On the justification of the Reynolds equation, describing isentropic compressible flows through a tiny pore

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Abstract: We consider the isentropic compressible flow through a tiny pore. Our approach is to adapt the recent results by N. Masmoudi on the homogenization of compressible flows through porous media to our situation. The major difference is in the a priori estimates for the pressure field. We derive the appropriate ones and then the Masmoudi's results allow to conclude the convergence. In this way the compressible Reynolds equation in the lubrication theory is rigorously justified.

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1 Introduction

We consider the following semi-stationary model for isentropic compressible flows through the rectangular tiny pore $\Omega^\varepsilon = (0,1)\times(0,\varepsilon)$:

$$\varepsilon^2 \partial_t \rho_\varepsilon + \text{ div } (\rho_\varepsilon u_\varepsilon) = 0 \quad \text{in} \quad \Omega^\varepsilon \times (0,T) \tag{1}$$

$$-\mu \Delta u_\varepsilon - \xi \nabla \text{ div } u_\varepsilon + \nabla p_\varepsilon = \rho_\varepsilon f + g \quad \text{in} \quad \Omega^\varepsilon \times (0,T) \tag{2}$$

$$p_\varepsilon = \rho_\varepsilon^\gamma,\ \rho_\varepsilon \geq 0,\ \gamma > 1 \quad \text{in} \quad \Omega^\varepsilon \times (0,T) \tag{3}$$

$$u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega^\varepsilon \times (0,T) \tag{4}$$

$$\rho_\varepsilon|_{t=0} = \rho_{\varepsilon 0}, \quad \rho_\varepsilon u_\varepsilon|_{t=0} = m_{\varepsilon 0} \quad \text{in} \quad \Omega^\varepsilon \tag{5}$$

Here u_ε is the fluid velocity, p_ε the pressure and ρ_ε the density. μ and ξ are the viscosities, supposed to be positive constants. For simplicity we suppose the forces f and g independent of x_2, $f, g \in C^\infty([0,T]\times[0,1])^2$. Furthermore, we suppose $\rho_{\varepsilon 0} = \rho^0$ non-negative, independent of x_2 and being an element of $C^\infty[0,1]$. Finally, initial momentum $m_{\varepsilon 0}$ is for simplicity supposed to be equal to zero.

The system (1)-(5) describes an isentropic compressible flow, with negligible Reynolds and Strouhal numbers. For any given $\varepsilon > 0$, its theory is developed in [17] . Here we are interested in finding the asymptotic behavior when $\varepsilon \to 0$.

The corresponding homogenization for the porous media case is in the paper [19] by N. Masmoudi. Here we are concerned with obtaining the lubrication approximation. The system (1)-(5) is expected to give the 1D compressible Reynolds equation in the limit $\varepsilon \to 0$.

Clearly, the geometry is now much simpler than in the case of a porous medium and in the incompressible case passing to the limit was, in some sense, the simplified version of the obtention of the Darcy law for the filtration through porous media. Only difference was the anisotropy. It leads to better estimates for the velocity, since we control also the derivative in the vertical direction. Consequence is that the estimates for the pressure, through a duality argument, are in dual space of an anisotropic Sobolev space. In the incompressible case it did not matter, since the weak convergence in pressure was sufficient. Rigorous justification of Reynolds' equation for incompressible viscous flows through tiny domains, using weak convergences and dimension reduction, was undertaken in [13, 12, 5].

In the compressible case, the corresponding homogenization proof from [19] is derived essentially using a kind of " compensated compactness " for pressure/density. If one uses the classical dimension reduction of Ciarlet et al (see e.g. [11]), the rescaled u_ε is defined on $\Omega = \Omega^1$ and depends on $x_1, z = x_2/\varepsilon$ and t. Then the *a priori* estimate for the velocity is in the functional space $L^2(0,T;W)$, where

$$W = \{\varphi \in L^2(\Omega) \mid \frac{\partial \varphi}{\partial z} \in L^2(\Omega), \quad \varphi|_{z=0} = \varphi|_{z=1} = 0\} \tag{6}$$

Using duality, it is possible to obtain the *a priori* estimate for the rescaled pressure in $L^2((0,T)\times\Omega^1)$, but validity of the " compensated compactness " estimate in $L^2(0,T;H^1) + \varepsilon L^2((0,T)\times\Omega^1)$ from [19], necessary for the convergence, is not clear.

This motivates us to reduce our problem to a porous medium flow. Simplest porous media are bundles of parallel tubes. Nevertheless, the results from [19] are not directly applicable, since our porous medium is not *connected*. In the next section we generalize the proof from [19] to our situation and rigorously justify the 1D compressible Reynolds equation appearing in the lubrication theory.

