Polynomial filtration laws for low Reynolds number flows through porous media

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Abstract: In this work we use the method of homogenization to develop a filtration law in porous media that includes the effects of inertia at finite Reynolds numbers. The result is much different than the empirically observed quadratic Forchheimer equation. First, the correction to Darcy's

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law is initially cubic (not quadratic) for isotropic media. This is consistent with several other authors ([31], [45], [14], [36]) who have solved the Navier-Stokes equations analytically and numerically. Second, the resulting filtration model is an infinite series polynomial in velocity, instead of a single corrective term to Darcy's law.

Although the model is only valid up to the local Reynolds number at most of order 1, the findings are important from a fundamental perspective because it shows that the often-used quadratic Forchheimer equation is not a universal law for laminar flow, but rather an empirical one that is useful in a limited range of velocities. Moreover, as stated by Mei and Auriault in [31] and Barree and Conway in [4], even if the quadratic model were valid at moderate Reynolds numbers in the laminar flow regime, the permeability extrapolated on a Forchheimer plot would not be the intrinsic Darcy permeability.

A major contribution of this work is that the coefficients of the polynomial law can be derived a priori, by solving sequential Stokes problems. In each case, the solution to the Stokes problem is used to calculate a coefficient in the polynomial, and the velocity field is an input of the forcing function, F, to subsequent problems. While numerical solutions must be utilized to compute each coefficient in the polynomial, these problems are much simpler and robust than solving the full Navier-Stokes equations.

1 Introduction

Darcy's Law ($v = -\frac{K}{\mu}\nabla P$) adequately describes the slow flow of Newtonian fluids in porous media and is strictly valid for Stokes flow ($\mathbf{Re} = 0$), but is usually applicable in engineering applications for $\mathbf{Re} < 1$. While initially observed experimentally, Darcy's law can be recovered analytically or numerically by solving the steady-state Stokes equations. It is generally acceptable to use Darcy's Law for modeling flow in subsurface applications, such as reservoirs and aquifers, because the low matrix permeability results in low velocities. However, higher velocities are often observed in fractures and near wellbores; a more complicated model is necessary to describe flow in these cases.

Forchheimer's equation (see [21]) is an empirical extension to Darcy's law that is intended to capture nonlinearities that occur due to inertia in the laminar flow regime.

$$-\frac{\Delta P}{L} = \frac{\mu}{K}v + \rho\beta v^2 \quad (1)$$

The quadratic term is small compared to the linear term at low velocities and Darcy's law is often a good approximation. The constant, β, is referred to as the non-Darcy coefficient and, like permeability, is an empirical value that is specific to the porous medium. It is often found experimentally through data reduction. While often assumed a scalar, the non-Darcy coefficient is likely a tensor for anisotropic media (see [45] and [31]) since it is dependent on the medium morphology.

Usually Eq. (1) is rearranged to create a Forchheimer plot; relating $\frac{1}{K_{app}}$ versus $\frac{\rho v}{\mu}$ results in a straight line with slope β and an intercept $\frac{1}{K}$.

$$\frac{\Delta P}{L v \mu} = \frac{1}{K_{app}} = \frac{1}{K} + \beta \left(\frac{\rho v}{\mu} \right) \tag{2}$$

Forchheimer's equation has been found to fit some experimental data very well by Forchheimer in [21] and [22] and others ([1], [43], [12], [19], [9], [27] and [34]).However, the equation has been shown to be unacceptable for matching other experimental data ([25], [4] and [5])and even Forchheimer (see [22])added additional terms for some data sets.

Recently, Barree and Conway ([4] and [5])conducted experiments and produced data that did not follow the straight line in (2), suggesting that Forchheimer's equation is not valid over a large range of velocities. Their data is concave downward, which they explain is caused by streamlining and partitioning in porous media at higher velocities. Batenburg and Milton-Taylor produced in [7] data that disagreed with Barree and Conway and appeared to validate the Forchheimer model. However, Huang and Ayoub argue in [24] that the work of Barree and Conway [4] was partially in a turbulent flow regime and Batenburg and Milton-Taylor's data from [7] entirely in the turbulent regime. Nonlinearities associated with the Forchheimer equation occur at velocities well before, and not related to, turbulence. Regardless, the arguments made by Barree and Conway in [4] and [5] for a minimum-permeability plateau has validity and are supported theoretically and numerically by other authors in [18], [42] and [3]. In their paper [4], Barree and Conway have also suggested that the permeability obtained by extrapolation to the intercept in a Forchheimer plot is not the intrinsic, Darcy permeability.

Many attempts have been made to derive the quadratic, Forchheimer equation from first principles using homogenization. Attempts using the formal homogenization go back to 1978 and to the paper [29] by J.L. Lions and to the book [28], by the same author. Some other non-linear filtration laws could be found in the book [39]. The approach of J.L. Lions and E. Sanchez-Palencia is applicable for Reynolds' numbers smaller than a threshold value and it was observed by Auriault, Lévy, Mei and others that the obtained homogenized problem leads to polynomial filtration laws. This situation is called the "weakly non-linear" case and is studied in details in the papers [45], [31] and [35]. Homogenization derivation of Darcy's law is through a two-scale expansion for the velocity and for the pressure. It is an infinite series in ε being the ratio between the typical pore size ℓ and the reservoir size L. In the leading order we obtain the velocity and pressure approximations. Handling them requires an additional term, which is of next order and which contains second and higher order derivatives of the effective pressure. As proved in [32], in the absence of the inertia ($\mathbf{Re}= 0$) this leads to an approximation of the physical quantities which are of order ε. If we want to go further, then we see that the velocity approximation creates compressibility effects. Furthermore there is a force created by the lower order terms coming from the zero order approximation. In the fundamental paper [31] the local Reynolds number $\mathbf{Re}^{loc} = \varepsilon \ \ \mathbf{Re}$ was set to $\sqrt{\varepsilon}$. As a consequence, the $\sqrt{\varepsilon}$-correction to Darcy's law was quadratic filtration law. For an isotropic porous medium this contribution was proved to be zero. Then the next order correction is of order ε and contains simultaneously next order inertia contribution and the compressibility and forcing contributions, present in the Stokes flow case. Interaction of all these effects leads to an effective filtration law which is not polynomial. It is only with additional restrictions to the geometry that Mei and Auriault obtain the cubic filtration law. In [35] the local Reynolds number $\mathbf{Re}^{loc} = \varepsilon$, other effects appear immediately and the effective filtration law is a nonlinear differential operator and not a polynomial. Formal homogenization derivation was rigorously established in [10], by proving the error estimate for the whole range of Reynolds numbers in the weak inertia case. In [30] the general non-local filtration law for the threshold value of Reynolds' number was rigorously established in the homogenization limit when the pore size tends to zero. One of the important observations from [45], [31] and [35] was that for an isotropic porous medium the quadratic terms cancel and one has a cubic filtration law. This observation is confirmed analytically and numerically in the paper [20]. In [13] the authors claim that the non-linear filtration law is quadratic even for isotropic porous media but their conclusions seem to contradict the theory and numerical experiments.

Derivation using volume averaging was undertaken in [37], [38] and [44]. For related approaches we refer to [15] and [23]. In some cases the quadratic correction to Darcy's law is recovered. However, in [37], [38], Ruth and Ma explain that microscopic inertial effects are neglected in volume averaging techniques and therefore cannot be used to derive a macroscopic law. They point out that the Forchheimer equation is non-unique and any number of polynomials could be used to describe non-linear behavior due to inertia in laminar flow. This is confirmed in [10], where the nonlinear filtration law is obtained as an infinite entire series in powers of the local Reynolds number.

The cubic law has been verified by several authors numerically in simple porous media by solving the Navier-Stokes equations directly using the Finite Element Method or the Lattice-Boltzman method in [14], [36], [20] and [26]. In most cases, the cubic law is only valid at very low velocities (where Darcy's law is approximately valid anyway) and the quadratic Forchheimer equation appears applicable at more moderate velocities. Nonetheless, these findings are significant because they suggest that

1. Forchheimer's equation may not be universal and only valid in a limited range of velocities and

2. Permeability obtained by extrapolation to the intercept on a Forchheimer plot may not be the intrinsic, Darcy permeability (a point made by Barree and Conway in [4] as well as by Skjetne and Auriault in [41].

The objectives of this work are to

1. Derive a filtration law via homogenization of the Navier-Stokes equations to account for nonlinearities due to inertia at low local Reynolds number **Re** (<1),

2. Derive a procedure for determining the constants in the law without experiment or solving the full Navier-Stokes equations,

3. Validate the filtration law through comparison to numerical solution of the Navier-Stokes equations in simple porous media, and

4. Compare the derived law to existing models, such as the quadratic Forchheimer's equation or cubic law derived in [45], [31] and [35].

The paper is organized as follows. In section §2, homogenization is used on the steady state Navier Stokes equations to arrive at infinite series polynomial filtration law. In difference with the results in the article [31] and [35], we always get a polynomial filtration law, by establishing clearly its range of validity in terms of the local Reynolds number $\mathbf{Re}^{loc}$. This agrees with the result of Wodié and Lévy in [45]. Nevertheless, we propose a different two-scale expansion for the pressure. Our approach is rigorously justified by the error estimates from [10]. Furthermore, our approach permits to go to any order of approximation and the constants in the polynomial can be found *a priori*, by solving sequential Stokes flow problems. It is important to note that such systematic approach gave us explicitly the permeability in the cubic filtration law, which differs from Darcy's permeability by a contribution proportional to local Reynolds number squared. In section §3, an expansion is used to derive a law specifically valid for periodic, axisymmetric geometries which is simplification of the model in section §2. Such geometries give an isotropic porous medium and for them we were able to derive without cumbersome calculations the fifth order filtration law. In section §4 details of numerical techniques used to solve the full Navier-Stokes equations, as well as the Stokes flow problems used to determine the polynomial constants are discussed. The polynomial law is compared directly the numerical solution and good agreement is found. Conclusions of the work are summarized in Section §5.

2 Homogenization of the stationary Navier-Stokes equations and polynomial non-linear filtration laws

We consider the stationary incompressible viscous flow through a porous medium. The flow regime is assumed to be laminar through the fluid part of porous medium, which is considered as a network of interconnected channels.

In order to write the Navier-Stokes system with the viscosity μ and the density ρ in non-dimensional form, we introduce the macroscopic characteristic length L, the characteristic velocity V and the characteristic pressure $\mathcal{P}$. Flow is governed by a given pressure drop ΔP in the direction x_1. This pressure drop determines the characteristic volume force $\dfrac{\Delta P}{L}\mathbf{e}_1$. Then characteristic numbers are defined as follows:

- $\mathbf{Re} = \dfrac{VL\rho}{\mu}$ is the Reynolds number
- Froude's number is $\mathbf{Fr} = \dfrac{\rho V^2}{|\Delta P|}$.

