for $s = 0$, this corresponds to the settings $\Omega \subset \Omega^\varepsilon$ (see [7],[10]) or $\Omega^\varepsilon \subset \Omega$ (see [3]). A further possibility to compute β_i is to solve for the function*

$$\chi_i(\mathbf{x}',(\mathbf{y}',\mathbf{y}_n)) = \min\{\mathbf{y}_n, 0\} + \beta_i(\mathbf{x}',(\mathbf{y}',\mathbf{y}_n)) \tag{49}$$

instead which is a solution to the problem

$$\begin{aligned} -\operatorname{div}_{\mathbf{y}}\big(A_i(\mathbf{x}')\nabla_{\mathbf{y}}\chi_i(\mathbf{x}',\mathbf{y})\big) = 0\,, &\quad 0 \neq \mathbf{y}_n < \gamma_i(\mathbf{x}',\mathbf{y}')\,,\\ \chi_i(\mathbf{x}',\mathbf{y}) \text{ is 1-periodic in } &\mathbf{y}_1,\dots,\mathbf{y}_{n-1}.\\ \chi_i(\mathbf{x}',\mathbf{y}) = 0\,, &\quad \mathbf{y}_n \leq \gamma_i(\mathbf{x}',\mathbf{y}')\,,\\ |\nabla_{\mathbf{y}}(\chi_i(\mathbf{x}',\mathbf{y}) - \mathbf{y}_n)| \to 0\,, &\quad \mathbf{y}_n \to -\infty\,. \end{aligned} \tag{50}$$

Now, χ_i can be approximated well by functions χ_i^L, $L > M$, defined on the strip

$$\{(\mathbf{y}',\mathbf{y}_n) \in E'_{\mathbf{x}} : \gamma_i(\mathbf{x}',\mathbf{y}') > \mathbf{y}_n > -L\} \tag{51}$$

which are solutions to

$$
\begin{aligned}
-\operatorname{div}_{\mathbf{y}}\big(A_i(\mathbf{x}')\nabla_{\mathbf{y}}\chi_i^L(\mathbf{x}',\mathbf{y})\big) = 0\,, &\quad 0 \neq \mathbf{y}_n < \gamma_i(\mathbf{x}',\mathbf{y}')\,,\\
\chi_i^L(\mathbf{x}',\mathbf{y}) \text{ is 1-periodic in } &\mathbf{y}_1,\dots,\mathbf{y}_{n-1}.\\
\chi_i^L(\mathbf{x}',\mathbf{y}) = 0\,, &\quad \mathbf{y}_n \le \gamma_i(\mathbf{x}',\mathbf{y}')\,,\\
\frac{\partial\chi_i^L}{\partial\mathbf{y}_n}(\mathbf{x}',\mathbf{y}) = 1\,, &\quad \mathbf{y}_n = -L\,.
\end{aligned}
\tag{52}
$$

This is very similar to the cell problem suggested in [2] and [9]. However, it is important that L has to be chosen larger than $\frac{|\log\varepsilon|}{\lambda}$, because otherwise χ_i is approximated by χ_i^L only up to an ε-independent error of size $e^{-\lambda L}$.

5 Local boundary corrector

With the help of the solutions $\beta_i : U_i \times \mathbb{R}^n \to \mathbb{R}$ of the previous section and the map $\mathcal{T}_i$ from (11) we now construct a boundary corrector which is defined on

$$\Gamma^\varepsilon_{\delta,i} := \mathcal{T}_i(U_i \times (-\delta,\delta)) \cap \Omega^\varepsilon\,. \tag{53}$$

This corrector can be split in a smooth part and a rapidly oscillating part with exponential decay. The smooth part $\widetilde{c_i^{bl}} : \Gamma^\varepsilon_{\delta,i} \to \mathbb{R}$ is defined as

$$\widetilde{c_i^{bl}} \,:\, x = \mathcal{T}_i(\mathbf{x}',\mathbf{x}_n) \mapsto c_i^{bl}(\mathbf{x}') \tag{54}$$

where c_i^{bl} is the function defined in (39). The oscillating part $\widetilde{\beta_i^\varepsilon} : \Gamma^\varepsilon_{\delta,i} \to \mathbb{R}$ is defined as

$$\widetilde{\beta_i^\varepsilon} \,:\, x = \mathcal{T}_i(\mathbf{x}',\mathbf{x}_n) \mapsto \bar\beta_i^\varepsilon(\mathbf{x}',\mathbf{x}_n) \tag{55}$$

where $\bar\beta_i^\varepsilon$ is defined using $\bar\beta_i$ from (41) as

$$\bar\beta_i^\varepsilon : U_i \times \mathbb{R} \to \mathbb{R}\,, \quad (\mathbf{x}',\mathbf{x}_n) \mapsto \bar\beta_i(\mathbf{x}', \frac{\mathbf{x}'}{\varepsilon}, \frac{\mathbf{x}_n}{\varepsilon})\,. \tag{56}$$

The estimates for $\bar\beta_i$ from Theorem 4.3 yield the following estimates for $\widetilde{\beta_i^\varepsilon}$.

Theorem 5.1 *For an arbitrary choice*

$$\lambda \in (0, \min_{i=1,\dots,N} \lambda_i)\,, \tag{57}$$

we have the estimates

$$|D_x^{\vec{k}}\widetilde{\beta}_i^\varepsilon(x)| \lesssim \varepsilon^{-|\vec{k}|}e^{-\lambda\frac{d(x,\Gamma)}{\varepsilon}}\,, \qquad x \in \Gamma_{\delta,i}^\varepsilon \setminus \Gamma\,,\ \vec{k} \in \mathbb{N}^n \tag{58}$$

and

$$|\Delta\widetilde{\beta}_i^\varepsilon(x)| \lesssim \frac{1}{\varepsilon}e^{-\lambda\frac{d(x,\Gamma)}{\varepsilon}}\,, \qquad x \in \Gamma_{\delta,i}^\varepsilon \setminus \Gamma\,. \tag{59}$$