2 Rigorous justification of the compressible Reynolds equation

2.1 *A priori* estimate for the velocity and the density

This estimate follows the calculations from [19]. We note that all constants depend on T. We have

Proposition 1. *Let $\gamma \geq 2$, $\varepsilon > 0$ and let $\{u_\varepsilon, p_\varepsilon, \rho_\varepsilon\}$ be a variational solution to the system (1)-(5). Then we have*

$$\sup_{0\leq t\leq T} \int_{\Omega^\varepsilon} \rho_\varepsilon^\gamma(t)\ dx + \mu \int_0^T \int_{\Omega^\varepsilon} |\nabla \frac{u_\varepsilon}{\varepsilon}|^2\ dxdt +$$
$$\xi \int_0^T \int_{\Omega^\varepsilon} |\ div\ \frac{u_\varepsilon}{\varepsilon}|^2\ dxdt \leq C\varepsilon \big\{ \|f\|^2_{L^\infty(\Omega^\varepsilon)} + \|g\|^2_{L^\infty(\Omega^\varepsilon)} \big\} \tag{7}$$

Proof. As in [19], first we test the equation (2) by u_ε and get

$$\int_{\Omega^\varepsilon} \Big(\mu |\nabla u_\varepsilon|^2 + \xi |\ \text{div}\ u_\varepsilon|^2 \Big)\ dx + \gamma \int_{\Omega^\varepsilon} \rho_\varepsilon^{\gamma-1} \nabla \rho_\varepsilon u_\varepsilon\ dx = \int_{\Omega^\varepsilon} \Big(\rho_\varepsilon f + g \Big) u_\varepsilon\ dx. \tag{8}$$

Then the equation (1) is tested by $\gamma\rho_\varepsilon^{\gamma-1}$, leading to

$$\int_{\Omega^\varepsilon} \gamma\rho_\varepsilon^{\gamma-1}\nabla\rho_\varepsilon u_\varepsilon \; dx = -\int_{\Omega^\varepsilon}\Big(\varepsilon^2\partial_t\rho_\varepsilon + \text{ div } u_\varepsilon\rho_\varepsilon\Big)\gamma\rho_\varepsilon^{\gamma-1}\; dx =$$
$$-\varepsilon^2\partial_t\int_{\Omega^\varepsilon}\rho_\varepsilon^\gamma \; dx + \int_{\Omega^\varepsilon} u_\varepsilon\gamma\nabla\rho_\varepsilon^\gamma \; dx, \tag{9}$$

implying

$$\int_{\Omega^\varepsilon}\gamma(1-\gamma)\rho_\varepsilon^{\gamma-1}\nabla\rho_\varepsilon u_\varepsilon \; dx = -\varepsilon^2\partial_t\int_{\Omega^\varepsilon}\rho_\varepsilon^\gamma \; dx. \tag{10}$$

After inserting (10) into (8), we get the energy equality

$$\mu\int_{\Omega^\varepsilon}|\nabla u_\varepsilon|^2 \; dx + \xi\int_{\Omega^\varepsilon}| \text{ div } u_\varepsilon|^2 \; dx + \frac{\varepsilon^2}{\gamma-1}\partial_t\int_{\Omega^\varepsilon}\rho_\varepsilon^\gamma \; dx =$$
$$\int_{\Omega^\varepsilon}\rho_\varepsilon f u_\varepsilon \; dx + \int_{\Omega^\varepsilon} g u_\varepsilon \; dx. \tag{11}$$

Next Poincaré's inequality gives

$$\int_{\Omega^\varepsilon} u_\varepsilon^2 \; dx \le \frac{\varepsilon^2}{2}\int_{\Omega^\varepsilon}|\frac{\partial u_\varepsilon}{\partial x_2}|^2 \; dx. \tag{12}$$

and using (12) and Hölder's inequality, we estimate the right hand side in (11) as

$$|\int_{\Omega^\varepsilon}\rho_\varepsilon f u_\varepsilon \; dx| \le \frac{\mu}{\varepsilon^2}\int_{\Omega^\varepsilon} u_\varepsilon^2 \; dx + \frac{\varepsilon^2\|f\|^2_{L^\infty(\Omega^\varepsilon)}}{4\mu}\int_{\Omega^\varepsilon}\rho_\varepsilon^2 \; dx \le$$
$$\frac{\mu}{2}\int_{\Omega^\varepsilon}|\nabla u_\varepsilon|^2 \; dx + \frac{\varepsilon^3\|f\|^2_{L^\infty(\Omega^\varepsilon)}}{4\mu}\frac{\gamma-2}{\gamma} + \frac{\varepsilon^2\|f\|^2_{L^\infty(\Omega^\varepsilon)}}{2\mu\gamma}\int_{\Omega^\varepsilon}\rho_\varepsilon^\gamma \; dx. \tag{13}$$

After inserting (13) into (11) and applying Gronwall's lemma we obtain (7). We note that the constant in (7) grows in time with exponential rate. □