As customary in modeling the filtration using homogenization, we use the fact that the porous medium has a microscopic length scale ℓ (e.g. a typical pore size) which is small compared to the characteristic length L.

Therefore there is a small parameter $\varepsilon = \ell/L$ in the problem and we suppose that 2 introduced characteristic numbers behave as powers of ε.

With these conventions, the non-dimensional incompressible Navier-Stokes system is given by

$$-\frac{1}{\mathbf{Re}}\nabla^2\mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon\nabla)\mathbf{v}_\varepsilon + \frac{\mathcal{P}}{\rho V^2}\nabla p_\varepsilon = \frac{\text{sign }(-\Delta P)}{\mathbf{Fr}}\mathbf{e}_1 \ \text{ in } \ \Omega_\varepsilon \ , \tag{3}$$

$$\operatorname{div}\mathbf{v}_\varepsilon = 0 \ \text{ in } \ \Omega_\varepsilon \ , \tag{4}$$

where Ω_ε is the fluid part of the porous medium Ω, $\mathbf{v}_\varepsilon$ is the velocity and p_ε is the pressure.
For simplicity, we suppose that Ω is the cube $D = (0, L)^n,\ n = 2, 3$. Then Ω_ε is a bounded domain in $\mathbb{R}^n,\ n = 2, 3$. For simplicity we suppose it periodic but our approach would work also for a statistically homogeneous random porous medium.

Formal description of Ω_ε goes along the following lines:

First we define the geometrical structure inside the unit cell $Y = (0, 1)^n$, $n = 2, 3$. Let Y_s (the solid part) be a closed subset of $\bar{Y}$ and $Y_F = Y\backslash Y_s$ (the fluid part). Now we make the periodic repetition of Y_s all over $\mathbb{R}^n$ and set $Y_s^k = Y_s + k$, $k \in \mathbb{Z}^n$. Obviously the obtained set $E_s = \bigcup_{k\in\mathbb{Z}^n} Y_s^k$ is a closed subset of $\mathbb{R}^n$ and $E_F = \mathbb{R}^n\backslash E_s$ in an open set in $\mathbb{R}^n$. Following Allaire [2] we make the following assumptions on Y_F and E_F:

- (i) Y_F is an open connected set of strictly positive measure, with a Lipschitz boundary and Y_s has strictly positive measure in $\bar{Y}$, as well.
- (ii) E_F and the interior of E_s are open sets with the boundary of class $C^{0,1}$, which are locally located on one side of their boundary. Moreover E_F is connected.

Now we see that Ω is covered with a regular mesh of size ε, each cell being a cube Y_i^ε, with $1 \le i \le N(\varepsilon) = |\Omega|\varepsilon^{-n}[1 + O(1)]$. Each cube Y_i^ε is homeomorphic to Y, by linear homeomorphism Π_i^ε, being composed of translation and an homothety of ratio $1/\varepsilon$.

We define

$$Y_{S_i}^\varepsilon = (\Pi_i^\varepsilon)^{-1}(Y_s) \qquad \text{and} \qquad Y_{F_i}^\varepsilon = (\Pi_i^\varepsilon)^{-1}(Y_F)$$

For sufficiently small $\varepsilon > 0$ we consider the set

$$T_\varepsilon = \{k \in \mathbb{Z}^n | Y_{S_k}^\varepsilon \subset \Omega\}$$

and define

$$O_\varepsilon = \bigcup_{k\in T_\varepsilon} Y^\varepsilon_{S_k}, \quad S_\varepsilon = \partial O_\varepsilon, \quad \Omega_\varepsilon = \Omega\backslash O_\varepsilon = \Omega\cap\varepsilon E_F$$

Obviously, $\partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon$. The domains O_ε and Ω_ε represent, respectively, the solid and fluid parts of a porous medium Ω. For simplicity we suppose $L/\varepsilon \in \mathbb{N}$.

Then for $n = 2, 3$ the classical theory gives the existence of at least one weak solution $(\mathbf{v}_\varepsilon, p_\varepsilon) \in V_{per}(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon)$ for the problem (3), (4) with the boundary conditions

$$\mathbf{v}_\varepsilon = 0 \ \text{ on } S_\varepsilon \ \ , \ \ (\mathbf{v}_\varepsilon, p_\varepsilon) \ \text{ is } L-\text{periodic} \tag{5}$$

and

$$V_{per}(\Omega_\varepsilon) = \{\mathbf{z} \in H^1(\Omega_\varepsilon)^n : \ \mathbf{z} = 0 \text{ on } S_\varepsilon, \ \mathbf{z} \text{ is } L\text{−periodic and } \operatorname{div}\mathbf{z} = 0 \text{ in } \Omega_\varepsilon\}.$$

Let us discuss the influence of the coefficients to the size of the solution: after testing (3) by $\mathbf{v}_\varepsilon$ and integrating over Ω_ε, we get

$$||\nabla\mathbf{v}_\varepsilon||^2_{L^2(\Omega_\varepsilon)^{n^2}} \leq \sqrt{\varphi}\frac{\mathbf{Re}}{\mathbf{Fr}}||\mathbf{v}_\varepsilon||_{L^2(\Omega_\varepsilon)^n}, \tag{6}$$

where $\varphi = |\Omega_\varepsilon|$ is the porosity. After recalling that in a periodic porous medium, with period ε, Poincaré's inequality gives $||\mathbf{v}_\varepsilon||_{L^2(\Omega_\varepsilon)^n} \leq \dfrac{\varepsilon}{\sqrt{2}}||\nabla\mathbf{v}_\varepsilon||_{L^2(\Omega_\varepsilon)^{n^2}}$, we find out that (6) yields

$$||\mathbf{v}_\varepsilon||_{L^2(\Omega_\varepsilon)^n} \leq \frac{\sqrt{\varphi}}{2}\varepsilon^2\frac{\mathbf{Re}}{\mathbf{Fr}} = \frac{\sqrt{\varphi}}{2}\varepsilon^2\frac{L|\Delta P|}{V\mu}. \tag{7}$$

Now we build into the model our dimensional requirements:

- Since the dimensionless velocity should be of order one, the estimate (7) allows to calculate the characteristic velocity and we find $V = \dfrac{\sqrt{\varphi}}{2}\varepsilon^2\dfrac{L|\Delta P|}{\mu}$, which agrees with the Poiseuille profile and with the corresponding discussion in [20]. The corresponding Reynolds number is then $\mathbf{Re}= \dfrac{\varepsilon^2 L^2\rho|\Delta P|}{\mu^2}\dfrac{\sqrt{\varphi}}{2}$.
- In order that the expansion leads to the nontrivial leading term, corresponding to the non-linear laminar flow, we require that at the pore scale the forcing term, caused by the pressure drop, and the viscous term in the fast variable are of the same order. This condition reads $\varepsilon^2 \ \mathbf{Re} = \mathbf{Fr}$ and follows from the above choice of the characteristic velocity.

- Next $\mathcal{P} = |\Delta P| \dfrac{\sqrt{\varphi}}{2}$, assuring the well-posedness of the leading equation for the zeroth order expansion term.

- Finally, if we want to remain in the stationary non-linear laminar flow regime, then the local Reynolds number $\mathbf{Re}^{loc} = \varepsilon \ \mathbf{Re}$ should be at most of order 1. This implies that our analysis applies to the flows such that

$$|\Delta P| \leq \frac{\mu^2}{\rho L^2 \varepsilon^3} \frac{2}{\sqrt{\varphi}} \quad \text{and} \quad V \leq \frac{\mu}{\rho \ell} \frac{2}{\sqrt{\varphi}}. \tag{8}$$

  Consequently we will use the local Reynolds number as expansion parameter.

We expect that being close to the critical value $\mathbf{Re}^{loc} = \varepsilon \ \ \mathbf{Re} \ = 1$ produces non-linear effects of a polynomial type. For this reason, we restrict our investigation to the weak nonlinear effects, i.e. we will keep $\mathbf{Re}^{loc}$ smaller, but or order one .

Presence of the constant forcing term will oversimplify the result. In order to be able to give non-linear filtration laws in the presence of gravity effects, source terms and wells, we suppose that instead of setting $\dfrac{\mathbf{Re}}{\mathbf{Fr}} \mathbf{e}_1 = \dfrac{2}{\sqrt{\varphi}} \dfrac{1}{\varepsilon^2}$ sign $(-\Delta P) \mathbf{e}_1$, we have for the forcing term $\dfrac{\mathbf{F}(x)}{\varepsilon^2}$, with $|\mathbf{F}(x)| \leq \dfrac{2}{\sqrt{\varphi}}$. In the end of the section we will state the results also for $\mathbf{F} = \dfrac{2}{\sqrt{\varphi}} \mathbf{e}_1$, which corresponds to our model.
According to the scaling of data, we seek an asymptotic expansion in powers of the local Reynolds number for $\{\mathbf{v}_\varepsilon, p_\varepsilon\}$ solution of (3)-(5).
If $\mathbf{Re}^{loc}$ is *sufficiently close* to 1 we set the following asymptotic expansion:

$$\begin{cases} (i) \quad \mathbf{v}_\varepsilon(x) & = \mathbf{v}^0(x,y) + \mathbf{Re}^{loc} \mathbf{v}^1(x,y) + (\mathbf{Re}^{loc})^2 \mathbf{v}^2(x,y) + \cdots + \\ & + \varepsilon \{ \mathbf{v}^{0,1}(x,y) + \mathbf{Re}^{loc} \mathbf{v}^{1,1}(x,y) + \cdots \} + \cdots \\ (ii) \quad p_\varepsilon(x) & = p^0(x,y) + \mathbf{Re}^{loc} p^1(x,y) + (\mathbf{Re}^{loc})^2 p^2(x,y) + \cdots \\ & + \varepsilon \{ p^{0,1}(x,y) + \mathbf{Re}^{loc} p^{1,1}(x,y) + \cdots \quad , \end{cases} \tag{9}$$

where $y = x/\varepsilon$.
We insert the expansions (9) into the system (3)-(5), now written in the fast

and slow variables:

$$\mathbf{Re}^{loc}\Big(\big(\mathbf{v}^0(x,y)+\mathbf{Re}^{loc}\mathbf{v}^1(x,y)+(\mathbf{Re}^{loc})^2\mathbf{v}^2(x,y)+\varepsilon\mathbf{v}^{0,1}(x,y)+\cdots\big)(\nabla_y+$$

