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A NONLINEAR EFFECTIVE SLIP INTERFACE LAW FOR TRANSPORT PHENOMENA BETWEEN A FRACTURE FLOW AND A POROUS MEDIUM

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Abstract. We present modeling of an incompressible viscous flow through a fracture adjacent to a porous medium. A fast stationary flow, predominantly tangential to the porous medium is considered. Slow flow in such setting can be described by the Beavers-Joseph-Saffman slip. For fast flows, a nonlinear filtration law in the porous medium and a non- linear interface law are expected. In this paper we rigorously derive a quadratic effective slip interface law which holds for a range of Reynolds numbers and fracture widths. The porous medium flow is described by the Darcy law. The result shows that the interface slip law can be nonlinear, independently of the regime for the bulk flow. Since most of the interface and boundary slip laws are obtained via upscaling of complex systems, the result indicates that studying the inviscid limits for the Navier-Stokes equations with linear slip law at the boundary should be rethought.

1. **Introduction.**

Coupling between a fast viscous incompressible fracture flow and an adjacent filtration through porous medium occurs in a wide range of industrial processes and natural phenomena. The classical approach is to model the fracture flow using the lubrication approximation and to include it as an interface condition. Subsequently, it is coupled with a porous medium flow, described for small Reynolds numbers by the Darcy's law and by the Forchheimer's law in the case of large Reynolds' number.

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Study of the coupling between a slow viscous incompressible fracture flow and a porous medium was undertaken in [3] and [4]. For the critical fracture width, the interface condition linked to the Reynolds' equation from lubrication was found.

To describe a contact between a porous medium and a large fracture with the width significantly larger than the pore size, the following effective slip interface law was established in the seminal work by Beavers and Joseph [2],

$$\sqrt{K}\frac{\partial v_\tau}{\partial \mathbf{n}} = \alpha_{BJ} v_\tau + O(K), \tag{1}$$

where α_{BJ} is a dimensionless parameter depending on the geometrical structure of the porous medium and K is the scalar permeability. v_τ is the tangential velocity and $\mathbf{n}$ is the unit normal exterior to the fluid region. Note that in the original version of the law (1), v_τ was replaced by the difference between v_τ and the tangential Darcy velocity at the interface. In [18], Saffman remarked that the tangential Darcy velocity at the interface is of order $O(K)$, hence of a lower order. Then, the slip law without the tangential Darcy velocity at the interface (1) became generally accepted.

The rigorous derivation of the law by Beavers and Joseph through the homogenization limit and by constructing the interface boundary layer was done by Jäger and colleagues in [10], [11] and [12]. The pressure jump at the interface was studied analytically in [16] and using numerical simulations in [6]. For the review of the results we refer to [13], [17] and [7].

Sahraoui and Kaviany investigated in [19] a flow at the interface between a fracture and a porous medium by direct numerical simulations. One of the questions they studied was about the interface laws in presence of large Reynolds' numbers. The interface slip behavior in that case turned out to be complex. It was concluded that the flow inertia effects appear independently from the bulk nonlinear filtration in the porous medium. If ε is a characteristic nondimensional pore size, then for longitudinal Reynolds' numbers of order $O(1/\varepsilon)$, numerical simulations indicate that the slip law ceases to be linear. The inertia forces at the interface become significant for Reynolds' numbers of order $O(0.1/\varepsilon)$. Then, the slip coefficient α_{BJ} increases. For the bulk porous medium flow, the nonlinear effects become visible only for Reynolds' numbers greater than $O(3/\varepsilon)$. Those observations led to a conclusion that α_{BJ} depends on the Reynolds' number, [14] and [9]. Similar conclusion is in [15].

However, it seems that a linear slip law, even with the slip coefficient depending on Reynolds' number, is not enough to get an accurate description of the observed phenomena and a nonlinear slip law has to be derived. We will justify it by constructing rigorously an accurate approximation to the velocity field and showing that it leads to a quadratic slip law.

In the present paper we aim to identify a setting corresponding to a nonlinear slip law. We show that for a range of values of Reynolds' number and fracture width, the homogenization leads to a nonlinear interface law, even though the bulk filtration remains of the Darcy type. To streamline the presentation, we focus on a mathematical model in a simple setting. We consider a constant driving force, present only in the fracture and, for simplicity, impose periodic longitudinal boundary conditions for the velocity and for the pressure. Such simplification allows to avoid handling the pressure field and the outer boundary layers. The general case of nonstationary flows with physical boundary conditions and forcing terms will be considered in forthcoming papers.

The paper is organized as follows: In section 2, we define the problem as a stationary incompressible Navier-Stokes flow with Reynolds' number of the order $\varepsilon^{-\gamma}$ and the fracture width of the order ε^{δ}. Assuming a relation between γ and δ, allows us to obtain an approximation which satisfies a nonlinear slip law (11), while keeping a linear filtration equation in a porous medium. In section 3 we construct the approximation and prove that it provides a higher order approximation to the original problem.

2. **Main result.**

2.1. **Geometry.** We consider a two dimensional periodic porous medium $\Omega_2 = (0,1) \times (-1,0)$ with a periodic arrangement of the pores. The formal description goes along the following lines:

First, we define the geometrical structure inside the unit cell $Y = (0,1)^2$. Let Y_s (the solid part) be a closed strictly included subset of $\bar{Y}$, and $Y_F = Y \backslash Y_s$ (the fluid part). Then, we introduce a periodic repetition of Y_s all over $\mathbb{R}^2$ and set $Y_s^k = Y_s + k$, $k \in \mathbb{Z}^2$. Obviously, the resulting set $E_s = \bigcup_{k\in\mathbb{Z}^2} Y_s^k$ is a closed subset of $\mathbb{R}^2$ and $E_F = \mathbb{R}^2 \backslash E_s$ in an open set in $\mathbb{R}^2$. We suppose that Y_s has a smooth boundary. Consequently, E_F is connected and E_s is not. Finally, we notice that Ω_2 is covered with a regular mesh of size ε, each cell being a cube Y_i^ε, with $1 \le i \le N(\varepsilon) = |\Omega_2|\varepsilon^{-2}[1 + o(1)]$. Each cube Y_i^ε is homeomorphic to Y, by linear homeomorphism Π_i^ε, being composed of translation and a homothety of ratio $1/\varepsilon$.

We define $Y_{S_i}^\varepsilon = (\Pi_i^\varepsilon)^{-1}(Y_s)$ and $Y_{F_i}^\varepsilon = (\Pi_i^\varepsilon)^{-1}(Y_F)$. For sufficiently small $\varepsilon > 0$, we consider a set $T_\varepsilon = \{k \in \mathbb{Z}^2 | Y_{S_k}^\varepsilon \subset \Omega_2\}$ and define

$$O_\varepsilon = \bigcup_{k\in T_\varepsilon} Y_{S_k}^\varepsilon, \quad S^\varepsilon = \partial O_\varepsilon, \quad \Omega_2^\varepsilon = \Omega_2 \backslash O_\varepsilon = \Omega_2 \cap \varepsilon E_F.$$

Obviously, $\partial\Omega_2^\varepsilon = \partial\Omega_2 \cup S^\varepsilon$. The domains O_ε and Ω_2^ε represent the solid and the fluid part of the porous medium Ω, respectively. For simplicity, we assume $1/\varepsilon \in \mathbb{N}$.