Remark 2. *We note that, as in [19] , the estimate for* $1 < \gamma < 2$ *requires the pressure estimate.*

2.2 *A priori* estimate for the pressure

Next step is to obtain an a priori estimate for the pressure $p_\varepsilon = \rho_\varepsilon^\gamma$. Working on Ω^ε isn't suitable any more, and if one uses the classical dimension reduction of Ciarlet et al, the rescaled p_ε is defined on $\Omega = \Omega^1$ and depends on $x_1, z = x_2/\varepsilon$ and t. But then, as already said, the *a priori* estimate for the velocity is in the functional space $L^2(0,T;W)$, where W is given by (6). By the classical duality argument (see e.g. [6]), it is possible to obtain the *a priori* estimate for the rescaled pressure in $L^2((0,T)\times\Omega^1)$, but Masmoudi's argument requires a " compensated compactness " estimate in $L^2(0,T;H^1) + \varepsilon L^2((0,T)\times\Omega^1)$.

The remedy is to embed our problem into a porous medium setting.

Let $Y = (0,1)\times(-1/2,3/2)$ and $Y_F = (0,1)\times(0,1)$. We consider the porous medium $\Omega_U = (0,1)\times(-1,1)$, with the fluid part

$$\Omega^\varepsilon_F = \left((0,1) \times \cup_{k\in\mathbb{Z}} \left\{ (0,\varepsilon) + 2k\varepsilon\vec{e}_2 \right\} \right) \cap \Omega \tag{14}$$

and the solid part $\Omega^\varepsilon_S = \Omega_U \setminus \overline{\Omega^\varepsilon}_F$.

We note that in our situation the fluid part **is not connected**, which means that the solution for $\forall \varepsilon > 0$ is a periodic repetition of the solution in Ω^ε. Hence we are mostly in the setting of [19] .

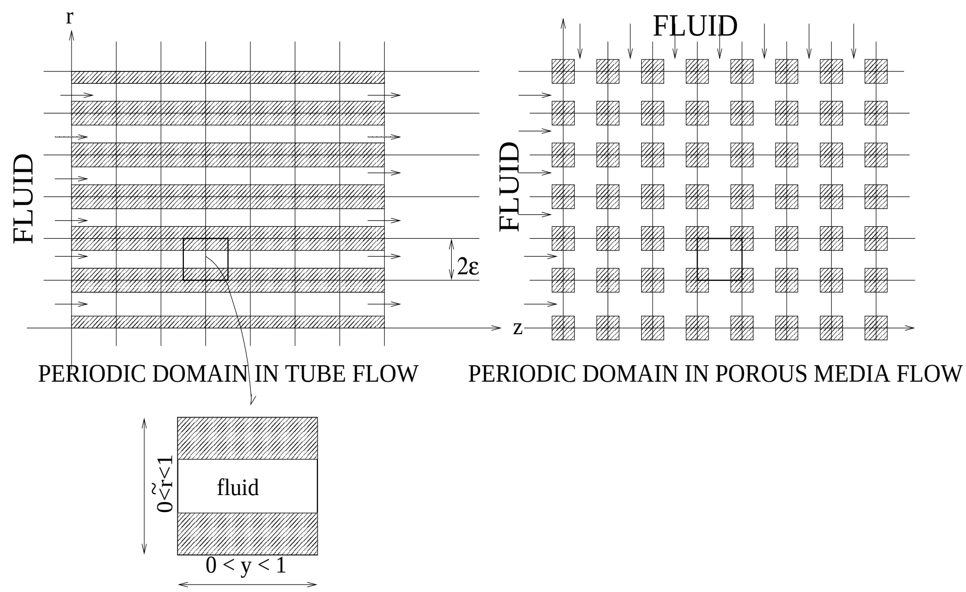

Figure 1: Homogenization domains for repeated tiny tubes and for porous media.

Nevertheless, the non-connectedness leads to complications. In order to apply the results from [19], we should revisit the construction of the Tartar's restriction operator and the ellipticity of the permeability tensor. First, the classical method of extending pressure in optimal way is to use the Tartar's construction from [22] (which is the appendix to [21]). This construction gives a restriction operator from the whole porous medium to its fluid part and it preserves the incompressiblity. Then the pressure is estimated through a duality argument. For the generalization of the construction to 3D geometries and detailed discussions, we refer to [2] and [3] . Unfortunately, these constructions relie on the connectivity of the fluid part of the porous medium. For porous media with non-connected pores, the construction of the restriction operator R_ε doesn't seem possible.

We will proceed differently and generalize an idea of V. Zhikov, published in [23] and further developed in [20] . We do not use any more the construction of an extension operator for connected sets from [1] and we proceed differently, by imposing more contraints.