$$\varepsilon\nabla_x)\Big)\big(\mathbf{v}^0(x,y)+\mathbf{Re}^{loc}\mathbf{v}^1(x,y)+(\mathbf{Re}^{loc})^2\mathbf{v}^2(x,y)+\varepsilon\mathbf{v}^{0,1}(x,y)+\cdots\big)=$$

$$-\frac{1}{\varepsilon}(\nabla_y+\varepsilon\nabla_x)\Big(p^0(x,y)+\mathbf{Re}^{loc}p^1(x,y)+(\mathbf{Re}^{loc})^2p^2(x,y)+\varepsilon p^{0,1}(x,y)+\cdots\Big)$$

$$+(\nabla_y^2+2\varepsilon\ \mathrm{div}\ _y\nabla_x+\varepsilon^2\nabla_x^2)\big(\mathbf{v}^0(x,y)+\mathbf{Re}^{loc}\mathbf{v}^1(x,y)+(\mathbf{Re}^{loc})^2\mathbf{v}^2(x,y)+$$

$$\varepsilon\mathbf{v}^{0,1}(x,y)+\cdots\big)+\mathbf{F}; \tag{10}$$

$$(\ \mathrm{div}\ _y+\varepsilon\ \mathrm{div}\ _x)\big(\mathbf{v}^0(x,y)+\mathbf{Re}^{loc}\mathbf{v}^1(x,y)+(\mathbf{Re}^{loc})^2\mathbf{v}^2(x,y)+$$

$$\varepsilon\mathbf{v}^{0,1}(x,y)+\cdots\big)=0 \tag{11}$$

After collecting equal powers of ε in (10)-(11), we obtain, as in [32], a sequence of the problems in $Y_F\times\Omega$.
First we have at the order $\mathcal{O}(\varepsilon^{-1})$ (and afterwards at orders $\mathcal{O}(\varepsilon^{-1}(\mathbf{Re}^{loc})^k)$

$$\begin{array}{l}\nabla_y p^0=0\ \ ,\ \ \text{i.e.}\ \ p^0=p^0(x)\\ \nabla_y p^1=0\ \ ,\ \ \text{i.e.}\ \ p^1=p^1(x)\ \ ,\end{array}$$

and in fact $p^k=p^k(x)$ for every k. Then at the order $\mathcal{O}(1)$

$$\left\{\begin{array}{l}-\nabla_y^2\mathbf{v}^0+\nabla_y p^{0,1}=\mathbf{F}-\nabla_x p^0\ \text{ in }\ Y_F\times\Omega\\ \mathrm{div}_y\mathbf{v}^0=0\ \text{ in }\ Y_F\times\Omega,\ \ \mathbf{v}^0=0\ \text{ on }\ S\times\Omega\\ \{\mathbf{v}^0,p^{0,1}\}\ \text{ is }\ Y-\text{periodic},\ \ \mathrm{div}_x\{\int_{Y_F}\mathbf{v}^0\ dy\}=0\ \text{ in }\ \Omega\\ \{\int_{Y_F}\mathbf{v}^0,p^0\}\ \text{ is }\ \Omega-\text{periodic},\end{array}\right. \tag{12}$$

and, at the arbitrary order $\mathcal{O}((\mathbf{Re}^{loc})^k)$, $k\geq 1$,

$$\left\{\begin{array}{l}-\nabla_y^2\mathbf{v}^k+\nabla_y p^{k,1}=-\displaystyle\sum_{\ell=0}^{k-1}(\mathbf{v}^\ell\nabla_y)\mathbf{v}^{k-1-\ell}-\nabla_x p^k\ \text{ in }\ Y_F\times\Omega\\ \mathrm{div}_y\mathbf{v}^k=0\ \text{ in }\ Y_F\times\Omega,\ \ \mathbf{v}^k=0\ \text{ on }\ S\times\Omega\\ \{\mathbf{v}^k,p^{k,1}\}\ \text{ is }\ Y-\text{periodic},\ \ \mathrm{div}_x\{\int_{Y_F}\mathbf{v}^k\}=0\ \text{ in }\ \Omega\\ \{\int_{Y_F}\mathbf{v}^k,p^k\}\ \text{ is }\ \Omega-\text{periodic}.\end{array}\right. \tag{13}$$

Problems (12)-(13) are standard Stokes problems in Y_F and the regularity of the solutions follows from the regularity of the geometry and of the data.

Using in (12)-(13) the classical separation of scales, as for instance in [39] or in [32], leads to the following formulas for $\mathbf{v}^0$ and $p^{0,1}$.

$$\mathbf{v}^0(x,y)=\sum_{i=1}^{n}\mathbf{w}^i(y)[F_i-\frac{\partial p^0}{\partial x_i}(x)];\; p^{0,1}(x,y)=\sum_{i=1}^{n}\pi^i(y)[F_i-\frac{\partial p^0}{\partial x_i}(x)],$$

where $(w^i,\pi^i)\in C^\infty(\cup_{k\in\mathbb{Z}^n}(k+\overline{Y_F}))^{n+1}$ is the Y-periodic solution of the auxiliary Stokes problem:

$$\begin{cases} -\nabla_y^2\mathbf{w}^i+\nabla_y\pi^i=\mathbf{e}_i \;\;,\;\; \mathrm{div}_y\mathbf{w}^i=0 \;\text{ in }\; Y_F \\ \mathbf{w}^i=0 \;\text{ on }\; S,\;\; \int_{Y_F}\pi^i=0\;\;. \end{cases} \tag{14}$$

In addition $(\mathbf{v}_F^0,p^0)\in C_{per}^\infty(\overline{\Omega})^{n+1}$, is the unique solution of :

$$\begin{cases} (i) \quad \mathrm{div}_x\mathbf{v}_F^0(x)=0 \;\text{ in }\; \Omega\;,\; \{\mathbf{v}_F^0,p^0\} \text{ is } \Omega-\text{periodic} \\ (ii) \quad \mathbf{v}_F^0(x)=K(\mathbf{F}-\nabla_x p^0)(x) \;\text{ in } \Omega,\; \int_\Omega p^0=0, \end{cases} \tag{15}$$

where K is the permeability tensor, defined by

$$K_{ij}=\int_{Y_F}\mathbf{w}_j^i(y)dy \quad i,j=1,\dots,n\;\;. \tag{16}$$

and $\mathbf{v}_F^0(x)=\int_{Y_F}\mathbf{v}^0(x,y)dy$ is Darcy's velocity.
Now we turn to the **corrections to Darcy's law coming from inertia effects**.

In function of the closeness of $\mathbf{Re}^{loc}$ to 1 we could continue with our approximations. Once Darcy's pressure p^0 calculated, the scale separation for the problem (13) gives

$$\mathbf{v}^k(x,y)=\sum_{\ell=0}^{k-1}\sum_{i_1,\dots,i_{\ell+1}=1}^{n}\sum_{j_1,\dots,j_{k-\ell}=1}^{n}\prod_{m=1}^{\ell+1}[F_{i_m}-\frac{\partial p^0}{\partial x_{i_m}}(x)]\prod_{r=1}^{k-\ell}[F_{j_r}-\frac{\partial p^0}{\partial x_{j_r}}(x)]\mathbf{u}^{i_1,\dots,i_{\ell+1},j_1,\dots,j_{k-\ell}}(y)-\sum_{i=1}^{n}\mathbf{w}^i(y)\frac{\partial p^k}{\partial x_i}(x)$$

$$p^{k,1}(x,y)=\sum_{\ell=0}^{k-1}\sum_{i_1,\dots,i_{\ell+1}=1}^{n}\sum_{j_1,\dots,j_{k-\ell}=1}^{n}\prod_{m=1}^{\ell+1}[F_{i_m}-\frac{\partial p^0}{\partial x_{i_m}}(x)]\prod_{r=1}^{k-\ell}[F_{j_r}-\frac{\partial p^0}{\partial x_{j_r}}(x)]\Lambda^{i_1,\dots,i_{\ell+1},j_1,\dots,j_{k-\ell}}(y)-\sum_{i=1}^{n}\pi^i(y)\frac{\partial p^k}{\partial x_i}(x)$$

where $(\mathbf{u}^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}}, \Lambda^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}}) \in C^\infty(\cup_{k\in\mathbb{Z}^n}(k+\overline{Y_F}))$ is the Y-periodic solution of the auxiliary Stokes problem:

$$\begin{cases} -\nabla_y^2 \mathbf{u}^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}} + \nabla_y \Lambda^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}} = \\ -(\mathbf{u}^{i_1,\dots,i_{\ell+1}}\nabla_y)\mathbf{u}^{j_1,\dots,i_{k-\ell}} \text{ in } Y_F \\ \operatorname{div}_y \mathbf{u}^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}} = 0 \text{ in } Y_F \\ \mathbf{u}^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}} = 0 \text{ on } S \ , \ \int_{Y_F} \Lambda^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}} \, dy = 0 \ . \end{cases}$$

In addition $(\mathbf{v}^{k,F}, p^k) \in C^\infty_{per}(\overline{\Omega})^{n+1}$ is the solution of

(i) $\operatorname{div}_x \mathbf{v}^{k,F} = 0$ in Ω

$$(ii)\ \mathbf{v}^{k,F} = -K\nabla p^k + \sum_{\ell=0}^{k-1} \sum_{i_1,\dots,i_{\ell+1}=1}^{n} \sum_{j_1,\dots,j_{k-\ell}=1}^{n} \mathbf{M}^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}} \cdot$$
$$\prod_{m=1}^{\ell+1}[F_{i_m} - \frac{\partial p^0}{\partial x_{i_m}}(x)] \prod_{r=1}^{k-\ell}[F_{j_r} - \frac{\partial p^0}{\partial x_{j_r}}(x)] \tag{17}$$

(iii) $\{\mathbf{v}^{k,F}, p^k\}$ is $\Omega -$ periodic, $\displaystyle\int_\Omega p^k = 0$,

where $\mathbf{M}^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}}$ is defined by

$$\mathbf{M}^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}} = \int_{Y_F} \mathbf{u}^{i_1,\dots,i_{\ell+1},j_1,\dots,i_{k-\ell}}(y)dy, \quad i,j,k = 1,\dots,n \tag{18}$$

and $\mathbf{v}^{k,F}(x) = \int_{Y_F} \mathbf{v}^k(x,y)\ dy$.