Let $0 < \delta < 1$. We set $\Sigma = (0,1) \times \{0\}$, $\Omega_1^{\varepsilon,\delta} = (0,1) \times (0, \varepsilon^\delta)$ and $\Omega = (0,1) \times (-1, \varepsilon^\delta)$. Furthermore, let $\Omega^\varepsilon = \Omega_2^\varepsilon \cup \Sigma \cup \Omega_1^{\varepsilon,\delta}$.

In such geometry, homogenization of the Stokes equation with no-slip boundary conditions on S^ε leads to Darcy law (see [1], [8], [20] and [21]). In the presence of inertia, nonlinear corrections to Darcy law arise, as studied in [5].

2.2. **Position of the problem and the nonlinear slip law.** Let $0 < \gamma < 3/2$ and let F be a constant. In Ω^ε we study the following stationary Navier-Stokes equation

$$-\varepsilon^\gamma \Delta \mathbf{v}^\varepsilon + (\mathbf{v}^\varepsilon \nabla)\mathbf{v}^\varepsilon + \nabla p^\varepsilon = F\mathbf{e}^1 \mathbb{1}_{\{x_2>0\}} \qquad \text{in} \quad \Omega^\varepsilon \tag{2}$$

$$\operatorname{div} \mathbf{v}^\varepsilon = 0 \qquad \text{in} \quad \Omega^\varepsilon, \qquad \int_{\Omega_1^{\varepsilon,\delta}} p^\varepsilon \, dx = 0, \tag{3}$$

$$\mathbf{v}^\varepsilon = 0 \quad \text{on} \quad \partial\Omega^\varepsilon \setminus \Big(\{x_1 = 0\} \cup \{x_1 = 1\} \Big), \quad \{\mathbf{v}^\varepsilon, p^\varepsilon\} \quad \text{is } 1-\text{periodic in } x_1. \tag{4}$$

Remark 1. We skip here a discussion of modeling aspects. We only mention that ε^γ stands for the inverse of Reynolds' number and that the small fracture width ε^δ prevents creation of the Prandtl's boundary layer.

In order to simplify calculations we take a constant F. It corresponds to an affine pressure drop. Additionally, we assume its presence only in the fracture $\Omega_1^{\varepsilon,\delta}$. Let

$$W^\varepsilon = \{\mathbf{z} \in H^1(\Omega^\varepsilon)^2,\ \mathbf{z} = 0 \ \text{ on } \ \partial\Omega^\varepsilon \setminus \Big(\{x_1 = 0\} \cup \{x_1 = 1\} \Big) \\ \text{and } \ \mathbf{z} \ \text{ is } \ 1-\text{periodic in } x_1\}. \tag{5}$$

The variational form of problem (2)-(4) reads:

Find $\mathbf{v}^\varepsilon \in W^\varepsilon$, div $\mathbf{v}^\varepsilon = 0$ in Ω^ε and $p^\varepsilon \in L^2(\Omega^\varepsilon)$ such that

$$\int_{\Omega^\varepsilon} \varepsilon^\gamma \nabla \mathbf{v}^\varepsilon \nabla \varphi \, dx + \int_{\Omega^\varepsilon} (\mathbf{v}^\varepsilon \cdot \nabla)\mathbf{v}^\varepsilon \varphi \, dx - \int_{\Omega^\varepsilon} p^\varepsilon \ \text{div} \ \varphi \, dx = \int_{\Omega_1^{\varepsilon,\delta}} F \varphi_1 \, dx, \ \forall \varphi \in W^\varepsilon. \tag{6}$$

Theory of the stationary Navier-Stokes equations with homogeneous boundary conditions results in existence of the least one smooth velocity field $\mathbf{v}^\varepsilon \in W^\varepsilon$, div $\mathbf{v}^\varepsilon = 0$ in Ω^ε, which solves (6) for every $\varphi \in W^\varepsilon$, div $\varphi = 0$ in Ω^ε. The construction of the pressure field goes through De Rham's theorem. For more details we refer to the classical Temam's book [22].

Now we make assumptions on the parameters δ and γ.

(H1): $2\gamma < 3\delta$,
(H2): $0 < \delta < 1$ and $0 < \gamma < 3/2$,
(H3): $4\delta < 2\gamma + 1$.

and formulate the main result

Theorem 2.1. *Let us suppose the hypothesis (**H1**)-(**H3**) and let $\mathcal{U}^{2,\varepsilon}$ be defined by*

$$\mathcal{U}^{2,\varepsilon} = \mathbf{v}^\varepsilon + \varepsilon^{2\delta-\gamma} \frac{F}{2} \frac{x_2^+}{\varepsilon^\delta} (\frac{x_2}{\varepsilon^\delta} - 1)\mathbf{e}^1 + \frac{F}{2} \varepsilon^{\delta+1-\gamma} \beta^{bl}(\frac{x}{\varepsilon}) - \frac{F}{2} \varepsilon^{\delta+1-\gamma} C_1^{bl} \frac{x_2^+}{\varepsilon^\delta} \mathbf{e}^1 \\ - \frac{F}{2} C_1^{bl} \varepsilon^{2-\gamma} \beta^{bl}(\frac{x}{\varepsilon}) + \frac{F}{2} \varepsilon^{2-\gamma} (C_1^{bl})^2 \frac{x_2^+}{\varepsilon^\delta} \mathbf{e}^1 + (\frac{F}{2})^2 \varepsilon^{2\delta+3-3\gamma} \beta^{1,bl}(\frac{x}{\varepsilon}) - \\ (\frac{F}{2})^2 \varepsilon^{2\delta+3-3\gamma} C_{11}^{bl} \frac{x_2^+}{\varepsilon^\delta} \mathbf{e}^1, \tag{7}$$

where the boundary layer functions β^{bl} and $\beta^{1,bl}$ are defined, respectively, by (41)-(44) and (63)-(66). The constant $C_1^{bl} < 0$ is the stabilization constant for β_1^{bl} when $y_2 \to +\infty$. Similarly C_{11}^{bl} is the stabilization constant for $\beta_1^{1,bl}$ when $y_2 \to +\infty$.

Then, the following estimate holds

$$\varepsilon \|\nabla \mathcal{U}^{2,\varepsilon}\|_{L^2(\Omega^\varepsilon)^4} + \|\mathcal{U}^{2,\varepsilon}\|_{L^2(\Omega_2^\varepsilon)^2} + \varepsilon^{1/2} \|\mathcal{U}^{2,\varepsilon}\|_{L^2(\Sigma)^2} + \\ \varepsilon^{1-\delta} \|\mathcal{U}^{2,\varepsilon}\|_{L^2(\Omega_1^{\varepsilon,\delta})^2} \le C \varepsilon^{7/2-\delta-\gamma}. \tag{8}$$

Remark 2. The rigorous result from Theorem 2.1, showing that $\mathcal{U}^{2,\varepsilon}$ is of order $O(\varepsilon^{3-\delta-\gamma})$ on Σ, allows justifying a nonlinear interface law. Contrary to the classical situation, when Saffman's modification of the linear slip law by Beavers and Joseph (see [2] and [18]) is used, the nonlinear interface laws are rarely derived in the literature. However, they are supposed to be appropriate for fast flows.