We start by generalizing the Lemma 2.1, pages 133-137, from [20] . We have the following result

Proposition 3. *Let $\kappa \in L^2(\Omega_U)$, such that supp $\kappa \subset \Omega_F^\varepsilon$ and $\int_{\Omega^\varepsilon_{F,k}} \kappa \, dx = 0$, $\forall k \in \mathbb{Z}$, where $\Omega^\varepsilon_{F,k} = (0,1) \times (2k\varepsilon, (2k+1)\varepsilon)$. Then we have*

a) *The function κ admits the representation $\kappa =$ div F^ε in Ω, with*

$$F^\varepsilon \in L^2(\Omega_U)^2, \qquad \int_{\Omega_U} |F^\varepsilon|^2 \, dx \leq C \sum_k \|\kappa\|^2_{(H^1(\Omega^\varepsilon_{F,k}))'} \tag{15}$$

b) *Furthermore, $F^\varepsilon \in H^1(\Omega_U)^2$ and*

$$\int_{\Omega_U} |\nabla F^\varepsilon|^2 \, dx \leq \frac{C_1}{\varepsilon^2} \Big\{ \sum_k \|\kappa\|^2_{(H^1(\Omega^\varepsilon_{F,k}))'} + \varepsilon^2 \int_{\Omega^\varepsilon_F} \kappa^2 \, dx \Big\} \tag{16}$$

Proof. It follows the lines of the proof of Lemma 2.1. from [20] .

Step 1. Let u^ε_k be the solution for

$$-\Delta u^\varepsilon_k = \kappa \ \text{ in } \ \Omega^\varepsilon_{F,k}; \quad \frac{\partial u^\varepsilon_k}{\partial \nu} = 0 \ \text{ on } \partial \Omega^\varepsilon_{F,k} \tag{17}$$

Since $\kappa \in L^2(\Omega^\varepsilon_{F,k})$, and $\int_{\Omega^\varepsilon_{F,k}} \kappa \, dx = 0$, the problem (17) has a unique solution $u^\varepsilon_k \in H^1(\Omega^\varepsilon_{F,k}), \int_{\Omega^\varepsilon_{F,k}} u^\varepsilon_k \, dx = 0$. Let $F^\varepsilon = \nabla u^\varepsilon_k$ in $\Omega^\varepsilon_{F,k}$ and 0 elsewhere. The function F^ε satisfies the estimate (15) and has the properties required in **a)**.

Step 2. Now we consider a smooth partition of the unity $\{E^j, \psi^j\}$, related to $I\!R_+ \times (0,1)$. Let $\psi^j_{\varepsilon,k}(x) = \psi^j(x_1/\varepsilon, x_2/\varepsilon - 2k)$. Then $\{E^j_{\varepsilon,k}, \psi^j_{\varepsilon,k}\}$, $E^j_{\varepsilon,k} = \varepsilon E^j + 2k\vec{e}_2\varepsilon$, is the decomposition of unity related to $\Omega^\varepsilon_{F,k}$. We have

$$\kappa = \text{ div } F^\varepsilon = \sum_j \text{ div } (\psi^j_{\varepsilon,k} F^\varepsilon) = \sum_j \kappa^\varepsilon_j.$$

Furthermore, since $F^\varepsilon \cdot \nu = 0$ on $\partial\Omega^\varepsilon_{F,k}$ and $\psi^\varepsilon_j = 0$ on $\Omega^\varepsilon_{F,k} \cap \partial E^j_{\varepsilon,k}$, we have $\int_{E^j_{\varepsilon,k}} \kappa^\varepsilon_j = 0$.

Now we use the surjectivity of the divergence operator and conclude that there is a vector field $z^j_\varepsilon \in H^1_0(E^j_{\varepsilon,k})^2$ such that

$$\left\{ \begin{array}{l} \text{div}_x z^j_\varepsilon = \kappa^\varepsilon_j \quad \text{ in } E^j_{\varepsilon,k} \quad \text{ and } , \\ \int_{E^j_{\varepsilon,k}} |\nabla_x z^j_\varepsilon|^2 \, dx \le C_0 \varepsilon^{-2} \int_{E^j_{\varepsilon,k}} \Big\{ |F^\varepsilon|^2 + \varepsilon^2 |\kappa|^2 \Big\} \, dx, \end{array} \right. \tag{18}$$

For detailed calculations we refer to [20], page 135.

Step 3. We set $z^\varepsilon|_{\Omega^\varepsilon_{F,k}} = \sum_j z^j_\varepsilon$. Then z^ε satisfies (15)-(16). □

Now we will estimate the pressure without using the restriction operator R_ε.