The above expressions lead to the following algorithm for describing flows by **polynomial laws of any order**:

- Let the local Reynolds number $\mathbf{Re}^{loc}$ be smaller or equal to ε. Then, after [10] and [32], the effective filtration is described by Darcy's law (15) and we have

$$\int_{\Omega_\varepsilon} \left\{ |\mathbf{v}_\varepsilon(x) - \mathbf{v}^0(x, \frac{x}{\varepsilon})|^2 + |p_\varepsilon(x) - p^0(x)|^2 \right\} dx \le C\varepsilon^2. \tag{19}$$

  The estimate (19) clarifies in which sense the filtration velocity $\mathcal{V}^0 := \mathbf{v}^{0,F} = \displaystyle\int_{Y_F} \mathbf{v}^0(x,y)\ dy$ approximates the physical velocity $\mathbf{v}_\varepsilon$.

- Next let $\varepsilon <\mathbf{Re}^{loc} \leq \sqrt{\varepsilon}$. Then we set $k = 1$ and use (17) to calculate $\{\mathbf{v}^{1,F},\ p^1\}$. It gives us $\mathbf{v}^1$ and auxiliary Stokes problems give us $\mathbf{u}^{i,j}$. Note that for this, knowledge of the solutions to all auxiliary Stokes problems from previous steps was necessary. Then, after [10] and [32], we have

$$\int_{\Omega_\varepsilon} \Big\{ |\mathbf{v}_\varepsilon(x) - \mathbf{v}^0(x, \frac{x}{\varepsilon}) - \mathbf{Re}^{loc}\mathbf{v}^1(x, \frac{x}{\varepsilon}))|^2 + \\ |p_\varepsilon(x) - p^0(x) - \mathbf{Re}^{loc}p^1(x)|^2 \Big\}\ dx \leq C\varepsilon^2. \qquad (20)$$

  From this estimate we will obtain in subsection 2.1 the quadratic filtration law.

- Next let $\sqrt{\varepsilon} <\mathbf{Re}^{loc} \leq \varepsilon^{1/3}$. Then we set $k = 2$ and use (17) to calculate $\{\mathbf{v}^{2,F},\ p^2\}$. It gives us $\mathbf{v}^2$ and auxiliary Stokes problems give us $\mathbf{u}^{i,j,k}$. Again, knowledge of the solutions to all auxiliary Stokes problems from previous steps was necessary. Then, after [10] and [32], we have

$$\int_{\Omega_\varepsilon} \Big\{ |\mathbf{v}_\varepsilon(x) - \mathbf{v}^0(x, \frac{x}{\varepsilon}) - \mathbf{Re}^{loc}\mathbf{v}^1(x, \frac{x}{\varepsilon})) - (\mathbf{Re}^{loc})^2\mathbf{v}^2(x, \frac{x}{\varepsilon}))|^2 + \\ |p_\varepsilon(x) - p^0(x) - \mathbf{Re}^{loc}p^1(x) - (\mathbf{Re}^{loc})^2 p^2(x)|^2 \Big\}\ dx \leq C\varepsilon^2. \qquad (21)$$

  From this estimate we will obtain in subsection 2.2 the cubic filtration law.

- This way we arrive at the range $\varepsilon^{1/(k-1)} <\mathbf{Re}^{loc} \leq \varepsilon^{1/k}$. For given k we use (17) to calculate $\{\mathbf{v}^{k,F},\ p^k\}$. It gives us $\mathbf{v}^k$ and auxiliary Stokes problems give us $\mathbf{u}^{i_1,\cdots,i_k}$. Again, knowledge of the solutions to all auxiliary Stokes problems from previous steps was necessary. Then, after [10] and [32], we have

$$\int_{\Omega_\varepsilon} \Big\{ |\mathbf{v}_\varepsilon(x) - \sum_{j=0}^{k-1} \mathbf{v}^j(x, \frac{x}{\varepsilon})(\mathbf{Re}^{loc})^j|^2 + \\ |p_\varepsilon(x) - \sum_{j=0}^{k-1} p^j(x)(\mathbf{Re}^{loc})^j|^2 \Big\}\ dx \leq C\varepsilon^2. \qquad (22)$$

  From this estimate we are able to obtain the kth order polynomial filtration law, for any k.

We define the effective filtration velocity and for the effective pressure by the following formula

$$\mathcal{V}^{k+1} := \sum_{\ell=0}^{k+1} (\mathbf{Re}^{loc})^{\ell} \mathbf{v}^{\ell,F} \quad , \quad \Pi^{k+1} := \sum_{\ell=0}^{k+1} (\mathbf{Re}^{loc})^{\ell} p^{\ell} \ ,$$

where $\mathbf{v}^{\ell,F} = \displaystyle\int_{Y_F} \mathbf{v}^{\ell}(x,y)\ dy$. The estimate (22) clarifies in which sense the filtration velocity $\mathcal{V}^k$ approximates the physical velocity $\mathbf{v}_\varepsilon$. Furthermore, using the two-scale filtration laws (17), we obtain that $\mathcal{V}^{k+1}$ is a polynomial of order $k+1$ in $\nabla \Pi^{k+1}$. The filtration law of order $k+1$ is obtain from the law of order k by a recursive procedure. We can write the coefficients as functions of the vector $M^{i_1,\dots,i_l}$, but it leads to very cumbersome recursion relations. We prefer to give expressions for several interesting cases.

2.1 The quadratic filtration law

Truncation of the infinite series polynomial to only two terms results in a quadratic correction to Darcy's law. At first glance, the homogenization may seem to be in agreement with Forchheimer's empirically-observed quadratic equation. However, it has been shown (see [31], [45] and [20]) the quadratic term vanishes for isotropic media and the first correction is cubic. This has been verified both numerically and experimentally. The quadratic behavior observed by Forchheimer and others likely occurs at more moderate Re, outside the limits of this homogenization.

Similarly to the Darcy law, by separation of scales, we have $\mathbf{v}^1$ and $p^{1,1}$ given by :

$$\mathbf{v}^1(x,y) = \sum_{i,j=1}^{n} \mathbf{u}^{ij}(y)[F_i - \frac{\partial p^0}{\partial x_i}(x)]\ [F_j - \frac{\partial p^0}{\partial x_j}(x)] - \sum_{i=1}^{n} \mathbf{w}^i(y) \frac{\partial p^1}{\partial x_i}(x)$$

$$p^{1,1}(x,y) = \sum_{i,j=1}^{n} \Lambda^{ij}(y)[F_i - \frac{\partial p^0}{\partial x_i}(x)]\ [F_j - \frac{\partial p^0}{\partial x_j}(x)] - \sum_{i=1}^{n} \pi^i(y) \frac{\partial p^1}{\partial x_i}(x),$$

where $(\mathbf{u}^{ij}, \Lambda^{ij}) \in C^\infty(\cup_{k\in\mathbb{Z}^n}(k+\overline{Y_F}))^{n+1}$ is the Y-periodic solution of the auxiliary Stokes problem:

$$\left\{ \begin{array}{l} -\nabla_y^2 \mathbf{u}^{ij} + \nabla_y \Lambda^{ij} = -(\mathbf{w}^i \nabla_y)\mathbf{w}^j = -\ \mathrm{Div}_y\ (\mathbf{w}^i \otimes \mathbf{w}^j) \ \text{ in } \ Y_F \\ \mathrm{div}_y \mathbf{u}^{ij} = 0 \ \text{ in } \ Y_F; \quad \mathbf{u}^{ij} = 0 \ \text{ on } \ S, \ \int_{Y_F} \Lambda^{ij}\ dy = 0. \end{array} \right. \tag{23}$$

In addition $(\mathbf{v}^{1,F}, p^1) \in C^\infty_{per}(\overline{\Omega})^{n+1}$ is the solution of

$$
\begin{aligned}
&(i)\ \operatorname{div}_x \mathbf{v}^{1,F} = 0 \ \text{ in } \ \Omega; \quad \{\mathbf{v}^{1,F}, p^1\} \ \text{ is } \Omega - \text{periodic } \ , \int_\Omega p^1 = 0, \\
&(ii)\ v_k^{1,F} = \sum_{i,j=1}^n M_k^{ij} \{F_i - \frac{\partial p^0}{\partial x_i}\} \ \{F_j - \frac{\partial p^0}{\partial x_j}\} - \sum_{j=1}^n K_{kj} \frac{\partial p^1}{\partial x_j}
\end{aligned}
\tag{24}
$$

where $\mathbf{M}^{ij}$ is defined by

$$
\mathbf{M}^{ij} = \int_{Y_F} \mathbf{u}^{ij}(y) dy \quad i, j = 1, \ldots, n \tag{25}
$$

and $\mathbf{v}^{1,F}(x) = \int_{Y_F} \mathbf{v}^1(x,y) \ dy$.

The above expansions allow us to write the **quadratic filtration law**.

Now we introduce the averaged velocity $\mathcal{V}^1$ and the averaged pressure Π^1 by

$$
\mathcal{V}^1 = \mathbf{v}_F^0 + \mathbf{Re}^{loc} \mathbf{v}^{1,F}; \quad \Pi^1 = p^0 + \mathbf{Re}^{loc} p^1 \tag{26}
$$

and with these notations, (15) (ii) and (24) (ii) can be summarized in:

$$
\mathcal{V}^1 = K(\mathbf{F} - \nabla \Pi^1) + \mathbf{Re}^{loc} \sum_{i,j=1}^n \mathbf{M}^{ij} (F_i - \frac{\partial \Pi^1}{\partial x_i})(F_j - \frac{\partial \Pi^1}{\partial x_j}). \tag{27}
$$

or equivalently, with the error of order $\mathcal{O}((\mathbf{Re}^{loc})^2)$, as

$$
\mathbf{F} - \nabla \Pi^1 = K^{-1} \mathcal{V}^1 - \mathbf{Re}^{loc} K^{-1} \sum_{i,j=1}^n \mathbf{M}^{ij} (K^{-1} \mathcal{V}^1)_i (K^{-1} \mathcal{V}^1)_j. \tag{28}
$$

The quadratic expression entering in equation (27) starts to be important when $\mathbf{Re}^{loc}$ is close to 1. The filtration law (28) corresponds to the classical form of non-Darcian filtration law. Nevertheless, the quadratic term is not monotone. For this reason, we believe that the form (27) is more useful for numerical calculations. Formal derivation of the law (27) using homogenization was undertaken in [45], [31] and [35]. For the rigorous justification, with correct choice of the pressure field, see the article [10].