Furthermore, we have

$$\left[\frac{\partial\widetilde{\beta}_i^\varepsilon}{\partial\nu}(x)\right] = \frac{1}{\varepsilon}\,, \qquad x \in \Gamma_{\delta,i}^\varepsilon \cap \Gamma \tag{60}$$

and

$$\widetilde{\beta}_i^\varepsilon(x) = \begin{cases} -\widetilde{c_i^{bl}}(x) & x \in \Gamma_{\delta,i}^\varepsilon \cap (\partial\Omega^\varepsilon \setminus \Omega) \\ \frac{d(x,\Gamma)}{\varepsilon} - \widetilde{c_i^{bl}}(x) & x \in \Gamma_{\delta,i}^\varepsilon \cap (\partial\Omega^\varepsilon \cap \Omega) \end{cases}\,. \tag{61}$$

Proof: Using (42) and the chain rule of differentiation immediately yields

$$|D_{\mathbf{x}}^{\vec{k}}\bar{\beta}_i^\varepsilon(\mathbf{x}',\mathbf{x}_n)| \lesssim \varepsilon^{-|\vec{k}|}e^{\lambda_i\frac{\mathbf{x}_n}{\varepsilon}}\,, \qquad (\mathbf{x}',\mathbf{x}_n) \in U_i \times \mathbb{R} \tag{62}$$

and therefore also (58). Also (60) follows because $\frac{\partial\widetilde{\beta}_i^\varepsilon}{\partial\nu}(x)$ for $x \in \Gamma$ is equal to $\frac{1}{\varepsilon}\frac{\partial\bar{\beta}_i}{\partial\mathbf{y}_n}(\mathbf{x}',\frac{\mathbf{x}'}{\varepsilon},0)$ in (36). It remains to show (59). We have

$$\begin{aligned} \Delta_x\widetilde{\beta}_i^\varepsilon &= \sum_{j=1}^n \frac{\partial^2}{\partial x_j\partial x_j}\Big(\bar{\beta}_i^\varepsilon \circ \mathcal{T}_i^{-1}\Big) \\ &= \sum_{j,k=1}^n \frac{\partial}{\partial x_j}\left(\big(\frac{\partial}{\partial\mathbf{x}_k}\bar{\beta}_i^\varepsilon\big) \circ \mathcal{T}_i^{-1}\frac{\partial(\mathcal{T}_i^{-1})_k}{\partial x_j}\right) \\ &= \sum_{j,k,l=1}^n \big(\frac{\partial^2}{\partial\mathbf{x}_l\partial\mathbf{x}_k}\bar{\beta}_i^\varepsilon\big) \circ \mathcal{T}_i^{-1}\frac{\partial(\mathcal{T}_i^{-1})_k}{\partial x_j}\frac{\partial(\mathcal{T}_i^{-1})_l}{\partial x_j} \\ &\quad + \sum_{k=1}^n \big(\frac{\partial}{\partial\mathbf{x}_k}\bar{\beta}_i^\varepsilon\big) \circ \mathcal{T}_i^{-1}\Delta_x(\mathcal{T}_i^{-1})_k\,. \end{aligned}$$

Because of (62), the second summand can be estimated as desired. With respect to the first summand, we note that the smoothness of $\mathcal{T}_i$ and the equation

$$\nabla\mathcal{T}_i(\mathbf{x}',0) = \big(D\varphi_i(\mathbf{x}'),\nu(\varphi_i(\mathbf{x}'))\big) \tag{63}$$

imply that

$$
\begin{aligned}
(\nabla \mathcal{T}_i^{-1}(x))(\nabla \mathcal{T}_i^{-1}(x))^t &= \big((\nabla \mathcal{T}_i(\mathbf{x}))^t(\nabla \mathcal{T}_i(\mathbf{x}))\big)^{-1} \\
&= \big((\nabla \mathcal{T}_i(\mathbf{x}',0))^t(\nabla \mathcal{T}_i(\mathbf{x}',0))\big)^{-1} + S(\mathbf{x}',\mathbf{x}_n) \\
&= A_i(\mathbf{x}') + S(\mathbf{x}',\mathbf{x}_n)
\end{aligned}
$$

where $\|S(\mathbf{x}',\mathbf{x}_n)\|_\infty \lesssim \mathbf{x}_n$ and

$$
A_i(\mathbf{x}') = \begin{pmatrix} D\varphi_i^t(\mathbf{x}')D\varphi_i(\mathbf{x}') & 0 \\ 0^t & 1 \end{pmatrix}^{-1} \tag{64}
$$

is the matrix introduced in (35).

Therefore, we have to estimate

$$
\sum_{k,l=1}^{n} \frac{\partial^2 \bar{\beta}_i^\varepsilon}{\partial \mathbf{x}_l \partial \mathbf{x}_k}(\mathbf{x})(A_i(\mathbf{x}'))_{kl} + \sum_{k,l=1}^{n} \frac{\partial^2 \bar{\beta}_i^\varepsilon}{\partial \mathbf{x}_l \partial \mathbf{x}_k}(\mathbf{x})(S(\mathbf{x}',\mathbf{x}_n))_{kl}\,. \tag{65}
$$

Using again (62), the second summand can be estimated as

$$
\sum_{k,l=1}^{n} \frac{\partial^2 \bar{\beta}_i^\varepsilon}{\partial \mathbf{x}_l \partial \mathbf{x}_k}(\mathbf{x})(S(\mathbf{x}',\mathbf{x}_n))_{kl} \lesssim \frac{1}{\varepsilon^2}|\mathbf{x}_n| e^{\lambda_i \frac{\mathbf{x}_n}{\varepsilon}} \lesssim \frac{1}{\varepsilon} e^{\lambda \frac{\mathbf{x}_n}{\varepsilon}} \tag{66}
$$

for λ from (57).