Proposition 4. *(L^2- a priori estimate for the pressure field). Let $\{u_\varepsilon, p_\varepsilon, \rho_\varepsilon\}$ be the solution for the problem (1)-(5), extended by periodicity to $(0,T) \times \Omega^\varepsilon_F$. Then we have*

$$\|p_\varepsilon - \frac{1}{\varepsilon} \sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2) \int_0^1 \int_{2k\varepsilon}^{(2k+1)\varepsilon} p_\varepsilon \, dx \|_{L^2(\Omega^\varepsilon_F \times (0,T))} \le C \tag{19}$$

Proof. We have

$$\int_0^T \int_{\Omega^\varepsilon_F} |p_\varepsilon - \frac{1}{\varepsilon} \sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2) \int_0^1 \int_{2k\varepsilon}^{(2k+1)\varepsilon} p_\varepsilon \, dx|^2 dx_1 dx_2 dt = \int_0^T \int_{\Omega^\varepsilon_F} p_\varepsilon (p_\varepsilon$$
$$- \frac{1}{\varepsilon} \sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2) \int_0^1 \int_{2k\varepsilon}^{(2k+1)\varepsilon} p_\varepsilon \, dx) dx_1 dx_2 dt = \int_0^T \int_{\Omega^\varepsilon_F} p_\varepsilon \text{ div } F^\varepsilon \, dx dt$$

where F^ε satisfies (15)-(16) with

$$\kappa = p_\varepsilon - \frac{1}{\varepsilon} \sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2) \int_0^1 \int_{2k\varepsilon}^{(2k+1)\varepsilon} p_\varepsilon \, dx$$

Hence

$$\int_0^T \int_{\Omega_F^\varepsilon} p_\varepsilon \text{ div } F^\varepsilon \, dxdt = -\int_0^T < \nabla p_\varepsilon, F^\varepsilon >_{\Omega_F^\varepsilon} \, dt =$$

$$\int_0^T \int_{\Omega_F^\varepsilon} \Big(\mu \nabla u_\varepsilon \nabla F^\varepsilon + \xi \text{ div } u_\varepsilon \text{ div } F^\varepsilon - (\rho_\varepsilon f + g) F^\varepsilon \Big) \, dxdt. \tag{20}$$

Using (7) we conclude that

$$\int_0^T \int_{\Omega_F^\varepsilon} |\nabla \frac{u_\varepsilon}{\varepsilon}|^2 \, dxdt \leq C \tag{21}$$

and after insertion of (21) into (20), (19) follows. □

Now let us obtain the pressure estimate as in [19] . First we note that the pressure average

$$\int_{\Omega_{F,k}^\varepsilon} p_\varepsilon = \int_{\Omega^\varepsilon} p_\varepsilon = \int_{\Omega^\varepsilon} \rho_\varepsilon^\gamma$$

is uniformly bounded by the *a priori* estimate (7). Hence it is enough to deal with $p_\varepsilon - \frac{1}{\varepsilon} \sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2) \int_0^1 \int_{2k\varepsilon}^{(2k+1)\varepsilon} p_\varepsilon \, dx$. We have

Theorem 5.

$$\| p_\varepsilon \|_{L^2(0,T;H^1(\Omega_F^\varepsilon)) + \varepsilon L^2((0,T)\times \Omega_F^\varepsilon)} \leq C \tag{22}$$

Proof. Let $\kappa \in L^2(\Omega_F^\varepsilon)$. Then we have

$$\int_{\Omega_F^\varepsilon} (p_\varepsilon - \frac{1}{\varepsilon} \sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2) \int_0^1 \int_{2k\varepsilon}^{(2k+1)\varepsilon} p_\varepsilon \, dy) \kappa \, dx =$$

$$\int_{\Omega_F^\varepsilon} p_\varepsilon (\kappa - \frac{1}{\varepsilon} \sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2) \int_0^1 \int_{2k\varepsilon}^{(2k+1)\varepsilon} \kappa \, dy)$$

Since $\kappa - \frac{1}{\varepsilon} \sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2) \int_0^1 \int_{2k\varepsilon}^{(2k+1)\varepsilon} \kappa \, dx$ satisfies the assumptions of the Proposition 3. Consequently, there is $F^\varepsilon \in H^1(\Omega_F^\varepsilon)$ such that div $F^\varepsilon = \kappa - \frac{1}{\varepsilon} \sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2) \int_0^1 \int_{2k\varepsilon}^{(2k+1)\varepsilon} \kappa \, dy$ and the estimates (15)-(16) hold. Hence

$$| \int_{\Omega_F^\varepsilon} (p_\varepsilon - \frac{1}{\varepsilon} \sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2) \int_0^1 \int_{2k\varepsilon}^{(2k+1)\varepsilon} p_\varepsilon \, dy) \kappa \, dx| =$$

$$| - \int_{\Omega_F^\varepsilon} \nabla p_\varepsilon F^\varepsilon \, dx| \leq C \Big\{ \varepsilon \| \kappa \|_{L^2(\Omega_F^\varepsilon)} + \sqrt{\sum_k \| \kappa \|_{H^1(\Omega_{F,k}^\varepsilon)'} \|^2} \Big\}$$