2.2 The cubic filtration law

The cubic filtration law is obtained if we calculate the corresponding terms for $k = 2$. We note that in all one dimensional cases and in cases when the porous media satisfies some isotropy conditions, the quadratic term vanishes. For detailed discussion we refer to [20], were this important property is established under fairly realistic "reversibility" condition. This gives importance to the cubic filtration law.

Next, let us write explicitly the corresponding terms:

$$\mathbf{v}^2(x,y) = \sum_{i_1=1}^{n}\sum_{j_1,j_2=1}^{n} [F_{i_1} - \frac{\partial p^0}{\partial x_{i_1}}(x)][F_{j_1} - \frac{\partial p^0}{\partial x_{j_1}}(x)][F_{j_2} - \frac{\partial p^0}{\partial x_{j_2}}(x)]\mathbf{u}^{i_1,j_1,j_2}(y)$$
$$-\sum_{i=1}^{n}\mathbf{w}^i(y)\frac{\partial p^2}{\partial x_i}(x)(y)$$
$$p^{2,1}(x,y) = \sum_{i_1=1}^{n}\sum_{j_1,j_2=1}^{n} [F_{i_1} - \frac{\partial p^0}{\partial x_{i_1}}(x)][F_{j_1} - \frac{\partial p^0}{\partial x_{j_1}}(x)][F_{j_2} - \frac{\partial p^0}{\partial x_{j_2}}(x)]\Lambda^{i_1,j_1,j_2}(y)$$
$$-\sum_{i=1}^{n}\pi^i(y)\frac{\partial p^2}{\partial x_i}(x)$$

where $(\mathbf{u}^{i_1,j_1,j_2}, \Lambda^{i_1,j_1,j_2}) \in C^\infty(\cup_{k\in\mathbb{Z}^n}(k+\overline{Y_F}))$ is the Y-periodic solution of the auxiliary Stokes problem:

$$\begin{cases} -\nabla_y^2\mathbf{u}^{i_1,j_1,j_2} + \nabla_y\Lambda^{i_1,j_1,j_2} = -(\mathbf{w}^{i_1}\nabla_y)\mathbf{u}^{j_1,j_2} - (\mathbf{u}^{j_1,j_2}\nabla_y)\mathbf{w}^{i_1} \text{ in } Y_F \\ \mathrm{div}_y\mathbf{u}^{i_1,j_1,j_2} = 0 \text{ in } Y_F \\ \mathbf{u}^{i_1,j_1,j_2} = 0 \text{ on } S \ , \ \int_{Y_F}\Lambda^{i_1,j_1,j_2}\, dy = 0 \ . \end{cases} \tag{29}$$

In addition $(\mathbf{v}^{2,F}, p^2) \in C^\infty_{per}(\overline{\Omega})^{n+1}$ is the solution of

$$(i) \ \mathrm{div}_x \ \mathbf{v}^{2,F} = 0 \text{ in } \Omega; \quad \{\mathbf{v}^{2,F}, p^2\} \text{ is } \Omega-\text{periodic} \ , \int_\Omega p^2 = 0,$$

$$(ii)\mathbf{v}^{2,F} = \sum_{i_1,i_2,i_3=1}^{n} [F_{i_1} - \frac{\partial p^0}{\partial x_{i_1}}(x)][F_{i_2} - \frac{\partial p^0}{\partial x_{i_2}}(x)][F_{i_3} - \frac{\partial p^0}{\partial x_{i_3}}(x)]\mathbf{M}^{i_1,i_2,i_3} - K\nabla p^2, \tag{30}$$

where $\mathbf{M}^{i_1,i_2,i_3}$ is defined by

$$\mathbf{M}^{i_1,i_2,i_3} = \int_{Y_F}\mathbf{u}^{i_1,i_2,i_3}(y)dy \quad i_1,i_2,i_3 = 1,\dots,n \tag{31}$$

and $\mathbf{v}^{2,F}(x) = \int_{Y_F}\mathbf{v}^2(x,y)\ dy$.

The above expansions allow us to write the **cubic filtration law**.

Now we introduce the averaged velocity $\mathcal{V}^2$ and the averaged pressure Π^2 by

$$\mathcal{V}^2 = \mathbf{v}_F^0 + \mathbf{Re}^{loc}\mathbf{v}^{1,F} + (\mathbf{Re}^{loc})^2\mathbf{v}^{2,F}; \quad \Pi^2 = p^0 + \mathbf{Re}^{loc}p^1 + (\mathbf{Re}^{loc})^2 p^2$$

and with these notations, (15) (ii), (24) (ii) and (30) (ii) can be summarized, with the error of order $\mathcal{O}((\mathbf{Re}^{loc})^3)$, in:

$$\mathcal{V}^2 = K(\mathbf{F} - \nabla\Pi^2) + \mathbf{Re}^{loc}\sum_{i,j=1}^{n}\mathbf{M}^{ij}(F_i - \frac{\partial p^0}{\partial x_i})(F_j - \frac{\partial p^0}{\partial x_j}) +$$
$$(\mathbf{Re}^{loc})^2 \sum_{i_1,i_2,i_3=1}^{n} [F_{i_1} - \frac{\partial p^0}{\partial x_{i_1}}(x)][F_{i_2} - \frac{\partial p^0}{\partial x_{i_2}}(x)][F_{i_3} - \frac{\partial p^0}{\partial x_{i_3}}(x)]\mathbf{M}^{i_1,i_2,i_3}. \quad (32)$$

This in not yet a filtration law, since it should be expressed in terms of the effective pressure Π^2. In the case of the quadratic filtration law it was enough to replace p^0 by the effective pressure Π^1, but it is not the case any more. Rewriting (32) in terms of the effective pressure Π^2 gives

$$\mathcal{V}^2 = K(\mathbf{F} - \nabla\Pi^2) + \mathbf{Re}^{loc}\sum_{i,j=1}^{n}\mathbf{M}^{ij}(F_i - \frac{\partial \Pi^2)}{\partial x_i})(F_j - \frac{\partial \Pi^2)}{\partial x_j}) +$$
$$(\mathbf{Re}^{loc})^2 \Big\{ \sum_{i_1,i_2,i_3=1}^{n} [F_{i_1} - \frac{\partial \Pi^2)}{\partial x_{i_1}}(x)][F_{i_2} - \frac{\partial \Pi^2)}{\partial x_{i_2}}(x)][F_{i_3} - \frac{\partial \Pi^2)}{\partial x_{i_3}}(x)]\mathbf{M}^{i_1,i_2,i_3} +$$
$$\sum_{i,j=1}^{n}(\mathbf{M}^{ij} + \mathbf{M}^{ji})\frac{\partial p^1}{\partial x_i}(F_j - \frac{\partial \Pi^2)}{\partial x_j})\Big\} + \mathcal{O}((\mathbf{Re}^{loc})^3) = (K + (\mathbf{Re}^{loc})^2\mathcal{C})(\mathbf{F}$$
$$-\nabla\Pi^2) + \mathbf{Re}^{loc}\sum_{i,j=1}^{n}\mathbf{M}^{ij}(F_i - \frac{\partial \Pi^2)}{\partial x_i})(F_j - \frac{\partial \Pi^2)}{\partial x_j}) + (\mathbf{Re}^{loc})^2 \sum_{i_1,i_2,i_3=1}^{n}[F_{i_1} -$$
$$\frac{\partial \Pi^2)}{\partial x_{i_1}}(x)][F_{i_2} - \frac{\partial \Pi^2)}{\partial x_{i_2}}(x)][F_{i_3} - \frac{\partial \Pi^2)}{\partial x_{i_3}}(x)]\mathbf{M}^{i_1,i_2,i_3}, \quad (33)$$

where

$$\mathcal{C}_{kj} = \sum_{i=1}^{n}(M_k^{ij} + M_k^{ji})\frac{\partial p^1}{\partial x_i}, \quad k,j = 1,\dots n. \quad (34)$$

We note that the pressure p^1, from the quadratic perturbation, is given by (24) and depends non-locally on the permeability K and the geometry, linearly on $\mathbf{M}^{ij}$ and quadratically on ∇p^0.

Finally, we write the filtration law in the form which generalizes cubic expressions for Fochheimer's law:

$$\mathbf{F} - \nabla \Pi^2 = (K + (\mathbf{Re}^{loc})^2 \mathcal{C})^{-1} \mathcal{V}^2 - \mathbf{Re}^{loc} K^{-1} \sum_{i,j=1}^{n} \mathbf{M}^{ij} (K^{-1}\mathcal{V}^2)_i (K^{-1}\mathcal{V}^2)_j$$
$$-(\mathbf{Re}^{loc})^2 \sum_{i_1,i_2,i_3=1}^{n} \mathbf{B}^{i_1,i_2,i_3} (K^{-1}\mathcal{V}^2)_{i_1} (K^{-1}\mathcal{V}^2)_{i_2} (K^{-1}\mathcal{V}^2)_{i_3} + \mathcal{O}((\mathbf{Re}^{loc})^3), \tag{35}$$

where the vectors $\mathbf{B}^{i_1,i_2,i_3}$ are given by

$$\mathbf{B}^{i_1,i_2,i_3} = K^{-1}\mathbf{M}^{i_1,i_2,i_3} - \sum_{j=1}^{n} K^{-1}(\mathbf{M}^{i_1 j} + \mathbf{M}^{j i_1})\big(K^{-1}\mathbf{M}^{i_2 i_3}\big)_{i_1}. \tag{36}$$

Therefore we arrived at the following conclusions:

- The cubic filtration law (35) describes the effective filtration through porous media with the precision of order $\mathcal{O}((\mathbf{Re}^{loc})^3)$. It is different from the effective law obtained by Rasoloarijaona and Auriault in [35], which contains higher order derivatives of the pressure.
- The obtained law (35) is close to the result of Wodié and Lévy in [45]. Nevertheless, there is a difference in the expansion for the pressure.
- It is interesting to note the change in the permeability due to the high local Reynolds number. It was observed before in [41].
- In [10] it is proved that the difference between the physical dimensionless velocity $\mathbf{v}_\varepsilon$ and the upscaled velocity, satisfying the cubic filtration law (35), is of order $\mathcal{O}((\mathbf{Re}^{loc})^3)$ in the energy (L^2) norm. This rigorously establishes (35) as the correct filtration law.

We conclude section §2 by writing the cubic filtration law (35) in the dimensional form. We note that our procedure could be continued to any order of precision. But for general media higher order expansions start to be cumbersome and we prefer giving them for the particular case of the constricted tubes in section §3.

2.3 Dimensional form of the cubic filtration law for constant pressure drop

Now se consider again the case $\mathbf{F} = \text{ sign } (-\Delta P) \dfrac{2}{\sqrt{\varphi}} \mathbf{e}_1$.