Denoting by ξ resp. ζ the first and second variable of $\bar{\beta}_i : U_i \times \mathbb{R}^n \to \mathbb{R}$, the first summand can be written as

$$
\begin{aligned}
\sum_{k,l=1}^{n} (A_i(\mathbf{x}'))_{kl} \frac{\partial^2 \bar{\beta}_i^\varepsilon}{\partial \mathbf{x}_l \partial \mathbf{x}_k}(\mathbf{x}) &= \sum_{k,l=1}^{n} (A_i(\mathbf{x}'))_{kl} \frac{\partial^2}{\partial \mathbf{x}_l \partial \mathbf{x}_k}(\bar{\beta}_i(\mathbf{x}', (\frac{\mathbf{x}'}{\varepsilon}, \frac{\mathbf{x}_n}{\varepsilon}))) \\
&= \frac{1}{\varepsilon^2} \sum_{k,l=1}^{n} (A_i(\mathbf{x}'))_{kl} \frac{\partial^2 \bar{\beta}_i}{\partial \zeta_k \partial \zeta_l}\Big(\mathbf{x}', (\frac{\mathbf{x}'}{\varepsilon}, \frac{\mathbf{x}_n}{\varepsilon})\Big) \\
&\quad + \frac{2}{\varepsilon} \sum_{k=1}^{n-1} \sum_{l=1}^{n} (A_i(\mathbf{x}'))_{kl} \frac{\partial^2 \bar{\beta}_i}{\partial \xi_k \partial \zeta_l}\Big(\mathbf{x}', (\frac{\mathbf{x}'}{\varepsilon}, \frac{\mathbf{x}_n}{\varepsilon})\Big) \\
&\quad + \sum_{k,l=1}^{n-1} (A_i(\mathbf{x}'))_{kl} \frac{\partial^2 \bar{\beta}_i}{\partial \xi_k \partial \xi_l}\Big(\mathbf{x}', (\frac{\mathbf{x}'}{\varepsilon}, \frac{\mathbf{x}_n}{\varepsilon})\Big)
\end{aligned}
$$

Here, the first summand on the right-hand side vanishes because of (36) and the remaining summands can be estimated using (42) by $\varepsilon^{-1} e^{\lambda \frac{\mathbf{x}_n}{\varepsilon}}$ resp. $e^{\lambda \frac{\mathbf{x}_n}{\varepsilon}}$. This completes the proof. $\square$

6 Global boundary corrector

For $i = 1, \dots, N$, let $\psi_i \in C^\infty(\Gamma, [0,1])$ with $\mathrm{supp}(\psi_i) \subset V_i$ be a partition of unity subordinate to the covering $\{V_i\}_{i=1,\dots,N}$ of $\Gamma = \partial\Omega$. With this partition of unity we can define

$$c^{bl}(x) = \sum_{i=1}^{N} \psi_i(x)\widetilde{c_i^{bl}}(x)\,, \qquad x \in \Gamma \tag{67}$$

Notation 6.1 *In the following we will use for a point $x \in \mathcal{T}^\Gamma$ the representation in the coordinates of the tubular neighborhood $x = \mathcal{T}(\mathrm{x}', \mathrm{x}_n)$, with $(\mathrm{x}', \mathrm{x}_n) \in \Gamma \times (-\delta, \delta)$.*

Then we can define a global boundary corrector $\widetilde{\beta}^\varepsilon$ on the set

$$\Gamma_\delta^\varepsilon = \mathcal{T}^\Gamma \cap \Omega^\varepsilon \tag{68}$$

by

$$\widetilde{\beta}^\varepsilon(x) = \sum_{i=1}^{N} \psi_i(\mathrm{x}')\widetilde{\beta}_i^\varepsilon(x)\,, \qquad x \in \Gamma_\delta^\varepsilon\,. \tag{69}$$

We see that c^{bl} and $\widetilde{\beta}^\varepsilon$ depend on the partition of unity ψ_i. Nevertheless, the difference between c_i^{bl} and c_j^{bl} (as well as between $\widetilde{\beta}_i^\varepsilon$ and $\widetilde{\beta}_j^\varepsilon$) on the overlap regions $V_i \cap V_j$ are small enough to ensure that the local estimates proved for the $\widetilde{\beta}_i^\varepsilon$ in Theorem 5.1 lead to analogous global estimates for $\widetilde{\beta}^\varepsilon$.

Theorem 6.2 *With $\widetilde{\beta}^\varepsilon$ from (69), c^{bl} from (67), $\Gamma_\delta^\varepsilon$ from (68), and λ from (57) we have*

$$|D^{\vec{k}}\widetilde{\beta}^\varepsilon(x)| \lesssim \varepsilon^{-|\vec{k}|}e^{-\lambda\frac{d(x,\Gamma)}{\varepsilon}}\,, \qquad x \in \Gamma_\delta^\varepsilon \setminus \Gamma\,,\ \vec{k} \in \mathbb{N}^n \tag{70}$$

as well as

$$|\Delta\widetilde{\beta}^\varepsilon(x)| \lesssim \frac{1}{\varepsilon}e^{-\lambda\frac{d(x,\Gamma)}{\varepsilon}}\,, \qquad x \in \Gamma_\delta^\varepsilon \setminus \Gamma\,. \tag{71}$$

Furthermore, we have

$$\left[\frac{\partial\widetilde{\beta}^\varepsilon}{\partial\nu}(x)\right] = \frac{1}{\varepsilon}\,, \qquad x \in \Gamma \cap \Omega^\varepsilon\,, \tag{72}$$

and

$$\widetilde{\beta}^\varepsilon(x) = \begin{cases} -c^{bl}(x) & x \in \partial\Omega^\varepsilon \setminus \Omega \\ \frac{d(x,\Gamma)}{\varepsilon} - c^{bl}(x) & x \in \partial\Omega^\varepsilon \cap \Omega \end{cases}\,. \tag{73}$$

Proof: Let $\widetilde{\psi}_i \in C^\infty(\Gamma_\delta^\varepsilon)$ be defined as $\widetilde{\psi}_i(x) = \psi_i(\mathrm{x}')$. Then the product rule leads to

$$D^{\vec{k}}(\sum_{i=1}^N \widetilde{\psi}_i(x)\widetilde{\beta}_i^\varepsilon(x)) = \sum_{i=1}^N \widetilde{\psi}_i(x) D^{\vec{k}} \widetilde{\beta}_i^\varepsilon(x) + \ldots + \sum_{i=1}^N \widetilde{\beta}_i^\varepsilon(x) D^{\vec{k}} \widetilde{\psi}_i(x) \qquad (74)$$

Since $\psi_i \in C^\infty(\Gamma)$ and because of (58), the first term leads to an estimate of the form $\varepsilon^{-|\vec{k}|} e^{\lambda \frac{\mathrm{x}_n}{\varepsilon}}$ whereas the following terms are of lower order in ε. Thus, (70) is proved. (72) is obvious, and (71) follows by

$$\begin{aligned} &\sum_{i=1}^N \Delta(\widetilde{\psi}_i(x)\widetilde{\beta}_i^\varepsilon(x)) \\ &= \sum_{i=1}^N \widetilde{\psi}_i(x)\Delta\widetilde{\beta}_i^\varepsilon(x) + \sum_{i=1}^N 2\nabla\widetilde{\psi}_i(x)\nabla\widetilde{\beta}_i^\varepsilon(x) + \sum_{i=1}^N \widetilde{\beta}_i^\varepsilon(x)\Delta\widetilde{\psi}_i(x) \\ &\lesssim \varepsilon^{-1} e^{\lambda \frac{\mathrm{x}_n}{\varepsilon}} \end{aligned}$$

using (58), (59) and the product rule. □

The error estimate in the following section will be based on the estimates in the following two theorems.