Setting $\delta = 1 - 7\eta/12$ and $\gamma = 3/2 - \eta$, where $0 < \eta < 3/2$, which fulfills hypotheses (**H1**)-(**H3**), we obtain on the interface Σ

$$\begin{aligned} v_1(\varepsilon)|_\Sigma &= -\frac{F}{2}\varepsilon^{\delta+1-\gamma}(1 - C_1^{bl}\varepsilon^{1-\delta})\beta^{bl}(\frac{x}{\varepsilon})|_\Sigma - (\frac{F}{2})^2\varepsilon^{2\delta+3-3\gamma}\beta_1^{1,bl}(\frac{x}{\varepsilon})|_\Sigma \\ &= -\frac{F}{2}\sqrt{\varepsilon}\varepsilon^{5\eta/12}(1 - C_1^{bl}\varepsilon^{7\eta/12})\beta_1^{bl}(\frac{x}{\varepsilon})|_\Sigma - (\frac{F}{2})^2\sqrt{\varepsilon}\varepsilon^{11\eta/6}\beta_1^{1,bl}(\frac{x}{\varepsilon})|_\Sigma \end{aligned}$$

and for the average over the pore face on Σ

$$\langle v_1(\varepsilon)|_\Sigma\rangle = v_1^{eff} = -\frac{F}{2}\sqrt{\varepsilon}\varepsilon^{5\eta/12}(1 - C_1^{bl}\varepsilon^{7\eta/12})C_1^{bl} - (\frac{F}{2})^2\sqrt{\varepsilon}\varepsilon^{11\eta/6}\langle\beta_1^{1,bl}(\frac{x}{\varepsilon})|_\Sigma\rangle. \quad (9)$$

Next, for the shear stress we have

$$\begin{aligned} \frac{\partial v_1(\varepsilon)}{\partial x_2}|_\Sigma = \varepsilon^{\delta-\gamma}\frac{F}{2} - \varepsilon^{\delta-\gamma}\frac{F}{2}\frac{\partial\beta_1^{bl}}{\partial y_2}|_{\Sigma,y=x/\varepsilon} + \varepsilon^{1-\gamma}\frac{F}{2}C_1^{bl} + \frac{F}{2}C_1^{bl}\varepsilon^{1-\gamma}\frac{\partial\beta_1^{bl}}{\partial y_2}|_{\Sigma,y=x/\varepsilon} \\ -\varepsilon^{2-\delta-\gamma}\frac{F}{2}(C_1^{bl})^2 - (\frac{F}{2})^2\varepsilon^{2\delta+2-3\gamma}\frac{\partial\beta_1^{1,bl}(\frac{x}{\varepsilon})}{\partial y_2}|_\Sigma + (\frac{F}{2})^2\varepsilon^{\delta+3-3\gamma}C_{11}^{bl}. \end{aligned}$$

After averaging over Σ with respect to y_1, we obtain

$$\begin{aligned} \langle\frac{\partial v_1(\varepsilon)}{\partial x_2}|_\Sigma\rangle = \frac{\partial v_1^{eff}}{\partial x_2}|_\Sigma = \frac{F}{2}\varepsilon^{-1/2+5\eta/12}(1 + \varepsilon^{7\eta/12}C_1^{bl} - \varepsilon^{7\eta/6}(C_1^{bl})^2) - \\ (\frac{F}{2})^2\varepsilon^{-1/2+11\eta/6}(\langle\frac{\partial\beta_1^{1,bl}(\frac{x}{\varepsilon})}{\partial y_2}|_\Sigma\rangle - \varepsilon^{7\eta/12}C_{11}^{bl}). \end{aligned} \quad (10)$$

Next, elimination of $F/2$ yields

$$\begin{aligned} v_1^{eff} = -C_1^{bl}\varepsilon\frac{\partial v_1^{eff}}{\partial x_2}\frac{1 - C_1^{bl}\varepsilon^{7\eta/12}}{1 + C_1^{bl}\varepsilon^{7\eta/12}(1 - C_1^{bl}\varepsilon^{7\eta/12})} \\ -\varepsilon^{3/2+\eta}\langle\beta^{1,bl}(\frac{x}{\varepsilon})|_\Sigma\rangle(\frac{\partial v_1^{eff}}{\partial x_2})^2 + O(\varepsilon^{3/2+29\eta/12}(\frac{\partial v_1^{eff}}{\partial x_2})^2). \end{aligned} \quad (11)$$

The above formula results in Saffman' version of the law by Beavers and Joseph, if only the first term at the right hand-side is taken into consideration. For small η, we obtain a significant deviation of the law by Beavers and Joseph from [18] and [2]. We are not aware of any rigorous derivation of a nonlinear interface law for the unconfined fluid flow coupled to the porous media flow.

3. **Rigorous justification of the nonlinear slip law, generalizing the law by Beavers and Joseph.** In this section we extend the justification of the law of Beavers and Joseph from [11] to the case of nonlinear laminar flows. In the proofs we apply the following variant of Poincaré's inequality:

Lemma 3.1. *(see e.g. [20]) Let* $\varphi \in V(\Omega_2^\varepsilon) = \{\varphi \in H^1(\Omega_2^\varepsilon) \;|\varphi = 0 \;\; \text{on } S^\varepsilon\}$ *and* $\psi \in H^1(\Omega_1^{\varepsilon,\delta})$ *such that* $\psi|_{\{x_2=\varepsilon^\delta\}} = 0$*. Then, it holds*

$$\|\varphi\|_{L^2(\Sigma)} \le C\varepsilon^{1/2}\|\nabla_x\varphi\|_{L^2(\Omega_2^\varepsilon)^2}, \quad (12)$$

$$\|\varphi\|_{L^2(\Omega_2^\varepsilon)} \le C\varepsilon\|\nabla_x\varphi\|_{L^2(\Omega_2^\varepsilon)^2}, \quad (13)$$

$$\|\psi\|_{L^2(\Sigma)} \le C\varepsilon^{\delta/2}\|\nabla_x\psi\|_{L^2(\Omega_1^{\varepsilon,\delta})^2}, \quad (14)$$

$$\|\psi\|_{L^2(\Omega_1^{\varepsilon,\delta})} \le C\varepsilon^{\delta}\|\nabla_x\psi\|_{L^2(\Omega_1^{\varepsilon,\delta})^2}. \quad (15)$$

3.1. **The impermeable interface approximation.** Intuitively, the main flow is in the fracture $\Omega_1^{\varepsilon,\delta}$. Following the approach from [11] we study the problem

$$-\varepsilon^\gamma \triangle \mathbf{v}^0 + (\mathbf{v}^0\nabla)\mathbf{v}^0 + \nabla p^0 = F\mathbf{e}^1 \qquad \text{in } \Omega_1^{\varepsilon,\delta}, \tag{16}$$

$$\text{div } \mathbf{v}^0 = 0 \qquad \text{in } \Omega_1^{\varepsilon,\delta}, \tag{17}$$

$$\mathbf{v}^0 = 0 \qquad \text{on } \partial\Omega_1^{\varepsilon,\delta} \setminus \Big(\{x_1 = 0\} \cup \{x_1 = 1\} \Big) \quad , \tag{18}$$

$$\{\mathbf{v}^0, p^0\} \qquad \text{is } 1-\text{periodic in } x_1, \qquad \int_{\Omega_1^{\varepsilon,\delta}} p^0 \, dx = 0. \tag{19}$$

Therefore, as in [11] and [13], for the lowest order approximation $\{\mathbf{v}^0, p^0\}$ we impose on the interface the no-slip condition

$$\mathbf{v}^0 = 0 \qquad \text{on} \quad \Sigma. \tag{20}$$

Such choice leads to a cut-off of the shear and it introduces an error.