and (22) follows. $\square$

Remark 6. *It would be more confortable to work with the pressure field p_ε defined on Ω_U. Obviously, it is optimal to extend*

$$p_\varepsilon - \frac{1}{\varepsilon}\sum_k \chi_{(2k\varepsilon,(2k+1)\varepsilon)}(x_2)\int_0^1\int_{2k\varepsilon}^{(2k+1)\varepsilon} p_\varepsilon\, dx$$

by zero to the solid part. This implies that we extend the pressure following the extension by Lipton and Avellaneda from [18] :

$$\hat{p}_\varepsilon(x,t)) = \begin{cases} p_\varepsilon(x,t) \ \ in \ \Omega_F^\varepsilon, \\ \frac{1}{\varepsilon}\int_0^1\int_{2k\varepsilon}^{(2k+1)\varepsilon} p_\varepsilon\, dx, \ for \ x_2/\varepsilon \in ((-1/2,0)\cup(1,3/2)) + 2k \end{cases} \tag{23}$$

Then $\hat{p}_\varepsilon$ also satisfies (22).

Remark 7. *By a slight change of the argument from [19] , we get the a priori s estimate (7) for $1 < \gamma < 2$. Then the estimate (22) is straightforward.*

2.3 The convergence proof

We proceed as in [19]. First we introduce 2 extensions: the extension by zero to the solid part of ϕ denoted by $\tilde{\phi}$ and the extension of Lipton and Avellaneda $\hat{\phi}$, given by (23). Then, following [19], we have

Lemma 8. *The extension $\tilde{\rho}_\varepsilon$ satisfies the equation*

$$\varepsilon^2\partial_t\tilde{\rho}_\varepsilon + \ \ div\ (\tilde{\rho}_\varepsilon\tilde{u}_\varepsilon) = 0 \quad in \quad I\!R^2. \tag{24}$$

Obtained *a priori* estimates, give the weak compactness and we have existence of cluster points $\{u,\rho,p\}$ such that

$$\begin{aligned} \tilde{\rho}_\varepsilon &\rightharpoonup \vartheta\rho \quad \text{weakly and weak-* in } L^{2\gamma}((0,T)\times\Omega_U)\cap L^\infty(0,T;L^\gamma(\Omega_U)); \\ \hat{\rho}_\varepsilon &\rightharpoonup \rho \quad \text{weakly and weak-* in } L^{2\gamma}((0,T)\times\Omega_U)\cap L^\infty(0,T;L^\gamma(\Omega_U)); \\ \hat{p}_\varepsilon &\rightharpoonup p \quad \text{weakly in } L^2((0,T)\times\Omega_U),\ p\in L^2(0,T;H^1(\Omega_U)); \\ \frac{\tilde{u}_\varepsilon}{\varepsilon^2} &\rightharpoonup u \quad \text{weakly in } L^2((0,T)\times\Omega_U)^2. \end{aligned} \tag{25}$$

Next, since $\tilde{\rho}_\varepsilon$ and $\tilde{u}_\varepsilon/\varepsilon^2$ are uniformly bounded in $L^2((0,T)\times\Omega_U$, lemma 8 implies that $\partial_t\tilde{\rho}_\varepsilon$ is uniformly bounded in $L^1(0,T;W^{-1,1}(\Omega_U))$. Then, exactly as in [19], we use Lemma 5.1 of [17] to conclude that $\tilde{\rho}_\varepsilon \hat{p}_\varepsilon$ converge to $\vartheta \rho p$ weakly. It implies that $\hat{\rho}_\varepsilon \hat{p}_\varepsilon$ converge to ρp weakly and the strong convergence of $\hat{\rho}_\varepsilon$ is concluded as in [19], page 895. We summarize the convergence results in the following proposition:

Proposition 9. *There are subsequences of $\{u_\varepsilon, \rho_\varepsilon, p_\varepsilon\}$ of the solutions and cluster points such that*

$$\tilde{\rho}_\varepsilon \to \vartheta \rho \quad \text{weakly in} \quad L^r(0,T;L^\gamma(\Omega_U)) \cap L^{2\gamma}((0,T)\times\Omega_U) \tag{26}$$

$$\hat{\rho}_\varepsilon \to \rho \quad \text{strongly in} \quad L^r(0,T;L^\gamma(\Omega_U)) \cap L^{\gamma+1}((0,T)\times\Omega_U) \tag{27}$$

$$\frac{\tilde{u}_\varepsilon}{\varepsilon^2} \to u \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega_U))^2 \tag{28}$$

for all $r < \infty$, where $\rho \in L^{2\gamma}((0,T)\times\Omega_U)$, $\rho^\gamma \in L^2(0,T;H^1(\Omega_U))$ and for all $\varphi \in C_0^\infty((-1,T)\times \mathbb{R}^2)$, ρ satisfies the equation

$$-\int_0^T \int_{\Omega_U} \vartheta \rho \partial_t \varphi - \int_0^T \int_{\Omega_U} \rho u \nabla \varphi = \int_{\Omega_U} \frac{1}{2}\rho^0|_{t=0}. \tag{29}$$