Theorem 6.3 *With $\widetilde{\beta}^\varepsilon$ from (69) and $\Gamma_\delta^\varepsilon$ from (68), we have for $1 \le p < \infty$:*

$$\|\widetilde{\beta}^\varepsilon\|_{L^p(\Gamma_\delta^\varepsilon)} \lesssim \varepsilon^{\frac{1}{p}}\,. \qquad (75)$$

Proof: (75) follows from (70) because of

$$\|\widetilde{\beta}^\varepsilon\|^p_{L^p(\Gamma_\delta^\varepsilon)} \lesssim \int_{-\delta}^{M\varepsilon} e^{\lambda p \frac{\mathrm{x}_n}{\varepsilon}}\, d\mathrm{x}_n \lesssim \varepsilon\,.$$

□

Theorem 6.4 *Let $\widetilde{\beta}^\varepsilon$ and $\Gamma_\delta^\varepsilon$ be defined as in (69),(68), let $\varphi \in H_0^1(\Omega^\varepsilon)$ and assume that $\chi \in W^{1,\infty}(\Gamma_\delta^\varepsilon)$ satisfies $\chi(x) = 0$ for $x \in \partial\Gamma_\delta^\varepsilon \setminus \partial\Omega^\varepsilon$. Then, for $1 \le p < \infty$, we have*

$$\int_{\Gamma_\delta^\varepsilon} \chi \nabla\widetilde{\beta}^\varepsilon \nabla\varphi\, dx = -\frac{1}{\varepsilon}\int_{\Gamma\cap\Omega^\varepsilon} \chi\varphi\, dx' + O(\varepsilon^{\frac{1}{2}} \|\chi\|_{W^{1,\infty}(\Gamma_\delta^\varepsilon)} \|\nabla\varphi\|_{L^2(\Gamma_\delta^\varepsilon)})\,. \qquad (76)$$

Proof: By partial integration and use of (72) we obtain

$$\begin{aligned}\int_{\Gamma_\delta^\varepsilon} \chi\nabla\widetilde{\beta}^\varepsilon\nabla\varphi\,dx &= -\int_{\Gamma_\delta^\varepsilon} \nabla\chi\nabla\widetilde{\beta}^\varepsilon\varphi\,dx - \int_{\Gamma_\delta^\varepsilon\setminus\Gamma} \chi\Delta\widetilde{\beta}^\varepsilon\varphi\,dx - \frac{1}{\varepsilon}\int_{\Gamma\cap\Omega^\varepsilon} \chi\varphi\,dx' \\ &= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})\,.\end{aligned}$$

Here, we can estimate (I) as follows. With M being the constant from (6) and denoting by $\widetilde{\varphi}$ the trivial extension of φ to $\Gamma_\delta^\varepsilon$ we have for $-\delta < \mathrm{x}_n \leq M\varepsilon$

$$\int_\Gamma |\widetilde{\varphi}\circ\mathcal{T}(\mathrm{x}',\mathrm{x}_n)|d\mathrm{x}' \lesssim \sqrt{M\varepsilon - \mathrm{x}_n}\|\nabla\varphi\|_{L^2(\Gamma_\delta^\varepsilon)}\,,\ \varphi\in H_0^1(\Omega^\varepsilon)\,, \tag{77}$$

such that using (70) we obtain

$$\begin{aligned}(\mathrm{I}) &= \int_{\Gamma_\delta^\varepsilon} \nabla\chi\nabla\widetilde{\beta}^\varepsilon\varphi\,dx \\ &\lesssim \|\chi\|_{W^{1,\infty}(\Gamma_\delta^\varepsilon)}\varepsilon^{-1}\int_{-\delta}^{M\varepsilon} e^{\lambda\frac{\mathrm{x}_n}{\varepsilon}}\int_\Gamma |\widetilde{\varphi}\circ\mathcal{T}(x',\mathrm{x}_n)|\;d\mathrm{x}'\,d\mathrm{x}_n \\ &\lesssim \varepsilon^{-1}\|\chi\|_{W^{1,\infty}(\Gamma_\delta^\varepsilon)}\|\nabla\varphi\|_{L^2(\Gamma_\delta^\varepsilon)}\int_{-\infty}^{M\varepsilon} e^{\lambda\frac{\mathrm{x}_n}{\varepsilon}}\sqrt{M\varepsilon-\mathrm{x}_n}\,d\mathrm{x}_n \\ &\lesssim \varepsilon^{-1}\|\chi\|_{W^{1,\infty}(\Gamma_\delta^\varepsilon)}\|\nabla\varphi\|_{L^2(\Gamma_\delta^\varepsilon)}\int_{-\infty}^{0} e^{\lambda\frac{\mathrm{x}_n}{\varepsilon}}\sqrt{-\mathrm{x}_n}\,d\mathrm{x}_n \\ &\lesssim \varepsilon^{\frac{1}{2}}\|\chi\|_{W^{1,\infty}(\Gamma_\delta^\varepsilon)}\|\nabla\varphi\|_{L^2(\Gamma_\delta^\varepsilon)}\,.\end{aligned}$$

(II) can be estimated using (71) in the same way, such that the theorem is proven. □