A unique solution of problem (16)-(19) is the classic Poiseuille flow in $\Omega_1^{\varepsilon,\delta}$, satisfying the no-slip condition at Σ. It is given by

$$\mathbf{v}^0 = -\varepsilon^{2\delta-\gamma}\frac{F}{2}\frac{x_2}{\varepsilon^\delta}(\frac{x_2}{\varepsilon^\delta} - 1)\mathbf{e}^1 \text{ for } \quad 0 \le x_2 \le \varepsilon^\delta; \quad p^0 = 0 \text{ for } \quad 0 \le x_1 \le 1. \tag{21}$$

Concerning the normal derivative of the tangential velocity on Σ, we obtain

$$\frac{\partial v_1^0}{\partial x_2} = -\varepsilon^{\delta-\gamma}\frac{F}{2}(\frac{2x_2}{\varepsilon^\delta} - 1); \qquad \frac{\partial v_1^0}{\partial x_2}|_\Sigma = \varepsilon^{\delta-\gamma}\frac{F}{2}. \tag{22}$$

We extend $\mathbf{v}^0$ to Ω_2 by setting $\mathbf{v}^0 = 0$ for $-1 \le x_2 < 0$. p^0 is extended by 0 to Ω_2. The question is in which sense this solution approximates the solution $\{\mathbf{v}^\varepsilon, p^\varepsilon\}$ of the original problem (2)-(4).

A direct consequence of the weak formulation (6) is that the difference $\mathbf{v}^\varepsilon - \mathbf{v}^0$ satisfies the following variational equation

$$\int_{\Omega^\varepsilon} \varepsilon^\gamma \nabla(\mathbf{v}^\varepsilon - \mathbf{v}^0)\nabla\varphi \, dx + \int_{\Omega^\varepsilon} \Big(v_1^0 \frac{\partial(\mathbf{v}^\varepsilon - \mathbf{v}^0)}{\partial x_1} + (v_2^\varepsilon - v_2^0)\frac{\partial \mathbf{v}^0}{\partial x_2} + \\ ((\mathbf{v}^\varepsilon - \mathbf{v}^0)\nabla)(\mathbf{v}^\varepsilon - \mathbf{v}^0)\Big)\varphi \, dx - \int_{\Omega^\varepsilon} p^\varepsilon \text{ div } \varphi = \int_\Sigma \varepsilon^\gamma \frac{\partial v_1^0}{\partial x_2}\varphi_1 \, dS, \quad \forall \varphi \in W^\varepsilon. \tag{23}$$

It leads to the following result, which is a generalization of the result proved in [11]:

Proposition 1. *Let us assume that* (**H1**)*-*(**H2**) *are satisfied. Let* $\{\mathbf{v}^\varepsilon, p^\varepsilon\}$ *be a solution of (2)-(4) and* $\{\mathbf{v}^0, p^0\}$ *defined by (21). Then, it holds for* $\varepsilon \le \varepsilon_0$

$$\sqrt{\varepsilon}\|\nabla(\mathbf{v}^\varepsilon - \mathbf{v}^0)\|_{L^2(\Omega^\varepsilon)^4} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{v}^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\mathbf{v}^\varepsilon\|_{L^2(\Sigma)} + \\ \varepsilon^{1/2-\delta}\|\mathbf{v}^\varepsilon - \mathbf{v}^0\|_{L^2(\Omega_1^{\varepsilon,\delta})^2} \le C\varepsilon^{\delta-\gamma+1} \tag{24}$$

Proof. We test (23) with $\varphi = \mathbf{v}^\varepsilon - \mathbf{v}^0$ and obtain

$$\int_{\Omega^\varepsilon} \varepsilon^\gamma |\nabla(\mathbf{v}^\varepsilon - \mathbf{v}^0)|^2 \, dx = \int_{\Omega^\varepsilon} (v_1^0 - v_1^\varepsilon)(v_2^\varepsilon - v_2^0)\frac{\partial v_1^0}{\partial x_2} \, dx + \int_\Sigma \varepsilon^\gamma \frac{\partial v_1^0}{\partial x_2}(v_1^\varepsilon - v_1^0) \, dS. \tag{25}$$

Applying Lemma 3.1 and formula (22) yield

$$|\int_{\Omega^\varepsilon} (v_1^\varepsilon - v_1^0)(v_2^\varepsilon - v_2^0)\frac{\partial v_1^0}{\partial x_2}\, dx| \le C\varepsilon^{3\delta-\gamma}\|\nabla(\mathbf{v}^\varepsilon - \mathbf{v}^0)\|^2_{L^2(\Omega_1^{\varepsilon,\delta})^4},$$

$$|\int_\Sigma \varepsilon^\gamma \frac{\partial v_1^0}{\partial x_2}(v_1^\varepsilon - v_1^0)\, dS| \le C\varepsilon^{\delta+1/2}\|\nabla(\mathbf{v}^\varepsilon - \mathbf{v}^0)\|_{L^2(\Omega_2^\varepsilon)^4}.$$

Using hypothesis (**H1**) and above estimates lead to

$$\int_{\Omega^\varepsilon} \varepsilon^\gamma |\nabla(\mathbf{v}^\varepsilon - \mathbf{v}^0)|^2\, dx \le C\varepsilon^{\delta+1/2}\|\nabla(\mathbf{v}^\varepsilon - \mathbf{v}^0)\|_{L^2(\Omega_2^\varepsilon)^4}.$$

We apply once more Lemma 3.1 and (24) follows. □

This provides the uniform a priori estimates for $\{\mathbf{v}^\varepsilon, p^\varepsilon\}$. Moreover, we have found that the viscous flow in $\Omega_1^{\varepsilon,\delta}$ corresponding to an impermeable wall is an $O(\varepsilon^{2\delta-\gamma+1/2})$ L^2-approximation for $\mathbf{v}^\varepsilon$. The slip law, generalizing Beavers and Joseph's law, should correspond to the next order velocity correction. Since the Darcy velocity is of order $O(\varepsilon^{\delta-\gamma+3/2})$, we justify Saffman's observation that the bulk filtration effects are negligible at this stage.