Then we see immediately that

$$p^k = 0, \quad \text{for every} \quad k; \quad \mathbf{v}_F^0 = K \text{ sign } (-\Delta P)\frac{2}{\sqrt{\varphi}}\mathbf{e}_1; \quad \mathbf{v}^{1,F} = \frac{4}{\varphi}\mathbf{M}^{11};$$

$$\mathbf{v}^{2,F} = \frac{8}{\varphi\sqrt{\varphi}} \text{ sign } (-\Delta P)\mathbf{M}^{1,1,1}.$$

Now, using that $\mathbf{v}^{phys,0} = \mathbf{v}_F^0 \frac{\sqrt{\varphi}}{2}\varepsilon^2 \frac{L|\Delta P|}{\mu}$ and that physical permeability K^{phys} is equal to $\varepsilon^2 L^2 K$, we obtain in the zero order Darcy's law in its dimensional form:

$$\mathbf{v}^{phys,0} = -\frac{K^{phys}}{\mu}\frac{\Delta P}{L}\mathbf{e}_1.$$

Next for $\mathcal{V}^1 = \mathbf{v}_F^0 + \mathbf{Re}^{loc}\mathbf{v}^{1,F} = K \text{ sign } (-\Delta P)\frac{2}{\sqrt{\varphi}}\mathbf{e}_1 + \mathbf{Re}^{loc}\frac{4}{\varphi}\mathbf{M}^{11}$ we use the equation (28) and obtain

$$\frac{\Delta P}{L}\mathbf{e}_1 = -\mu(K^{phys})^{-1}\mathcal{V}^{phys,1} + \rho(\varepsilon L)^5((K^{phys})^{-1}\mathcal{V}^{phys,1})_1^2(K^{phys})^{-1}\mathbf{M}^{11}.$$

We note that

$$M_1^{11} = \int_{Y_F} u_1^{11}\ dy = \int_{Y_F} \nabla_y \mathbf{w}^1 \nabla_y \mathbf{u}^{11}\ dy = -\int_{Y_F} (\mathbf{w}^1\nabla_y)\mathbf{w}^1\mathbf{w}^1\ dy = 0.$$

Hence for scalar permeability there is no quadratic contribution in the direction of the flow. In general $(K^{phys})^{-1}\mathcal{V}^{phys,1})_1$ satisfies classical Darcy's law. We will see in next section that for constricted tubes the quadratic terms vanishes completely.

Finally, we switch to the cubic filtration law. Now the non-dimensional velocity is $\mathcal{V}^2 = \mathbf{v}_F^0 + \mathbf{Re}^{loc}\mathbf{v}^{1,F} + (\mathbf{Re}^{loc})^2\mathbf{v}^{2,F} = K \text{ sign } (-\Delta P)\frac{2}{\sqrt{\varphi}}\mathbf{e}_1 + \mathbf{Re}^{loc}\frac{4}{\varphi}\mathbf{M}^{11} + (\mathbf{Re}^{loc})^2\frac{8}{\varphi\sqrt{\varphi}} \text{ sign } (-\Delta P)\mathbf{M}^{1,1,1}$ and using (35) we get

$$\begin{aligned}\frac{\Delta P}{L} = &-\frac{\mu}{\varepsilon^2 L^2}(K^{-1}\mathcal{V}^{phys,2})_1 + \frac{\rho}{\varepsilon L}(K^{-1}\mathcal{V}^{phys,2})_1^2(K^{-1}\mathbf{M}^{11})_1 + \\ &\frac{\rho^2}{\mu}(K^{-1}\mathcal{V}^{phys,2})_1^3((K^{-1}\mathbf{M}^{1,1,1})_1 - 2(K^{-1}\mathbf{M}^{11})_1^2).\end{aligned} \tag{37}$$

We note that

$$\begin{aligned}M_1^{1,1,1} &= \int_{Y_F} u_1^{1,1,1}(y)\ dy = \int_{Y_F} \nabla_y \mathbf{w}^1 \nabla_y \mathbf{u}^{1,1,1}\ dy = -\int_{Y_F} (\mathbf{w}^1\nabla_y)\mathbf{u}^{1,1}\mathbf{w}^1\ dy \\ &= \int_{Y_F} (\mathbf{w}^1\nabla_y)\mathbf{w}^1\mathbf{u}^{1,1}\ dy = -\int_{Y_F} |\nabla_y \mathbf{u}^{11}|^2\ dy < 0.\end{aligned} \tag{38}$$

3 Non-linear filtration laws for flows through constricted tubes with axial variations in diameter

In this section we study inertia effects for viscous incompressible flows through a porous medium being a bundle of parallel axially symmetric constricted tubes. We suppose that the porous medium is periodic, obtained by periodic repetition of the tubes $\tilde{\Omega}_\delta = \{x_1 \in (0, L_1);\ 0 < \tilde{r} < \delta\tilde{R}(x_1/\delta)\}$, where $\delta > 0$ is a small parameter, representing the ratio between the pore size and the size of the domain. The porous medium contains a large number of tubes, proportional to C/δ^2. Function $\tilde{R}$ is periodic with the period L. This means that the constriction repeats periodically a number of times proportional to $1/\delta$. In order to avoid boundary layer effects, we suppose that number an integer.

Flow is governed by a given pressure drop ΔP in the direction x_1. This pressure drop determines the characteristic volume force $\dfrac{\Delta P}{L_1}\mathbf{e}_1$ and we suppose the flow periodic in x_1 direction, with the period L_1.

Since we consider the stationary incompressible viscous flow through a porous medium, it is described by the incompressible Navier-Stokes system

$$-\mu\nabla^2\mathbf{v} + \rho(\mathbf{v}\nabla)\mathbf{v} + \nabla p = \frac{-\Delta P}{L_1}\ \mathbf{e}_1 \ \text{ in } \ \Omega_F\ , \tag{39}$$

$$\operatorname{div}\mathbf{v} = 0 \ \text{ in } \ \Omega_F\ , \qquad \mathbf{v} = 0 \ \text{ on } \ (\partial\Omega_F) \setminus (\{x_1 = 0\} \cup \{x_1 = L_1\})\ , \tag{40}$$

where Ω_F is the fluid part of the porous medium, i.e. union of the constricted separated tubes $\tilde{\Omega}_\delta$.

Due to the particular periodic geometry and to the constant forcing term, it is obvious that it is sufficient to solve the problem in just one tube. We choose the tube with the axis $\tilde{r} = 0$. Furthermore, the Navier-Stokes system (39)-(40) is invariant to the following change of variables and unknowns:

$$\mathbf{x} = \delta\mathbf{x}; \qquad \mathbf{v} = \frac{\mathbf{v}}{\delta}; \qquad p = \frac{p}{\delta^2}, \tag{41}$$

and it is sufficient to consider the problem

$$-\mu\nabla^2\mathbf{v} + \rho(\mathbf{v}\nabla)\mathbf{v} + \nabla p = \frac{-\Delta P}{L_1}\delta^3\ \mathbf{e}_1 \ \text{ in } \ \tilde{\Omega}\ , \tag{42}$$

$$\operatorname{div}\mathbf{v} = 0 \ \text{ in } \ \tilde{\Omega}, \quad \mathbf{v} \ \text{ and } \ p \ \text{ are } \ L-\text{periodic in } \ x_1, \tag{43}$$

$$\mathbf{v} = 0 \ \text{ on } \ \partial\tilde{\Omega} \setminus (\{x_1 = 0\} \cup \{x_1 = L\}), \tag{44}$$

where $\tilde{\Omega}$ is the canonic axially symmetric constricted tube $\tilde{\Omega} = \{x_1 \in (0, L);\ 0 < \tilde{r} < \tilde{R}(x_1)\}$. Without loosing generality we can suppose $L = L_1$.

Due to the above proved equivalence the direct numerical simulations in §4 will be performed for a single tube with the pressure drop modified on the way we saw above.

As we deal with a porous medium, we will apply the results from §2. We note that the fluid part of the porous medium now is not connected. As we will see this leads to many simplifications. Mathematical theory of the homogenization process could be generalized to such situation (see [33]). Also pipes could be considered as thin domains and then there is a direct analysis by Bourgeat and Marušić-Paloka in [11].

Our goal is to find an expansion of the velocity and pressure fields in terms of the local Reynolds number and to obtain from it polynomial non-Darcian filtration laws, giving a relationship between the pressure drop and the effective volumetric flow for this simple geometry.

3.1 Darcy permeability and the quadratic correction

Hence we consider a porous medium which is a bundle of capillary tubes. Each tube is obtained by a periodic repetition of the dilated unit tube $Y_F ==\{y_1 \in (0,1)\ :\ 0 < r < R(y_1) = \tilde{R}(y_1)/L\}$, contained in a cell $Y = (0,1)^3$. Y_F has as a boundary an axially symmetrical surface of revolution S, given by

$$r = R(y_1), \quad r^2 = y_2^2 + y_3^2. \tag{45}$$

The internal boundary does not intersect the boundary of Y, except at $y_1 = 0, 1$, where we have the inlet and outlet boundaries. In such geometry, our expressions simplify considerably.