It remains to obtain the filtration law. Using the energy method and regularization of the pressure field, as in [19], pages 896-898, we have

Proposition 10. *The weak limit u of $\frac{\tilde{u}_\varepsilon}{\varepsilon^2}$, satifies the following filtration law*

$$\rho u = \frac{1}{\mu} A\left(\rho^2 f + \rho g - \frac{\gamma}{\gamma+1}\nabla \rho^{\gamma+1}\right) \tag{30}$$

where A is the permeability tensor given by

$$\begin{cases} A_{ij} = \frac{1}{2}\int_{Y_F} v^i_j(y)\; dy_1 dy_2, \; i,j = 1,2; \\ -\Delta v^i + \nabla q^i = \vec{e}_i \;\; \text{in} \;\; Y_F \\ \; div\; v^i = 0 \;\; \text{in} \;\; Y_F, \quad v^i = 0 \;\; \text{on} \;\; y_2 = 0,1 \\ \{v^i, q_i\} \;\; \text{are} \;\; Y-\text{periodic.} \end{cases} \tag{31}$$

Contrary to the situation in [19], the permeability tensor is not positive definite. Since our geometry is very simple, let us calculate it.

- For $i = 1$, we have $v^1 = (\frac{1}{2}y_2(1-y_2), 0)$ and $q_1 = 0$.
- For $i = 2$, we have $v^2 = (0,0)$ and $q_2 = y_2$.

Consequently,

$$A = \frac{1}{24}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{32}$$

Corollary 11. *The filtration law (30) reads*

$$\begin{cases} \rho u_1 = \dfrac{1}{24\mu}\left(\rho^2 f_1 + \rho g_1 - \dfrac{\gamma}{\gamma+1}\dfrac{\partial}{\partial x_1}\rho^{\gamma+1}\right) \quad in \ (0,1)\times(0,T), \\ \rho u_2 = 0 \quad on \ (0,1)\times(0,T). \end{cases} \tag{33}$$

Let us summarize our results

Theorem 12. *The cluster point for the density, ρ, is solution to the problem*

$$\frac{\partial \rho}{\partial t} - \frac{\gamma}{12\mu(\gamma+1)}\frac{\partial^2}{\partial x_1^2}\rho^{\gamma+1} = -\frac{1}{12\mu}\frac{\partial}{\partial x_1}(\rho^2 f_1 + \rho g_1) \quad in \ (0,1)\times(0,T) \tag{34}$$

$$24\mu\rho u_1 = \rho^2 f_1 + \rho g_1 - \frac{\gamma}{\gamma+1}\frac{\partial}{\partial x_1}\rho^{\gamma+1} = 0 \quad for \ \ x_1 = 0,1,\ t\in(0,T) \tag{35}$$

$$\rho|_{t=0} = \rho^0 \quad in \ (0,1) \tag{36}$$

The filtration velocity u is calculated using (33) and $p = \rho^\gamma$.

Theorem 13. *Let $g_1 = 0$ for $\gamma < 2$. Then problem (34)-(36) has a unique solution. Consequently,*

$$\frac{1}{\varepsilon}\int_0^T \int_{\Omega^\varepsilon} |\rho_\varepsilon - \rho|^\gamma \, dx \to 0 \quad as \ \varepsilon \to 0 \tag{37}$$

$$\frac{1}{\varepsilon}\int_0^T \int_{\Omega^\varepsilon} \frac{u_\varepsilon}{\varepsilon^2}\varphi(x_1,t)\, dxdt \to \int_0^T\int_0^1 u\varphi \, dx_1 dt, \ \ as \ \varepsilon \to 0,$$
$$\forall \varphi \in C_0^\infty((0,1)\times(0,T)) \tag{38}$$

Proof. Let $w = \rho^{1+\gamma}$. Then $\partial_{x_1} w \in L^{2\gamma/(1+\gamma)}((0,1)\times(0,T))$ and $w \in L^{2\gamma/(1+\gamma)}(0,T; L^\infty(0,1))$.

Next we write problem (34)-(36) as

$$\begin{cases} \dfrac{\partial}{\partial t} w^{1/(1+\gamma)} - \dfrac{\gamma}{12\mu(\gamma+1)}\dfrac{\partial^2}{\partial x_1^2} w = \dfrac{\partial}{\partial x_1} F \ \text{ in } \ (0,1)\times(0,T), \\ -F - \dfrac{\gamma}{12\mu(\gamma+1)}\dfrac{\partial}{\partial x_1} w = 0, \ \text{ for } \ x_1 = 0,1,\ t\in(0,T) \\ w|_{t=0} = w^0 = (\rho^0)^{1+\gamma} \ \text{ in } \ (0,1), \end{cases} \tag{39}$$

where $F = -\frac{1}{12\mu}(\rho^2 f_1 + \rho g_1) \in L^\gamma(0,T; L^\infty(0,1))$. Then it is easy to see that $\frac{\partial}{\partial x_1}\int_0^t w \in L^{2\gamma}(0,T; L^\infty(0,1))$.