3.2. **Justification of the nonlinear slip law.** We denote the jump on Σ by $[\cdot]$. At the interface Σ the approximation from Subsection 3.1 leads to the shear stress jump equal to $\varepsilon^\gamma \frac{\partial v_1^0}{\partial x_2}|_\Sigma = \frac{F}{2}\varepsilon^\delta$. We correct the jump by constructing the corresponding boundary layer.

The natural stretching variable is given by the geometry and reads $y = \frac{x}{\varepsilon}$. Then the correction $\{\mathbf{w}, p_w\}$ of the shear stress jump is given by

$$-\varepsilon^{\gamma-2}\triangle_y \mathbf{w} + \varepsilon^{-1}(\mathbf{w}\nabla_y)\mathbf{w} + \varepsilon^{-1}\nabla_y p_w = 0 \qquad \text{in} \quad \Omega_1^{\varepsilon,\delta}/\varepsilon \cup \Omega_2^\varepsilon/\varepsilon, \tag{26}$$

$$\mathrm{div}_y \mathbf{w} = 0 \qquad \text{in} \quad \Omega_1/\varepsilon \cup \Sigma/\varepsilon \cup \Omega_2^\varepsilon/\varepsilon, \tag{27}$$

$$\big[\mathbf{w}\big](\cdot,0) = 0; \quad \big[p_w\big](\cdot,0) = 0 \quad \text{and}$$

$$\big[-\varepsilon^{\gamma-1}\frac{\partial w_1}{\partial y_2}\big](\cdot,0) = \varepsilon^\gamma \frac{\partial v_1^0}{\partial x_2}|_\Sigma = \frac{F}{2}\varepsilon^\delta \quad \text{on} \quad \frac{\Sigma}{\varepsilon}, \tag{28}$$

$$\nabla_y \mathbf{w} \in L^2(\Omega^\varepsilon/\varepsilon)^4 \quad \text{and} \quad \{\mathbf{w}, p_w\} \text{ is } 1/\varepsilon - \text{periodic in } y_1. \tag{29}$$

It is natural to rescale $\mathbf{w}$ and p_w by setting

$$\mathbf{w} = -\varepsilon^{\delta+1-\gamma}\frac{F}{2}\beta(y) \quad \text{and} \quad p_w = -\varepsilon^\delta \pi(y)\frac{F}{2}.$$

Using periodicity of the geometry and independence of $\frac{\partial v_1^0}{\partial x_2}|_\Sigma$ of y, we obtain

$$-\triangle_y \beta + \nabla_y \pi = \frac{F}{2}\varepsilon^{\delta-2\gamma+2}(\beta\nabla_y)\beta \qquad \text{in} \quad \Omega_1^{\varepsilon,\delta}/\varepsilon \cup \Omega_2^\varepsilon/\varepsilon, \tag{30}$$

$$\mathrm{div}_y \beta = 0 \qquad \text{in} \quad \Omega_1/\varepsilon \cup \Sigma/\varepsilon \cup \Omega_2^\varepsilon/\varepsilon, \tag{31}$$

$$\big[\beta\big](\cdot,0) = 0; \quad \big[\pi\big](\cdot,0) = 0 \quad \text{and} \quad \big[\frac{\partial \beta_1}{\partial y_2}\big](\cdot,0) = 1 \quad \text{on} \quad \Sigma/\varepsilon, \tag{32}$$

$$\nabla_y \beta \in L^2(\Omega^\varepsilon/\varepsilon)^4 \quad \text{and} \quad \{\beta, \pi\} \text{ is } 1/\varepsilon - \text{periodic in } y_1. \tag{33}$$

We do not use directly the nonlinear boundary layer problem (30)-(33). Since by (**H2**) we have $\delta - 2\gamma + 2 > 0$, we approximate $\{\beta, \pi\}$ with $\{\beta^0 + \frac{F}{2}\varepsilon^{\delta-2\gamma+2}\beta^1, \pi^0 +$

$\frac{F}{2}\varepsilon^{\delta-2\gamma+2}\pi^1\}$, where the new functions are given through the following problems

$$-\triangle_y\beta^0+\nabla_y\pi^0=0 \qquad \text{in} \quad \Omega_1^{\varepsilon,\delta}/\varepsilon\cup\Omega_2^\varepsilon/\varepsilon, \tag{34}$$

$$\text{div}_y\beta^0=0 \qquad \text{in} \quad \Omega_1/\varepsilon\cup\Sigma/\varepsilon\cup\Omega_2^\varepsilon/\varepsilon, \tag{35}$$

$$\big[\beta^0\big](\cdot,0)=0; \quad \big[\pi^0\big](\cdot,0)=0 \quad \text{and} \quad \big[\frac{\partial\beta_1^0}{\partial y_2}\big](\cdot,0)=1 \quad \text{on} \quad \Sigma/\varepsilon, \tag{36}$$

$$\nabla_y\beta^0\in L^2(\Omega^\varepsilon/\varepsilon)^4 \quad \text{and} \quad \{\beta^0,\pi^0\} \text{ is } 1/\varepsilon-\text{periodic in } y_1 \tag{37}$$

and

$$-\triangle_y\beta^1+\nabla_y\pi^1=(\beta^0\nabla_y)\beta^0 \qquad \text{in} \quad \Omega_1^{\varepsilon,\delta}/\varepsilon\cup\Sigma/\varepsilon\cup\Omega_2^\varepsilon/\varepsilon, \tag{38}$$

$$\text{div}_y\beta^1=0 \qquad \text{in} \quad \Omega_1/\varepsilon\cup\Sigma/\varepsilon\cup\Omega_2^\varepsilon/\varepsilon, \tag{39}$$

$$\nabla_y\beta^1\in L^2(\Omega^\varepsilon/\varepsilon)^4 \quad \text{and} \quad \{\beta^1,\pi^1\} \text{ is } 1/\varepsilon-\text{periodic in } y_1. \tag{40}$$

Because of the 1-periodicity of the geometry with respect to y_1, we search for $\{\beta^0,\pi^0\}$ and $\{\beta^1,\pi^1\}$ which are $1-$periodic in y_1. Then problems (34)-(37) and (38)-(40) reduce to boundary layer problems introduced in [10].

The boundary value problem for $\beta^0=\beta^{bl}$ and $\pi^0=\pi^{bl}$ reads as follows: We introduce the interface $S=(0,1)\times\{0\}$, the semi-infinite slab $Z^+=(0,1)\times(0,+\infty)$ and the semi-infinite porous slab $Z^-=\cup_{k=1}^\infty(Y_F-\{0,k\})$. The flow region is then $Z_{BL}=Z^+\cup S\cup Z^-$.