First, it is easy to study the problem (14) and get

$$\mathbf{w}^i = 0, \quad \pi^i = y_i - \frac{1}{|Y_F|}\int_{Y_F} y_i\ dy,\ i = 2,3;\ K^{ij} = \delta_{1j}\delta_{i1}K^{11} > 0, \tag{46}$$

$$\varphi = \int_0^1 \pi R^2(y_1)\ dy_1 \quad K^{11} = 2\pi\int_0^{R(0)} w_1^1(0,r)\ rdr, \tag{47}$$

$$\mathbf{v}_F^0 = K^{11}\ \mathrm{sign}\ (-\Delta P)\frac{2}{\sqrt{\varphi}}\mathbf{e}_1, \quad p^0(x) = 0 \tag{48}$$

We recall that $\mathbf{w}^1$ is the 1-periodic in y_1 solution for the problem (14) with $i = 1$

$$\left\{\begin{array}{l} -\nabla_y^2\mathbf{w}^1 + \nabla_y\pi^1 = \mathbf{e}^1\ ,\ \mathrm{div}_y\mathbf{w}^1 = 0\ \text{ in }\ Y_F \\ \mathbf{w}^1 = 0\ \text{ on }\ S,\ \int_{Y_F}\pi^1 = 0 \\ \{\mathbf{w}^1,\pi^1\} \quad \text{are} \quad 1-\text{periodic with respect to }\ y_1. \end{array}\right. \tag{49}$$

Now, using that $\mathbf{v}_{tube}^{phys,0} = \mathbf{v}_F^0 \frac{\sqrt{\varphi}}{2} \varepsilon^2 \frac{L|\Delta P|}{\mu}$ and that physical permeability K_{tube}^{phys} is equal to $\varepsilon^2 L^2 K^{11}$, we obtain in the zero order Darcy's law in its dimensional form:

$$\mathbf{v}_{tube}^{phys,0} = -\frac{K_{tube}^{phys}}{\mu} \frac{\Delta P}{L} \mathbf{e}_1.$$

Next step is to study the problem (23). Clearly, the solution is not zero only if $i = j = 1$. Hence only cell problem to be solved is

$$\begin{cases} -\nabla_y^2 \mathbf{u}^{11} + \nabla_y \Lambda^{11} = -(\mathbf{w}^1 \nabla_y)\mathbf{w}^1 \ \text{ in } \ Y_F \\ \text{div}_y \mathbf{u}^{11} = 0 \ \text{ in } \ Y_F \\ \mathbf{u}^{11} = 0 \ \text{ on } \ S \ , \ \int_{Y_F} \Lambda^{11} \ dy = 0 \\ \{\mathbf{u}^{11}, \Lambda^{11}\} \quad \text{are} \quad 1-\text{periodic with respect to } \ y_1. \end{cases} \tag{50}$$

Consequently, only $\mathbf{M}^{11}$ is potentially a non zero vector. Let us calculate its components:

$$M_1^{11} = \int_{Y_F} u_1^{11}(y) \ dy = \int_{Y_F} \nabla_y \mathbf{w}^1 \nabla_y \mathbf{u}^{11} \ dy = -\int_{Y_F} (\mathbf{w}^1 \nabla_y)\mathbf{w}^1 \mathbf{w}^1 \ dy = 0.$$

$$M_j^{11} = \int_{Y_F} u_j^{11}(y) \ dy = \ \text{by} \ \ (14) = 0, \quad j = 2, 3.$$

Hence **there is no quadratic correction in this particular geometry**. Hence $\mathbf{v}^{1,F} = 0$ and $p^1 = 0$.

For a geometry having the symmetry of mirror with respect to y_1 around $y_1 = 1/2$, it is easy to see that $-\big((\mathbf{w}^1 \nabla_y)\mathbf{w}^1\big)_1$ is uneven with respect to the plane of symmetry $y_1 = 1/2$. The components $-\big((\mathbf{w}^1 \nabla_y)\mathbf{w}^1\big)_j$, $j = 2, 3$ are even. Consequently, u_1^{11} is an uneven function and Λ^{11}, u_2^{11} and u_3^{11} are even.

3.2 Cubic filtration law

Now we study the problem (29). Clearly, the solution is not zero only if $i_1 = j_1 = j_2 = 1$. Hence only cell problem to be solved is

$$\begin{cases} -\nabla_y^2 \mathbf{u}^{1,1,1} + \nabla_y \Lambda^{1,1,1} = -(\mathbf{w}^1 \nabla_y)\mathbf{u}^{1,1} - (\mathbf{u}^{1,1} \nabla_y)\mathbf{w}^1 \ \text{ in } \ Y_F \\ \text{div}_y \mathbf{u}^{1,1,1} = 0 \ \text{ in } \ Y_F \\ \mathbf{u}^{1,1,1} = 0 \ \text{ on } \ S \ , \ \int_{Y_F} \Lambda^{1,1,1} \ dy = 0 \\ \{\mathbf{u}^{1,1,1}, \Lambda^{1,1,1}\} \quad \text{are} \quad 1-\text{periodic with respect to } \ y_1. \end{cases} \tag{51}$$

Consequently, only $M^{1,1,1}$ is potentially a non zero vector. Let us calculate its components:

As before, $M_j^{1,1,1} = 0$, $j = 2,3$ and by using (38), we have $M_1^{1,1,1} < 0$.

For a geometry having the symmetry of mirror with respect to y_1 around $y_1 = 1/2$, it is easy to see that $-\big((\mathbf{w}^1\nabla_y)\mathbf{u}^{1,1}+(\mathbf{u}^{1,1}\nabla_y)\mathbf{w}^1\big)_1$ is even with respect to the plane of symmetry $y_1 = 1/2$. The components $-\big((\mathbf{w}^1\nabla_y)\mathbf{u}^{1,1} + (\mathbf{u}^{1,1}\nabla_y)\mathbf{w}^1\big)_j$, $j = 2,3$ are uneven. Consequently, $u_1^{1,1,1}$ is an even function and $\Lambda^{1,1,1}$, $u_2^{1,1,1}$ and $u_3^{1,1,1}$ are uneven.

Now $\mathbf{v}^{2,F} = 2M_1^{1,1,1}\frac{8}{\varphi\sqrt{\varphi}}$ sign $(-\Delta P)\mathbf{e}_1$, $p^2 = 0$ and the dimensional filtration law (37) reads

$$\frac{\Delta P}{L} = -\frac{\mu}{\varepsilon^2 L^2 K^{11}} V^{Cubic} + \frac{\rho^2}{\mu}\frac{M_1^{111}}{(K^{11})^4}(V^{Cubic})^3. \tag{52}$$

The law (52) is the cubic non-Darcian law.

3.3 The 5th order filtration law

In order to get higher order filtration laws, we continue with the expansion and take $k = 3$. Then we have:

$$\begin{cases} -\nabla_y^2\mathbf{u}^{iv} + \nabla_y\Lambda^{iv} = -\Big((\mathbf{w}^1\nabla_y)\mathbf{u}^{1,1,1} + (\mathbf{u}^{1,1,1}\nabla_y)\mathbf{w}^1 + (\mathbf{u}^{1,1}\nabla_y)\mathbf{u}^{1,1}\Big) \ \text{ in } \ Y_F \\ \operatorname{div}_y\mathbf{u}^{iv} = 0 \ \text{ in } \ Y_F \\ \mathbf{u}^{iv} = 0 \ \text{ on } \ S \ , \ \int_{Y_F}\Lambda^{iv}\, dy = 0 \\ \{\mathbf{u}^{iv}, \Lambda^{iv}\} \quad \text{are} \quad 1-\text{periodic with respect to } \ y_1. \end{cases} \tag{53}$$

Concerning the permeability, we have as before $M_j^{iv} = 0$, $j = 2,3$. In general geometry M_1^{iv} does not seem to be zero.

Now $\mathbf{v}^{3,F} = \mathbf{M}_1^{iv}\frac{16}{\varphi^2}\mathbf{e}_1$ and $p^3 = 0$, where $M^{iv} = \displaystyle\int_{Y_F}\mathbf{u}^{iv}(y)\, dy$.

For a geometry having the symmetry of mirror with respect to y_1 around $y_1 = 1/2$, it is easy to see that $-\big((\mathbf{w}^1\nabla_y)\mathbf{u}^{1,1,1}+(\mathbf{u}^{1,1,1}\nabla_y)\mathbf{w}^1+(\mathbf{u}^{1,1}\nabla_y)\mathbf{u}^{1,1}\big)_1$ is uneven with respect to the plane of symmetry $y_1 = 1/2$. The components $-\big((\mathbf{w}^1\nabla_y)\mathbf{u}^{1,1,1} + (\mathbf{u}^{1,1,1}\nabla_y)\mathbf{w}^1 + (\mathbf{u}^{1,1}\nabla_y)\mathbf{u}^{1,1}\big)_j$, $j = 2,3$ are even. Consequently, u_1^{iv} is an uneven function and Λ^{iv}, u_2^{iv} and u_3^{iv} are even. For such geometries $M_1^{iv} = 0$.

In our numerical exemples we will only consider the constricted tubes having the symmetry of mirror with respect to y_1 around $y_1 = 1/2$. Hence $M_1^{iv} = 0$ and there is no 4th order term in the non-Darcy law.

We can switch now to the 5th order contribution.

Only auxiliary problem with solution which is not identically zero is the following:

$$\begin{cases} -\nabla_y^2 \mathbf{u}^v + \nabla_y \Lambda^v = -\Big((\mathbf{w}^1 \nabla_y)\mathbf{u}^{iv} + (\mathbf{u}^{1,1,1}\nabla_y)\mathbf{u}^{1,1} + \\ (\mathbf{u}^{1,1}\nabla_y)\mathbf{u}^{1,1,1} + (\mathbf{u}^{iv}\nabla_y)\mathbf{w}^1 \Big) \ \text{ in } \ Y_F \\ \mathrm{div}_y \mathbf{u}^v = 0 \ \text{ in } \ Y_F \\ \mathbf{u}^v = 0 \ \text{ on } \ S \ , \ \int_{Y_F} \Lambda^v \ dy = 0 \\ \{\mathbf{u}^v, \Lambda^v\} \quad \text{are} \quad 1-\text{periodic with respect to } \ y_1. \end{cases} \tag{54}$$

Now $\mathbf{v}^{4,F} = M_1^v$ sign $(-\Delta P)\frac{32}{\varphi^2\sqrt{\varphi}}\mathbf{e}_1$ and $p^4 = 0$, where $M^v = \displaystyle\int_{Y_F} \mathbf{u}^v(y)\ dy$. As before, $M_j^v = 0$, $j = 2, 3$.

For a geometry having the symmetry of mirror with respect to y_1 around $y_1 = 1/2$, it is easy to see that $-\big((\mathbf{w}^1\nabla_y)\mathbf{u}^{iv} + (\mathbf{u}^{1,1,1}\nabla_y)\mathbf{u}^{1,1} + (\mathbf{u}^{1,1}\nabla_y)\mathbf{u}^{1,1,1} + (\mathbf{u}^{iv}\nabla_y)\mathbf{w}^1\big)_1$ is even with respect to the plane of symmetry $y_1 = 1/2$. The components $-\big((\mathbf{w}^1\nabla_y)\mathbf{u}^{iv} + (\mathbf{u}^{1,1,1}\nabla_y)\mathbf{u}^{1,1} + (\mathbf{u}^{1,1}\nabla_y)\mathbf{u}^{1,1,1} + (\mathbf{u}^{iv}\nabla_y)\mathbf{w}^1\big)_j$, $j = 2, 3$ are uneven. Consequently, u_1^v is an even function and Λ^v, u_2^v and u_3^v are uneven.