7 Improved approximation

Using the boundary corrector $\widetilde{\beta}^\varepsilon$ constructed in Section 6, we approximate the corrector θ^ε from Section 3 as follows. Let $\eta : \Omega \to \mathbb{R}$ be defined as

$$\begin{aligned} -\Delta\eta(x) &= 0\,, \quad x\in\Omega\,, \\ \eta(x) &= c^{bl}(x)\tfrac{\partial}{\partial\nu}u(x)\,, \quad x\in\Gamma\,, \end{aligned} \tag{78}$$

where $\frac{\partial}{\partial\nu}$ is again the derivative in direction of the exterior normal field $\nu : \Gamma \to \mathbb{R}^n$. We extend η to Ω^ε by defining $\widetilde{\eta} : \Omega^\varepsilon \to \mathbb{R}$ as

$$\widetilde{\eta} :\, x \mapsto \begin{cases} \eta(x) & x\in\Omega \\ \eta(\mathrm{x}') & x = \mathcal{T}(\mathrm{x}',\mathrm{x}_n)\in\Omega^\varepsilon\setminus\Omega \end{cases}. \tag{79}$$

For estimating the energy error we need another correction term $\widetilde{\eta}^\varepsilon$ which is constructed as follows. Let $\widetilde{\frac{\partial}{\partial\nu}u}$ be the extension of $\frac{\partial}{\partial\nu}u|_\Gamma$ to Ω^ε given by

$$\widetilde{\frac{\partial}{\partial\nu}u} : \Omega^\varepsilon \to \mathbb{R}\,, \qquad x \mapsto \begin{cases} \rho(\mathrm{x}_n)\frac{\partial}{\partial\nu}u(\mathrm{x}') & x = \mathcal{T}(\mathrm{x}',\mathrm{x}_n) \in \Gamma_\delta^\varepsilon \\ 0 & x \notin \Gamma_\delta^\varepsilon \end{cases}\,, \tag{80}$$

where $\rho \in C^\infty(\mathbb{R},[0,1])$ is a cut-off function satisfying

$$\begin{aligned} \rho(\mathrm{x}_n) &\equiv 1\,, \qquad \mathrm{x}_n \geq -M\varepsilon\,, \\ \rho(\mathrm{x}_n) &\equiv 0\,, \qquad \mathrm{x}_n \leq -\delta\,. \end{aligned}$$

Then $\widetilde{\eta}^\varepsilon : \Omega^\varepsilon \to \mathbb{R}$ is defined using $\widetilde{\beta}^\varepsilon$ from (55) as

$$\widetilde{\eta}^\varepsilon(x) = \begin{cases} \widetilde{\frac{\partial}{\partial\nu}u}(x)\widetilde{\beta}^\varepsilon(x) & x \in \Gamma_\delta^\varepsilon \\ 0 & x \notin \Gamma_\delta^\varepsilon \end{cases}\,. \tag{81}$$

Now, the following theorem is the main result of this article.

Theorem 7.1 *Assume $f \in L^\infty(\Omega^\varepsilon)$ and $u \in W^{2,\infty}(\Omega)$ for the solution u of (17). With θ^ε from (20), $\widetilde{\eta}$ from (79) and $\widetilde{\eta}^\varepsilon$ from (81) we have*

$$\|\nabla(\theta^\varepsilon - \varepsilon(\widetilde{\eta} + \widetilde{\eta}^\varepsilon))\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^{\frac{3}{2}}\,. \tag{82}$$

Furthermore, we have

$$\|\theta^\varepsilon - \varepsilon(\widetilde{\eta} + \widetilde{\eta}^\varepsilon)\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^2\,. \tag{83}$$

Because of Theorem 3.1, this implies

$$\|\nabla(u^\varepsilon - \widetilde{u} - \varepsilon(\widetilde{\eta} + \widetilde{\eta}^\varepsilon))\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^{\frac{3}{2}} \tag{84}$$

and

$$\|u^\varepsilon - \widetilde{u} - \varepsilon(\widetilde{\eta} + \widetilde{\eta}^\varepsilon)\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^2\,. \tag{85}$$

Proof: First, we note that $\|\widetilde{\eta}\|_{W^{1,\infty}(\Omega^\varepsilon)}$ and $\|\widetilde{\frac{\partial}{\partial\nu}u}\|_{W^{1,\infty}(\Gamma_\delta^\varepsilon)}$ can be estimated in terms of $\|u\|_{W^{2,\infty}(\Omega)}$, such that these norms may appear in the constants of the following estimates. Next, we note that $\theta^\varepsilon - \varepsilon(\widetilde{\eta}+\widetilde{\eta}^\varepsilon)$ vanishes on $\partial\Omega^\varepsilon \setminus \Omega$, but in general not on $\partial\Omega^\varepsilon \cap \Omega$. Therefore, we introduce an additional corrector $\zeta^\varepsilon : \Omega^\varepsilon \to \mathbb{R}$ defined as the solution to

$$\begin{aligned} -\Delta\zeta^\varepsilon(x) &= 0\,, \qquad x \in \Omega^\varepsilon\,, \\ \zeta^\varepsilon(x) &= \widetilde{u} + \varepsilon(\widetilde{\eta} + \widetilde{\eta}^\varepsilon)(x)\,, \qquad x \in \Gamma^\varepsilon\,. \end{aligned} \tag{86}$$

Now, $\theta^\varepsilon - \varepsilon(\widetilde{\eta} + \widetilde{\eta}^\varepsilon) + \zeta^\varepsilon \in H_0^1(\Omega^\varepsilon)$ and the energy norm of ζ^ε can be shown to be of order $\varepsilon^{\frac{3}{2}}$ using Lemma 7.2 below. For showing (82), it is therefore sufficient to show

$$\begin{aligned}\int_{\Omega^\varepsilon} \nabla(\theta^\varepsilon - \varepsilon(\widetilde{\eta} + \widetilde{\eta}^\varepsilon) + \zeta^\varepsilon)\nabla\varphi\,dx &= \int_{\Omega^\varepsilon} \nabla(\theta^\varepsilon - \varepsilon(\widetilde{\eta} + \widetilde{\eta}^\varepsilon))\nabla\varphi\,dx \\ &\lesssim \varepsilon^{\frac{3}{2}}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)}\,, \qquad \varphi \in H_0^1(\Omega^\varepsilon)\,.\end{aligned} \tag{87}$$