Then the following problem is considered: Find $\{\beta^{bl},\omega^{bl}\}$ with square-integrable gradients satisfying

$$-\triangle_y\beta^{bl}+\nabla_y\omega^{bl}=0 \qquad \text{in } Z^+\cup Z^- \tag{41}$$

$$\text{div}_y\beta^{bl}=0 \qquad \text{in } Z^+\cup Z^- \tag{42}$$

$$\big[\beta^{bl}\big]_S(\cdot,0)=0 \quad \text{and} \quad \big[\{\nabla_y\beta^{bl}-\omega^{bl}I\}\mathbf{e}^2\big]_S(\cdot,0)=\mathbf{e}^1 \quad \text{on } S \tag{43}$$

$$\beta^{bl}=0 \quad \text{on } \cup_{k=1}^\infty(\partial Y_s-\{0,k\}), \qquad \{\beta^{bl},\omega^{bl}\} \text{ is } 1-\text{periodic in } y_1. \tag{44}$$

By Lax-Milgram's lemma, there is a unique $\beta^{bl}\in L^2_{loc}(Z_{BL})^2$, $\nabla_y\beta^{bl}\in L^2(Z_{BL})^4$ satisfying (41)-(44) and $\omega^{bl}\in L^2_{loc}(Z^+\cup Z^-)$, unique up to a constant and satisfying (41).

After [10], [11] and [12], we know that system (41)-(44) describes a boundary layer, i.e. that β^{bl} and ω^{bl} stabilize exponentially towards constants, when $|y_2|\to\infty$.

Since we are studying an incompressible flow, it is useful to recall properties of the conserved averages.

Proposition 2. *([10]). Let*

$$C_1^{bl}=\int_0^1\beta_1^{bl}(y_1,0)dy_1=-\int_{Z_{BL}}|\nabla\beta^{bl}(y)|^2\,dy. \tag{45}$$

Then for every $y_2\ge 0$ and $y_1\in(0,1)$, we have

$$|\beta^{bl}(y_1,y_2)-(C_1^{bl},0)|\le Ce^{-\delta y_2}, \quad \textit{for all} \quad \delta<2\pi. \tag{46}$$

Corollary 1. *([10]). Let*

$$C_\omega^{bl}=\int_0^1\omega^{bl}(y_1,0)\,dy_1. \tag{47}$$

Then for every $y_2 \geq 0$ *and* $y_1 \in (0,1)$, *we have*

$$| \, \omega^{bl}(y_1, y_2) - C^{bl}_\omega \, | \leq e^{-2\pi y_2}. \tag{48}$$

Proposition 3. *([10]). Let* β^{bl} *and* ω^{bl} *be defined by (41)-(44). Then there exist positive constants* C *and* γ_0, *such that*

$$|\nabla \beta^{bl}(y_1, y_2)| + |\nabla \omega^{bl}(y_1, y_2)| \leq C e^{-\gamma_0 |y_2|}, \qquad \textit{for every} \quad (y_1, y_2) \in Z^-. \tag{49}$$

$\beta^{bl,\varepsilon}(x) = \beta^{bl}(\frac{x}{\varepsilon})$ is extended by zero to $\Omega_2 \setminus \Omega^\varepsilon$. Let H be Heaviside's function. Then for every $q \geq 1$ we have

$$\|\beta^{bl,\varepsilon} - \varepsilon (C^{bl}_1, 0) H(x_2)\|_{L^q(\Omega_2 \cup \Omega_1^{\varepsilon,\delta})^2} + \|\omega^{bl,\varepsilon} - C^{bl}_\omega H(x_2)\|_{L^q(\Omega^\varepsilon)} + \\ \varepsilon \|\nabla \beta^{bl,\varepsilon}\|_{L^q(\Omega_2 \cup \Omega_1^{\varepsilon,\delta})^4} = C\varepsilon^{1/q}. \tag{50}$$

Hence, our correction is not concentrated around the interface and there are some nonzero stabilization constants. We will see that these constants are closely linked with our effective interface law.

As in [10] stabilization of $\beta^{0,\varepsilon}$ towards a nonzero constant velocity $C^{bl}_1 \mathbf{e}^1$, at the upper boundary, generates a counterflow. It is given by the two dimensional Couette flow $\mathbf{d} = C^{bl}_1 \frac{x_2^+}{\varepsilon^\delta} \mathbf{e}^1$.

Now, after [10], we expected that the approximation for the velocity reads

$$\mathbf{v}(\varepsilon) = \mathbf{v}^0 - \frac{F}{2} \varepsilon^{\delta+1-\gamma} \beta^{bl}(\frac{x}{\varepsilon}) + \frac{F}{2} \varepsilon^{\delta+1-\gamma} \mathbf{d} = \\ -\varepsilon^{2\delta-\gamma} \frac{F}{2} \frac{x_2^+}{\varepsilon^\delta} (\frac{x_2}{\varepsilon^\delta} - 1) \mathbf{e}^1 - \frac{F}{2} \varepsilon^{\delta+1-\gamma} \beta^{bl}(\frac{x}{\varepsilon}) + \frac{F}{2} \varepsilon^{\delta+1-\gamma} C^{bl}_1 \frac{x_2^+}{\varepsilon^\delta} \mathbf{e}^1. \tag{51}$$

Concerning the pressure, there are additional complications due to the stabilization of the boundary layer pressure to C^{bl}_ω, when $y_2 \to +\infty$. Consequently, $\omega^{bl,\varepsilon} - H(x_2) C^{bl}_\omega$ is small in $\Omega_1^{\varepsilon,\delta}$ and we should take into account the pressure stabilization effect.

At the flat interface Σ, the normal component of the normal stress reduces to the pressure field. Subtraction of the stabilization pressure constant at infinity leads to the pressure jump on Σ and the correct pressure approximation would be $p(\varepsilon) = -\frac{F}{2} \varepsilon^\delta \big(\omega^{bl}(\frac{x}{\varepsilon}) - C^{bl}_\omega H(x_2)\big)$. For the rigorous justification of the pressure approximation, involving the Darcy flow generated by the pressure jump, we refer to [16] . Numerical experiments, justifying independently the pressure jump are in [6]. In this article we concentrate on the slip law and do not derive a pressure error estimate. Consequently, for simplicity we take

$$p(\varepsilon) = -\frac{F}{2} \varepsilon^\delta \big(\omega^{bl}(\frac{x}{\varepsilon}) - C^{bl}_\omega\big). \tag{52}$$

We now make the velocity calculations rigorous. Let us define the errors in velocity and in the pressure:

$$\mathcal{U}^\varepsilon(x) = \mathbf{v}^\varepsilon - \mathbf{v}(\varepsilon), \qquad \mathcal{P}^\varepsilon(x) = p^\varepsilon - p(\varepsilon). \tag{53}$$

Remark 3. Rigorous argument, showing that $\mathcal{U}^\varepsilon$ is of order $O(\varepsilon^{2-\gamma})$, allows justifying Saffman's modification of the Beavers and Joseph law (see [2] and [18]): On the interface Σ we obtain

$$\frac{\partial v_1(\varepsilon)}{\partial x_2}|_\Sigma = -\varepsilon^{\delta-\gamma}\frac{F}{2}(\frac{2x_2}{\varepsilon^\delta}-1)|_\Sigma - \varepsilon^{\delta-\gamma}\frac{F}{2}\frac{\partial \beta_1^{bl}}{\partial y_2}|_{\Sigma, y=x/\varepsilon} + \varepsilon^{1-\gamma}\frac{F}{2}C_1^{bl}$$
$$\text{and} \quad \frac{v_1(\varepsilon)}{\varepsilon} = -\beta_1^{bl}(x_1/\varepsilon, 0)\varepsilon^{\delta-\gamma}\frac{F}{2}.$$

After averaging over Σ with respect to y_1, we obtain the Saffman version of the law by Beavers and Joseph

$$u_1^{eff} = -\varepsilon C_1^{bl}\frac{\partial u_1^{eff}}{\partial x_2} + O(\varepsilon^{2-\gamma}) \quad \text{on} \quad \Sigma, \tag{54}$$

where u_1^{eff} is the average of $v_1(\varepsilon)$ over the characteristic pore opening at the naturally permeable wall. The higher order terms are neglected. Nevertheless, for γ close to 1 the Beavers and Joseph slip law isn't satisfactory any more.