Let us prove that, for a given $F \in L^\gamma(0,T; L^\infty(0,1))$ and $w^0 \in L^\infty(0,1)$, the problem (39) has a unique solution w such that $\partial_{x_1} w \in L^{2\gamma/(1+\gamma)}((0,1)\times(0,T))$, $w^{1/(1+\gamma)} \in C([0,T]; (W^{1,2\gamma/(\gamma-1)}(0,1))')$ and $\frac{\partial}{\partial x_1}\int_0^t w \in L^{2\gamma}(0,T; L^\infty(0,1))$.

We suppose that there are 2 solutions w_1 and w_2 and set $w = w_1 - w_2$. Then we have

$$w_1^{1/(1+\gamma)} - w_2^{1/(1+\gamma)} - \frac{\gamma}{12\mu(\gamma+1)}\frac{\partial^2}{\partial x_1^2}\int_0^t w = 0. \tag{40}$$

After testing (40) with w and integrating, we get the equality

$$\int_0^t \int_0^1 (w_1^{1/(1+\gamma)} - w_2^{1/(1+\gamma)})(w_1 - w_2)\, dx_1 d\tau + \frac{\gamma}{24\mu(\gamma+1)}\int_0^1 (\frac{\partial}{\partial x_1}\int_0^t w)^2\, dx_1 = 0. \tag{41}$$

(41) implies $w = 0$.

Hence if we are able to get a smoother solution for (34)-(36), that solution will be equal to w. Now we use the theory of renormalized solutions of degenerate elliptic-parabolic problems by J. Carillo et al, developed in [9], [10]. By a straightforward verification we find out that there is a unique entropy solution v for (34)-(36), satisfying $\partial_t(|v|^{-\gamma/(1+\gamma)}v) \in L^2(0,T; H^{-1}(0,1))$ and $v \in L^2(0,T; H^1(0,1))$. Unfortunately, we have uniqueness only for problem (39) and before applying the theory of entropy solutions, we should prove that w is as regular as an entropy solution. This requires getting regularity for the degenerate elliptic-parabolic problem (39) . First, for $\gamma \in [1,2]$ and $g_1 = 0$, we see that $\frac{\partial}{\partial x_1}F \in L^1((0,1)\times(0,T))$ and the theory from [9], [10] applies. Hence in this situation w has the same regularity as the entropy solution and it coincides with the entropy solution to (34)-(36). This proves uniqueness for $\gamma \in [1,2]$ and $g_1 = 0$.

Next case is when $\gamma > 2$. Then $F \in L^2(0,T; L^\infty(0,1))$ and by a slight generalization of the classical theory from [4], we obtain that w has the same regularity as of the entropy solution. Hence w is a weak solution which is also an entropy solution of (34)-(36) and consequently it is unique.

Uniqueness guarantees convergence of the whole sequence and we get (37)-(38). □

3 Conclusion and physical interpretation of the obtained results

In this paper we study derivation of the Reynolds equation for a compressible lubricant, from the compressible Navier-Stokes system. The flow is isentropic and satisfies the no-slip conditions at the boundary. Furthermore, for simplicity we consider only the flow with negligible Reynolds numbers. The goal was to derive *rigorously* the compressible Reynolds equation, in the limit when the domain width tends to zero. Our result confirms the corresponding models from the lubrication literature (see e.g. [14] or [15]), since after plugging (33) into (29) we obtain the compressible Reynolds equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1}\left\{\rho^2 f_1 + \rho g_1 - \rho \frac{\partial p}{\partial x_1}\right\} = 0, \tag{42}$$

where $p(\rho) = \rho^\gamma$.

We find here complete analogy with the incompressible case. At the leading order we have oscillations in the velocity field. In the pressure and density, oscillations are present only at the next order. The effective equation (29) could be also written as

$$24\mu u_1 + \frac{\partial p}{\partial x_1} = \rho f_1 + g_1 \quad \text{on} \quad \{\rho > 0\} \tag{43}$$

and $\rho u_2 = 0$. (43) is identical to the effective momentum equation in the incompressible case. Difference comes from the Gibbs relation linking pressure and density.

Analogous reduction of the high Reynolds number flow through a compliant blood vessel is undertaken in [7] and [8]. The *a priori* estimate for the pressure, constructed in this paper, justifies at least partially the approach from [7] and proposes closure scheme for the high Reynolds blood flows through a deformable blood vessel.

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