This is done as follows. First, a standard trace estimate yields

$$\begin{aligned}\left|\varepsilon \int_{\Omega^\varepsilon} \nabla\widetilde{\eta}\nabla\varphi\,dx\right| &\le |\varepsilon \int_{\Gamma\cap\Omega^\varepsilon} \tfrac{\partial}{\partial\nu}\eta(\mathrm{x}')\varphi(\mathrm{x}')\,d\mathrm{x}'| + |\varepsilon \int_{\Omega^\varepsilon\setminus\Omega} \nabla\widetilde{\eta}\nabla\varphi\,dx| \\ &\lesssim \varepsilon^{3/2}\|\widetilde{\eta}\|_{W^{1,\infty}(\Gamma_\delta^\varepsilon)}\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)}\,.\end{aligned}$$

Next, we have

$$\begin{aligned}\varepsilon \int_{\Omega^\varepsilon} \nabla\widetilde{\eta}^\varepsilon\nabla\varphi\,dx &= \varepsilon \int_{\Gamma_\delta^\varepsilon} \nabla(\widetilde{\tfrac{\partial}{\partial\nu}u}\widetilde{\beta}^\varepsilon)\nabla\varphi\,dx \\ &= \varepsilon \int_{\Gamma_\delta^\varepsilon} (\nabla\widetilde{\tfrac{\partial}{\partial\nu}u})\widetilde{\beta}^\varepsilon\nabla\varphi\,dx + \varepsilon \int_{\Gamma_\delta^\varepsilon} \widetilde{\tfrac{\partial}{\partial\nu}u}(\nabla\widetilde{\beta}^\varepsilon)\nabla\varphi\,dx \\ &= (\mathrm{I}) + (\mathrm{II})\end{aligned}$$

Here, (I) can be estimated using (75) by

$$\begin{aligned}|(\mathrm{I})| &\lesssim \varepsilon\|\widetilde{\tfrac{\partial}{\partial\nu}u}\|_{W^{1,\infty}(\Gamma_\delta^\varepsilon)}\|\widetilde{\beta}^\varepsilon\|_{L^2(\Gamma_\delta^\varepsilon)}\|\nabla\varphi\|_{L^2(\Gamma_\delta^\varepsilon)} \\ &\lesssim \varepsilon^{\frac{3}{2}}\|\widetilde{\tfrac{\partial}{\partial\nu}u}\|_{W^{1,\infty}(\Gamma_\delta^\varepsilon)}\|\nabla\varphi\|_{L^2(\Gamma_\delta^\varepsilon)}\,,\end{aligned}$$

and (II) can be written using (76) as

$$(\mathrm{II}) = -\int_{\Gamma\cap\Omega^\varepsilon} \tfrac{\partial}{\partial\nu}u(\mathrm{x}')\varphi(\mathrm{x}')\,d\mathrm{x}' + O(\varepsilon^{\frac{3}{2}}\|\widetilde{\tfrac{\partial}{\partial\nu}u}\|_{W^{1,\infty}(\Gamma_\delta^\varepsilon)}\|\nabla\varphi\|_{L^2(\Gamma_\delta^\varepsilon)})\,.$$

Now, the first term on the right-hand side cancels with $\int_{\Omega^\varepsilon} \nabla\theta^\varepsilon\nabla\varphi\,dx$ due to (20), such that (87) is proved.

We now want to prove (83) by following the proof of (22). Setting $v = \theta^\varepsilon - \varepsilon(\widetilde{\eta}+\widetilde{\eta}^\varepsilon)$ and using Ω' from (25), we obtain easily the estimates $\|v\|_{L^2(\Omega^\varepsilon\setminus\Omega')} \lesssim \varepsilon^{\frac{5}{2}}$ and $\|v\|_{L^2(\partial\Omega')} \lesssim \varepsilon^2$. The application of Theorem 3.2 then yields (83). $\square$

Lemma 7.2 *The corrector ζ^ε defined by (86) satisfies*

$$\|\zeta^\varepsilon\|_{L^\infty(\Omega^\varepsilon)} \lesssim \varepsilon^2\|u\|_{W^{2,\infty}(\Omega)} \tag{88}$$

$$\|\nabla\zeta^\varepsilon\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^{\frac{3}{2}}\|u\|_{W^{2,\infty}(\Omega)} \tag{89}$$

Proof: Let S be the region from (34). Then setting again $x = \mathcal{T}(\mathrm{x}', \mathrm{x}_n)$ we define

$$v(x) = \widetilde{u}(x) + \varepsilon(\widetilde{\eta}(x) - \widetilde{\eta}(\mathrm{x}') - \frac{\mathrm{x}_n}{\varepsilon}\widetilde{\frac{\partial}{\partial\nu}u}(\mathrm{x}'))\,, \tag{90}$$

which satisfies

$$\begin{aligned}\|\nabla v\|_{L^\infty(S)} &\lesssim \|\nabla\widetilde{u} - \nu\widetilde{\frac{\partial}{\partial\nu}u}\|_{L^\infty(S)} + \varepsilon\|\nabla\widetilde{\eta}\|_{L^\infty(S)} + \varepsilon\|\nabla\widetilde{\frac{\partial}{\partial\nu}u}\|_{L^\infty(S)} \\ &\lesssim \varepsilon\|u\|_{W^{2,\infty}(\Omega)}\,. \end{aligned}\tag{91}$$

v vanishes on $\partial\Omega^\varepsilon \setminus \Omega$ and satisfies for $x \in \partial\Omega^\varepsilon \cap \Omega$ with $x = \mathcal{T}(x', t)$

$$\begin{aligned}|v(x)| &= \varepsilon|\eta(x) - \eta(\mathrm{x}')| - |\widetilde{u}(x) - \mathrm{x}_n\widetilde{\frac{\partial}{\partial\nu}u}(\mathrm{x}')| \\ &\lesssim M\varepsilon^2(\|\nabla\eta\|_{L^\infty(\Omega)} + \|u\|_{W^{2,\infty}(\Omega)}) \lesssim \varepsilon^2\|u\|_{W^{2,\infty}(\Omega)}\,.\end{aligned}$$

Thus, we conclude that

$$\|v\|_{L^\infty(S)} \lesssim \varepsilon^2\|u\|_{W^{2,\infty}(\Omega)}\,. \tag{92}$$

Since $\zeta^\varepsilon = v$ on $\partial\Omega^\varepsilon$, an application of the maximum principle yields (88).