Next, the variational equation for $\{\mathcal{U}^\varepsilon, \mathcal{P}^\varepsilon\}$ reads

$$\int_{\Omega^\varepsilon} \varepsilon^\gamma \nabla\mathcal{U}^\varepsilon : \nabla\varphi \, dx + \int_{\Omega^\varepsilon}\Big((\mathcal{U}^\varepsilon\nabla)\mathcal{U}^\varepsilon + (\mathcal{U}^\varepsilon\nabla)\mathbf{v}(\varepsilon) + (\mathbf{v}(\varepsilon)\nabla)\mathcal{U}^\varepsilon\Big)\varphi \, dx$$
$$- \int_{\Omega^\varepsilon} \mathcal{U}^\varepsilon \text{ div } \varphi \, dx = -\int_{\Omega^\varepsilon} (\mathbf{v}(\varepsilon)\nabla)\mathbf{v}(\varepsilon)\varphi \, dx - \int_\Sigma \varepsilon\varphi_1\frac{F}{2}C_1^{bl} \, dS, \; \forall\varphi \in W^\varepsilon. \tag{55}$$

Note that $\mathcal{U}^\varepsilon$ is divergence free and the approximation satisfies the outer boundary conditions. In analogy with Proposition 4, pages 1120-1121, from [11] we have

Theorem 3.2. *Let us suppose the hypotheses (**H1**)-(**H2**) and let $\mathcal{U}^\varepsilon$ and $\mathcal{P}^\varepsilon$ be defined by (53). Then, the following estimates hold*

$$\varepsilon\|\nabla\mathcal{U}^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|\mathcal{U}^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \varepsilon^{1/2}\|\mathcal{U}^\varepsilon\|_{L^2(\Sigma)^2} +$$
$$\varepsilon^{1-\delta}\|\mathcal{U}^\varepsilon\|_{L^2(\Omega_1^{\varepsilon,\delta})^2} \le C\varepsilon^{5/2-\gamma}. \tag{56}$$

Proof. We test (55) by $\mathcal{U}^\varepsilon$. Since div $\mathcal{U}^\varepsilon = 0$, $\mathcal{P}^\varepsilon$ is eliminated from the equality. Next, arguing as in the proof of Proposition 1, we see that under assumptions (**H1**)-(**H2**) the viscous terms controls the inertia terms. Therefore, it remains to estimate the forcing term and the interface term, coming from the counterflow. We have

$$(\mathbf{v}(\varepsilon)\nabla)\mathbf{v}(\varepsilon) = -\frac{F}{2}\varepsilon^{\delta+1-\gamma}\Bigg(\Big(-\varepsilon^{2\delta-\gamma}\frac{F}{2}\frac{x_2^+}{\varepsilon^\delta}(\frac{x_2}{\varepsilon^\delta}-1) + \frac{F}{2}\varepsilon^{\delta+1-\gamma}C_1^{bl}\frac{x_2^+}{\varepsilon^\delta}$$
$$-\frac{F}{2}\varepsilon^{\delta+1-\gamma}\beta_1^{bl}(\frac{x}{\varepsilon})\Big)\frac{\partial\beta^{bl}(\frac{x}{\varepsilon})}{\partial x_1} - \frac{F}{2}\varepsilon^{\delta+1-\gamma}\beta_2^{bl}(\frac{x}{\varepsilon})\frac{\partial\beta^{bl}(\frac{x}{\varepsilon})}{\partial x_2} +$$
$$\beta_2^{bl}(\frac{x}{\varepsilon})\mathbf{e}^1\frac{\partial}{\partial x_2}\big(-\varepsilon^{2\delta-\gamma}\frac{F}{2}\frac{x_2^+}{\varepsilon^\delta}(\frac{x_2}{\varepsilon^\delta}-1) + \frac{F}{2}\varepsilon^{\delta+1-\gamma}C_1^{bl}\frac{x_2^+}{\varepsilon^\delta}\big)\Bigg).$$

Since $\nabla_y\beta^{bl}$ decays exponentially in y_2 and the functions of x_2 behave as $x_2\varepsilon^{-\delta}$ for small x_2, we obtain

$$|\int_{\Omega^\varepsilon}\varepsilon^{3\delta+1-2\gamma}\frac{x_2^+}{\varepsilon^\delta}(\frac{x_2}{\varepsilon^\delta}-1)\frac{\partial\beta^{bl}(\frac{x}{\varepsilon})}{\partial x_1}\mathcal{U}^\varepsilon \, dx| =$$
$$|\int_{\Omega^{1,\delta}}\varepsilon^{3\delta+1-2\gamma}\frac{x_2^+}{\varepsilon^\delta}(\frac{x_2}{\varepsilon^\delta}-1)\frac{\partial\mathcal{U}^\varepsilon}{\partial x_1}(\beta^{bl}(\frac{x}{\varepsilon}) - (C_1^{bl},0)) \, dx| \le C\varepsilon^{2\delta-2\gamma+5/2}||\nabla\mathcal{U}^\varepsilon||_{L^2(\Omega^\varepsilon)^4} \tag{57}$$

Next in the term

$$\varepsilon^{\delta+1-\gamma}\int_{\Omega^\varepsilon}\beta_2^{bl}(\frac{x}{\varepsilon})\mathbf{e}^1\frac{\partial}{\partial x_2}\big(-\varepsilon^{2\delta-\gamma}\frac{F}{2}\frac{x_2}{\varepsilon^\delta}(\frac{x_2}{\varepsilon^\delta}-1)+\frac{F}{2}\varepsilon^{\delta+1-\gamma}C_1^{bl}\frac{x_2}{\varepsilon^\delta}\big)\varphi\ dx$$

we perform first the integration by parts with respect to x_2 and then use the incompressibility and the integration by parts with respect to x_1 to get the bound (57) also for it.