With ψ denoting the cut-off function from the proof of Theorem 3.3 we have

$$\begin{aligned}\|\nabla(\psi v)\|_{L^2(\Omega^\varepsilon)} &\leq \|(\nabla\psi)v\|_{L^2(S)} + \|\psi\nabla v\|_{L^2(S)} \\ &\lesssim \varepsilon^{-\frac{1}{2}}\|v\|_{L^\infty(S)} + \varepsilon^{\frac{1}{2}}\|\nabla v\|_{L^\infty(S)} \\ &\lesssim \varepsilon^{\frac{3}{2}}\|u\|_{W^{2,\infty}(\Omega)}\,,\end{aligned}$$

where we used (91) and (92) in the last step. Since ζ^ε is harmonic, we have $\|\nabla\zeta^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq \|\nabla(\psi v)\|_{L^2(\Omega^\varepsilon)}$, and (89) follows. □

8 Effective law

First, we note that a good approximation in $L^p(\Omega^\varepsilon)$ with $1 \leq p \leq 2$ can already be obtained using only the correction η. From (75), we see that under the assumptions of Theorem 7.1, we have

$$\|\varepsilon\widetilde{\eta}^\varepsilon\|_{L^p(\Omega^\varepsilon)} \lesssim \varepsilon^{1+\frac{1}{p}}\,,\ 1 \leq p < +\infty\,. \tag{93}$$

Because of Theorem 7.1 and Theorem 3.1 this implies

Corollary 8.1 *Under the assumptions of Theorem 7.1 we have for* $1 \le p \le 2$ *the estimates*

$$\|\theta^\varepsilon - \varepsilon\widetilde{\eta}\|_{L^p(\Omega^\varepsilon)} \lesssim \varepsilon^{1+\frac{1}{p}} \tag{94}$$

and

$$\|u^\varepsilon - \widetilde{u} - \varepsilon\widetilde{\eta}\|_{L^p(\Omega^\varepsilon)} \lesssim \varepsilon^{1+\frac{1}{p}}\,. \tag{95}$$

Alternatively, we can prove better L^2-error estimates for subdomains $\Omega' \subset \Omega \cap \Omega^\varepsilon$ which are strictly contained in Ω. The decay estimates (70) on $\widetilde{\beta}^\varepsilon$ imply that

$$\|\varepsilon\widetilde{\eta}^\varepsilon\|_{L^\infty(\Omega')} \lesssim \varepsilon e^{-\lambda\frac{d(\Omega',\partial\Omega)}{\varepsilon}} \tag{96}$$

and $\varepsilon e^{-\lambda\frac{d(\Omega',\partial\Omega)}{\varepsilon}} \le \varepsilon^2$ when $d(\Omega',\partial\Omega) \ge \varepsilon\frac{|\log\varepsilon|}{\lambda}$. Thus, we obtain

Corollary 8.2 *Let* $\Omega' \subset \Omega$ *with* $d(\Omega',\partial\Omega) \ge \max\{M\varepsilon, \varepsilon\frac{|\log\varepsilon|}{\lambda}\}$. *Under the assumptions of Theorem 7.1 we have also the estimates*

$$\|\theta^\varepsilon - \varepsilon\widetilde{\eta}\|_{L^2(\Omega')} \lesssim \varepsilon^2 \tag{97}$$

and

$$\|u^\varepsilon - \widetilde{u} - \varepsilon\widetilde{\eta}\|_{L^2(\Omega')} \lesssim \varepsilon^2\,. \tag{98}$$

In some cases, it is possible to compute an approximation to $u + \varepsilon\eta$ directly by changing the boundary condition of the effective equation. In the following, we show how this is done in the case $c^{bl} \le 0$.[1]

We set

$$\Gamma^0 := \{x \in \Gamma : c^{bl}(x) = 0\}\,,\ \Gamma^- := \{x \in \Gamma : c^{bl}(x) < 0\} \tag{99}$$

and consider the problem: find $u^{\mathit{eff}} : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u^{\mathit{eff}}(x) &= f(x)\,, \quad x \in \Omega\,, \\ u^{\mathit{eff}}(x) &= 0\,,\ x \in \Gamma^0\,, \\ u^{\mathit{eff}}(x) &= \varepsilon c^{bl}(x)\tfrac{\partial}{\partial\nu}u^{\mathit{eff}}(x)\,,\ x \in \Gamma^- \end{aligned} \tag{100}$$

[1] Note that for $c^{bl} > 0$, problem (100) can be ill-posed without further restictions on ε and c^{bl}.

where $\frac{\partial}{\partial \nu}$ denotes again the derivative in direction of the exterior normal of Γ. Existence and uniqueness of this problem in the Hilbert space

$$V = \big\{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma^0 , \\ \int_\Omega |\nabla \varphi(x)|^2 \,dx + \int_{\Gamma^-} \frac{-1}{\varepsilon c^{bl}(x)} |\varphi(x)|^2 \,ds < +\infty \big\} .$$

can then be established easily by the lemma of Lax-Milgram.

Theorem 8.3 *We assume that we have the setting of Theorem 7.1 with $c^{bl} \leq 0$, and that $u^{\text{eff}} \in C^1(\Omega)$ is the restriction of some function $\widetilde{u^{\text{eff}}} \in C^1(\mathbb{R}^n)$ with $\|\widetilde{u^{\text{eff}}}\|_{W^{1,\infty}(\mathbb{R}^n)} \lesssim \|u^{\text{eff}}\|_{W^{1,\infty}(\Omega)}$. Then*

$$\|u^\varepsilon - \widetilde{u^{\text{eff}}}\|_{L^p(\Omega^\varepsilon)} \lesssim \varepsilon^{1+\frac{1}{p}} (\|u\|_{W^{2,\infty}(\Omega)} + \|u^{\text{eff}}\|_{W^{1,\infty}(\Omega)}) \tag{101}$$

for all $1 \leq p \leq 2$. For domains $\Omega' \subset \Omega \cap \Omega^\varepsilon$ with $d(\Omega', \partial\Omega) \geq \max\{M\varepsilon, \varepsilon \frac{|\log \varepsilon|}{\lambda}\}$, we have the interior estimate

$$\|u^\varepsilon - u^{\text{eff}}\|_{L^2(\Omega')} \lesssim \varepsilon^2 \|u\|_{W^{2,\infty}(\Omega)} . \tag{102}$$