The leading order terms in $(\mathbf{v}(\varepsilon)\nabla)\mathbf{v}(\varepsilon)$ turns to be

$$\frac{F^2}{4}\varepsilon^{2\delta+2-2\gamma}(\beta^{bl}(\frac{x}{\varepsilon})\nabla_x)\beta^{bl}(\frac{x}{\varepsilon}),$$

which is estimated as

$$|\frac{F^2}{4}\varepsilon^{2\delta+2-2\gamma}\int_{\Omega^\varepsilon}(\beta^{bl}(\frac{x}{\varepsilon})\nabla_x)\beta^{bl}(\frac{x}{\varepsilon})\ dx|\le C\varepsilon^{3\delta-2\gamma+3/2}||\nabla\mathcal{U}^\varepsilon||_{L^2(\Omega^\varepsilon)^4}.$$

The above estimates yield

$$|\int_{\Omega^\varepsilon}(\mathbf{v}(\varepsilon)\nabla)\mathbf{v}(\varepsilon)\mathcal{U}^\varepsilon\ dx|\le C\varepsilon^{3\delta-2\gamma+3/2}||\nabla\mathcal{U}^\varepsilon||_{L^2(\Omega^\varepsilon)^4} \tag{58}$$

$$|\int_\Sigma\varepsilon\mathcal{U}_1^\varepsilon\frac{F}{2}C_1^{bl}\ dS|\le C\varepsilon^{3/2}||\nabla\mathcal{U}^\varepsilon||_{L^2(\Omega^\varepsilon)^4}. \tag{59}$$

Applying Lemma 3.1 yields the estimate (56). □

Before getting to the inertia term, it remains to correct the shear jump term $-\int_\Sigma\varepsilon\varphi_1\frac{F}{2}C_1^{bl}\ dS$. Only difference with correcting the jump term from equation (23) is that that it is now of order ε, instead of being of order ε^δ. Furthermore, $F/2$ is replaced by $-FC_1^{bl}/2$. We eliminate it by modifying slightly the velocity and pressure corrections:

Corollary 2. *Let assumptions (***H1***)-(***H3***) hold, and $\mathcal{U}^\varepsilon$, $\mathcal{P}^\varepsilon$ be defined by (53). Let*

$$\mathcal{U}^{1,\varepsilon}=\mathcal{U}^\varepsilon-\frac{F}{2}C_1^{bl}\varepsilon^{2-\gamma}\beta^{bl}(\frac{x}{\varepsilon})+\frac{F}{2}\varepsilon^{2-\gamma}(C_1^{bl})^2\frac{x_2^+}{\varepsilon^\delta}\mathbf{e}^1, \tag{60}$$

Then, the following estimate holds

$$\varepsilon\|\nabla\mathcal{U}^{1,\varepsilon}\|_{L^2(\Omega^\varepsilon)^4}+\|\mathcal{U}^{1,\varepsilon}\|_{L^2(\Omega_2^\varepsilon)^2}+\varepsilon^{1/2}\|\mathcal{U}^{1,\varepsilon}\|_{L^2(\Sigma)^2}+ \\ \varepsilon^{1-\delta}\|\mathcal{U}^{1,\varepsilon}\|_{L^2(\Omega_1^{\varepsilon,\delta})^2}\le C\varepsilon^{5/2+3\delta-3\gamma}. \tag{61}$$

The new shear stress jump term generated by correction (60) is given by

$$-\int_\Sigma\varepsilon^{2-\delta}\varphi_1\frac{F}{2}(C_1^{bl})^2\ dS.$$

Then, the corresponding estimate (59) in the proof of Theorem 3.2 takes the form

$$|\int_\Sigma\varepsilon^{2-\delta}\mathcal{U}_1^\varepsilon\frac{F}{2}(C_1^{bl})^2\ dS|\le C\varepsilon^{5/2-\delta}||\nabla\mathcal{U}^\varepsilon||_{L^2(\Omega^\varepsilon)^4}. \tag{62}$$

Due to hypothesis (**H3**), we have $5/2-\delta>3\delta-2\gamma+3/2$ and the new error terms are less important than the leading inertia terms.

Finally, we correct the inertia term effects. We note that it is multiplied by a small parameter $\varepsilon^{\delta-2\gamma+2}$. We follow the idea from [5] and expand the solutions to the nonlinear boundary layer problem (30)-(33) in powers of that parameter. As already explained in the beginning of the section, the solutions of 30)-(33) take the form $\{\beta^0 + \frac{F}{2}\varepsilon^{\delta-2\gamma+2}\beta^1 + \dots, \pi^0 + \frac{F}{2}\varepsilon^{\delta-2\gamma+2}\pi^1 + \dots\}$. Furthermore, the 1-periodicity of the geometry in y_1-direction allows to replace β^0 by β^{bl}. It is similar with β^1. We recall that the leading error term for $\mathcal{U}^{1,\varepsilon}$ results from $(\beta^{bl}\nabla)\beta^{bl}$. We introduce the boundary layer problem for $\beta^{1,bl}$:

$$-\triangle_y \beta^{1,bl} + \nabla_y \pi^{1,bl} = (\beta^{bl}\nabla_y)\beta^{bl} \qquad \text{in} \quad Z_{BL}, \tag{63}$$

$$\text{div}_y \beta^{1,bl} = 0 \qquad \text{in} \quad Z_{BL}, \tag{64}$$

$$\nabla_y \beta^{1,bl} \in L^2(Z_{BL})^4 \quad \text{and} \quad \beta^{1,bl} \in L^2_{loc}(Z_{BL})^2, \tag{65}$$

$$\beta^{1,bl} = 0 \quad \text{on} \cup_{k=1}^{\infty} (\partial Y_s - \{0,k\}) \quad \text{and} \quad \{\beta^{1,bl}, \pi^{1,bl}\} \text{ is } 1-\text{periodic in } y_1. \tag{66}$$

The forcing term decays exponentially. Following [10], we know that the system (63)-(66) describes a boundary layer, i.e. $\beta^{1,bl}$ and $\omega^{1,bl}$ stabilize exponentially towards $C^{bl}_{11}\mathbf{e}^1$ and $C_{\pi 1}$, when $|y_2| \to \infty$. Then, the correction reads

$$\mathcal{U}^{2,\varepsilon} = \mathcal{U}^{\varepsilon} - \frac{F}{2}C^{bl}_1 \varepsilon^{2-\gamma}\beta^{bl}(\frac{x}{\varepsilon}) + \frac{F}{2}\varepsilon^{2-\gamma}(C^{bl}_1)^2 \frac{x_2^+}{\varepsilon^\delta}\mathbf{e}^1 + \tag{67}$$

$$+(\frac{F}{2})^2 \varepsilon^{2\delta+3-3\gamma}\beta^{1,bl}(\frac{x}{\varepsilon}) - (\frac{F}{2})^2\varepsilon^{2\delta+3-3\gamma}C^{bl}_{11}\frac{x_2^+}{\varepsilon^\delta}\mathbf{e}^1, \tag{68}$$

In complete analogy with Theorem 3.2 we prove Theorem 2.1.

To obtain estimate (8) from Theorem 2.1, it is enough to note that after (57), the leading remaining inertia terms give a contribution bounded by

$$C\varepsilon^{2\delta+5/2-2\gamma}||\nabla \mathcal{U}^{2,\varepsilon}||_{L^2(\Omega^\varepsilon)^4}$$

Next, using hypothesis (**H1**), we obtain that $5/2-\delta < 2\delta - 2\gamma + 5/2$. Furthermore, the leading order term is the shear stress jump term

$$\int_\Sigma \varepsilon^{2-\delta}\varphi_1 \frac{F}{2}(C^{bl}_1)^2 \; dS.$$

It is estimated by (62), which yields (8).

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