Proof: The error $e = u^{\text{eff}} - u - \varepsilon\eta$ satisfies

$$\begin{aligned} -\Delta e &= 0 , \quad x \in \Omega , \\ e(x) &= \varepsilon c^{bl} \tfrac{\partial}{\partial \nu} e(x) + \varepsilon^2 c^{bl}(x) \tfrac{\partial}{\partial \nu} \eta(x) , \ x \in \Gamma . \end{aligned} \tag{103}$$

Because of the maximum principle, e must attain its absolute maximum at the boundary Γ. Therefore, let $x^* \in \Gamma$ be such that $|e(x^*)| = \max_{x\in\Omega} |e(x)|$. If $x^* \in \Gamma^0$, $e(x^*) = 0$ and we are done. In the case $x^* \in \Gamma^-$, due to the Hopf maximum principle, the derivative $\frac{\partial}{\partial \nu} e(x^*)$ is either zero or has the same sign as $e(x^*)$. Because of $c^{bl} < 0$, we obtain

$$|e(x^*)| \lesssim \varepsilon^2 c^{bl}(x^*) |\tfrac{\partial}{\partial \nu} \eta(x^*)| \lesssim \varepsilon^2 \|\eta\|_{W^{1,\infty}(\Omega)} \lesssim \varepsilon^2 \|u\|_{W^{2,\infty}(\Omega)} . \tag{104}$$

Thus, we have $\|e\|_{L^\infty(\Omega)} \lesssim \varepsilon^2$, and (102) is immediate from (98). For proving (101), we note that $\widetilde{e} := \widetilde{u^{\text{eff}}} - \widetilde{u} - \varepsilon\widetilde{\eta}$ satisfies $\|\nabla \widetilde{e}\|_{W^{1,\infty}(\mathbb{R}^n)} \lesssim \|u^{\text{eff}}\|_{W^{1,\infty}(\Omega)} + \|u\|_{W^{2,\infty}(\Omega)}$, from which we easily obtain that $\|\widetilde{e}\|_{L^p(\Omega^\varepsilon)} \lesssim \varepsilon^{1+\frac{1}{p}}$, which together with (95) implies (101). □

9 Unbounded domains

In some applications, see e.g. [1], Ω^ε and Ω are unbounded, although the boundaries Γ^ε and Γ are compact. In this section, we want to give a brief discussion how the above results change in this situation.

In general, for problems on unbounded domains one has to pose conditions at infinity to ensure uniqueness of the solution. In our case, this can be done by requiring that the solutions u^ε of (15), u of (17), θ^ε of (18), η of (78), and u^{eff} of (100) have bounded energy. The variational formulation of those problems then uses the spaces

$$V^\varepsilon = \left\{ \varphi \in H^1_{\mathrm{loc}}(\Omega^\varepsilon) : \varphi = 0 \text{ on } \Gamma^\varepsilon\,,\ \|\varphi\|^2_{V^\varepsilon} := \int_{\Omega^\varepsilon} |\nabla \varphi|^2\,dx < +\infty \right\} \tag{105}$$

instead of $H^1_0(\Omega^\varepsilon)$ and

$$V = \left\{ \varphi \in H^1_{\mathrm{loc}}(\Omega) : \varphi = 0 \text{ on } \Gamma\,,\ \|\varphi\|^2_{V} := \int_{\Omega} |\nabla \varphi|^2\,dx < +\infty \right\} \tag{106}$$

instead of $H^1_0(\Omega)$. Additionally, one has to pose further restrictions on the right-hand side f. For example, one can require that $f \in L^\infty(\mathbb{R}^n)$ has compact support which ensures that f induces a continuous functional on both V and V^ε. Thus, the Lax-Milgram lemma can be applied, and the existence of u^ε and u follows.

Then the energy norm estimates (21), (82), and (84) carry over verbatim. Also Sections 4, 5, 6 remain completely unchanged because the boundary correction occurs only in a neighborhood of the compact manifold Γ. However, the L^2-estimates in (22), (83), (85), (102) or the L^p-estimate in (101) are not true in this form, due to the fact that the error might stabilize to a constant at infinity. It is possible however, to obtain L^2 or L^p-estimates on bounded subdomains $\Omega^\varepsilon_R = \Omega^\varepsilon \cap B_R \subset \Omega^\varepsilon$, where B_R is a ball with radius $R > 0$ such that the whole region $\Gamma^\varepsilon_\delta$ defined in (68) is contained in B_R. For example, a simple approach is to apply Poincaré's inequality for obtaining estimates where the $L^2(\Omega^\varepsilon_R)$-norm is estimated as $O(\varepsilon^{\frac{3}{2}})$ with a constant depending on R.

10 Discussion

In this article, we have constructed a good approximation to the solution u^ε of problem (1) which is valid for both bounded and unbounded domains $\Omega^\varepsilon \subset$

$\mathbb{R}^n$ with curved boundary $\partial\Omega^\varepsilon$. In practice, it is probably most convenient to calculate first the solution u from (17), and then the solution η of (78). The computation of η is best done inside a so-called *heterogeneous multiscale method*, see [6], where the coefficient in the boundary condition of (78) is evaluated by computing a boundary layer problem. Using the solution u^{eff} of (100) instead of u and η is an alternative, if one can guarantee that $c^{bl} \leq 0$ along Γ (which is usually true if $\Omega \subset \Omega^\varepsilon$).

Now, let us point out some directions in which one can extend this work.

First, although the approximation of second order in ε is probably good enough for most applications, it is possible to construct approximations of even higher order, see [9]. Naturally, the resulting boundary layer problems and error estimates will then incorporate also higher order curvatures of the manifold Γ.

Second, applications may require non-smooth domains Ω^ε and Ω (e.g. domains with edges/corners) and/or abrupt changes in the oscillation pattern. Now, the case where those corners or edges occur only in regions where $\partial\Omega^\varepsilon$ coincides with $\partial\Omega$ (no oscillations) can easily be treated using the techniques from this article. However, if this is not the case, more elaborate techniques are necessary: for example, $u \in W^{2,\infty}(\Omega)$ would not be an appropriate assumption for such situations, and also the approximation order in the error estimates would get worse.

Finally, it is of uttermost importance to transfer these results to other types of equations. An obvious and easy extension would be to allow for a smoothly varying diffusion coefficient. More important, however, is the treatment of flow problems where additional difficulties arise. We will address this in future work.

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