Dimensional velocity is now given by

$$\varepsilon V^{Fifth} = -\frac{L^2 K^{11}}{\mu}\frac{\Delta P \varepsilon^3}{L} - \frac{L^8 \rho^2 M_1^{111}}{\mu^5}\Big(\frac{\Delta P \varepsilon^3}{L}\Big)^3 - \frac{R_0^{14}\rho^4 M_1^v}{\mu^9}\Big(\frac{\Delta P \varepsilon^3}{L}\Big)^5 \tag{55}$$

and we get the following fifth order non-Darcian law:

$$\frac{\Delta P}{L} = -\frac{\mu}{\varepsilon^2 L^2 K^{11}} V^{Fifth} + \frac{\rho^2}{\mu}\frac{M_1^{111}}{(K^{11})^4}(V^{Fifth})^3 - \frac{\rho^4 \varepsilon^2 L^2}{\mu^3}\frac{3(M_1^{111})^2 - K^{11}M_1^v}{(K^{11})^7}(V^{Fifth})^5. \tag{56}$$

The law (56) is the 5th order non-Darcian law.

4 Numerical simulations

Numerical simulations are performed here in simple porous medium to determine the coefficients of the polynomial; the resulting model is then compared to the numerical solution of the full Navier-Stokes equations. The sinusoidal geometry depicted in Figure 1 is periodic, isotropic and symmetric so the expansion solution derived in section §3 is applicable. One advantage of using

Figure 1

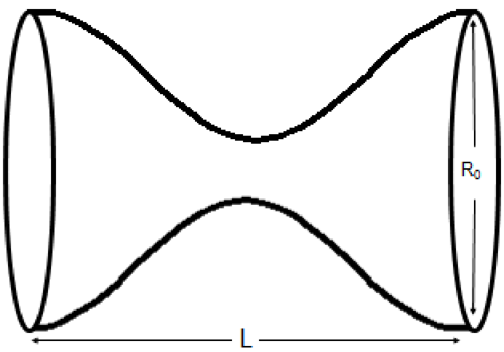

Figure 1: *The constricted tube*

the sinusoidal geometry is that a limited number of numerical simulations of the Navier-Stokes equations have been performed by Deiber and Schowalter in [16] and by Deiber and others in [17], which allowed for verification of our results. The variation in radius of a sinusoidal duct is given by the following equation:

$$R(z) = a - \gamma \cos\big(\frac{2\pi z}{L}\big). \tag{57}$$

The model geometry used has properties $a = 0.16$, $\gamma = 0.08$, and $L = 1$ and fluid properties viscosity and density are chosen arbitrarily as 1.0 Pa-sec and 1 kg/m3, respectively. The Stokes equations could be solved numerically in this porous medium using various methods and the Finite Element Method (FEM) is utilized here. In this work the FEM software COMSOL Multiphysics is used as a tool to solve the sequential Stokes problems as well as the full Navier-Stokes equations. The number of elements (10816) was chosen so that additional refinement did not result in an improvement in the solution. The resulting system of equations is solved using COMSOL's library of linear solvers.

The values of K^{11}, M^{111}, and M^{v} must be determined in order to calculate the coefficients of equation (56). As described in section §3, these parameters may be found from solving Stokes problems of the form:

$$-\nabla P + \mu \nabla^2 \mathbf{v} = \mathbf{F}; \quad \nabla \cdot \mathbf{v} = 0, \tag{58}$$

with velocity field $\mathbf{v} = 0$ on the lateral boundary S and with $\{\mathbf{v}, P\}$ being $L-$periodic in the axial variable. The forcing function, $\mathbf{F}$, is summarized in Table 1 and each subsequent function is dependent on the previous velocity fields.

Forcing Function, $\mathbf{F}$	Output velocity	Calculated value of the averaged quantity
$\mathbf{e}_1$	$\mathbf{w}^1$	$K^{11} = 3.411E - 4$
$(\mathbf{w}^1 \cdot \nabla)\mathbf{w}^1$	$\mathbf{u}^{11}$	$M_1^{11} = 0$
$(\mathbf{w}^1 \cdot \nabla)\mathbf{u}^{11} + (\mathbf{u}^{11} \cdot \nabla)\mathbf{w}^1$	$\mathbf{u}^{1,1,1}$	$M_1^{111} = -1.255E - 14$
$(\mathbf{w}^1 \cdot \nabla)\mathbf{u}^{1,1,1} + (\mathbf{u}^{1,1,1} \cdot \nabla)\mathbf{w}^1$ $+(\mathbf{u}^{11} \cdot \nabla)\mathbf{u}^{11}$	$\mathbf{u}^{iv}$	$M_1^{iv} = 0$
$(\mathbf{w}^1 \cdot \nabla)\mathbf{u}^{iv} + (\mathbf{u}^{1,1,1} \cdot \nabla)\mathbf{u}^{11}$ $+(\mathbf{u}^{11} \cdot \nabla)\mathbf{u}^{1,1,1} + (\mathbf{u}^{iv} \cdot \nabla)\mathbf{w}^1$	$\mathbf{u}^v$	$M_1^v = 1.314E - 24$

Table 1: *Forcing functions for Stokes problems and resulting coefficients obtained numerically*

Streamline plots of the velocity field are shown in Figure 2 for the first five problems and a few observations can be made from the qualitative flow patterns.

The velocity field is symmetric for the first, third, and fifth terms and flow is normal at the boundaries. This is consistent with the model developed in section §3, which leads to finite values of K^{11}, M_1^{111}, and M_1^v. These values are calculated using equations (49), (50), (51), (53) and (54) respectively. The 2nd order and 4th order terms are antisymmetric and flow does not enter or exit the periodic boundaries, resulting in the expected zero values of $\mathbf{M}^{11}$ (isotropic medium) and $\mathbf{M}^{iv}$ (symmetric and isotropic medium). Plugging in the values from Table 1 into equation (56), gives the following filtration law for this porous medium:

$$\frac{\Delta P}{L} = -2931.47V - 0.92705V^3 + 4.57415 \cdot 10^{-5} \cdot V^5 = P_{F5}(V). \tag{59}$$

The model predicts the first correction to Darcy's law is cubic. The magnitude of the coefficients diminishes greatly with order and the cubic term (and certainly higher order terms) may be negligible in practice, which explains Darcy's law often being an acceptable approximation at low Re.

Hence the momentum equation is replaced by the polynomial filtration law. Its coefficients are calculated using the auxiliary Stokes' problems (49), (50), (51), (53) and (54) and then we have a nonlinear scalar PDE for the pressure, valid in the whole reservoir. Its numerical solution is much cheaper than solving the full Navier-Stokes system in the complicated pore geometry. The approximation was validated theoretically by estimates (19)-(22), proved in [30]. Here we validate its accuracy by comparison to a direct numerical solution of the Navier-Stokes equations.

We will solve problem (42)-(44), with $\delta = 1$, for different pressure drops ΔP. Then we will compute the average of the velocity over the constricted tube and check whether it satisfies the nonlinear filtration law obtained by homogenization.

The full Navier-Stokes equations (42)-(44) are solved using the FEM in COMSOL as was done for the Stokes problems. The equations are solved at various applied pressure gradients to compute velocity fields and a macroscopic average velocity, V. Figure 3 shows streamlines of the velocity field at various pressure drops; slight changes can be observed as the Re increases which accounts for corrections to Darcy's law.

At the Table 2 we give the comparaison between pressure drops $\Delta P^{calc} = f_{F5}(V)$, evaluated using (59), and the given pressure drops:

Figures 4a and 4b compare the analytical model (Equation (59)) to the numerical results and show excellent agreement.

In Figure 4a a plot of pressure gradient versus velocity appears linear and the cubic behavior is difficult to distinguish.

Figure 4b is similar to a Forchheimer plot (Equation (2)) and the cubic correction to Darcy's law is more apparent. Moreover, the cubic coefficient is verified by the agreement. The fifth order contribution to pressure loss is not significant, and roundoff error makes it nearly impossible to observe on any plot.

5 Conclusions

Darcy's law states that velocity is proportional to pressure gradient in porous media; it is usually accepted for low velocities in the creeping flow regime ($\mathbf{Re}^{loc} << 1$). Darcy's law can be found analytically or numerically by solving the Stokes equations, but for finite $\mathbf{Re}^{loc}$ the inertial terms will add additional resistance in a porous medium and the relationship between pressure gradient and velocity is nonlinear. Experimentally, a quadratic correction to Darcy's law is often observed, but work by several authors ([31], [45], [14] , [36]) has suggested this functionality is not correct, particularly for $\mathbf{Re}^{loc} < 1$. In this work homogenization has been used to derive a general filtration law in porous media (for $\mathbf{Re}^{loc}$ less than unity) and it verifies that the first correction to Darcy's law is cubic for isotropic media. A novel finding of the homogenization (as well as the expansion for axisymmetric, periodic media) is that the filtration law is an infinite series polynomial and that the coefficients can be determined a priori by solving a series of successive Stokes flow problems. We note that we establish simultaneously the filtration velocity as a polynomial in the pressure gradient (see e.g. (55) and the inverse relationship (see e.g. (56)) where the pressure gradient is a polynomial in the velocity.

The Stokes problems were solved here for a specific medium (an axisymmetric, periodic, sinusoidal duct) and the first five coefficients of the infinite series polynomial were determined. As predicted by the model, the quadratic and 4th order terms are zero because of isotropy and symmetry, respectively. Excellent agreement between the analytical model and a numerical solution of the full Navier-Stokes equations is found, verifying the cubic functionality as well as the quantitative value of the cubic coefficient. Fifth (and higher) order terms are not significant and are not observable numerically. In practice even the cubic term can often be neglected, which verifies the use of Darcy's law at low Re.

The infinite series filtration model is only valid for $\mathbf{Re}^{loc} < 1$ and, as previously stated, corrections to Darcy's law are probably negligible in practice. Additional pressure loss due to inertia is most significant and of most interest in subsurface applications at higher $\mathbf{Re}^{loc}$ (but still in the laminar flow regime). Nonetheless, the results are important from several fundamental perspectives. First of all, it further suggests that Forchheimer's quadratic equation is only empirical and not a universal law valid within the entire laminar flow regime. Since the quadratic model may not be fundamental, deviations from Forchheimer at high $\mathbf{Re}^{loc}$ may also occur (and in fact are observed). Second, as other authors have observed, even if the quadratic law were valid at higher $\mathbf{Re}^{loc}$, extrapolation to the intercept on a Forchheimer plot (Equation 2) would not result in the intrinsic, Darcy permeability. Future work will focus on developing a more general model based on homogenization and/or numerical simulation applicable for $\mathbf{Re}^{loc} > 1$.

Nomenclature

$\mathbf{v}$	Physical velocity	$[L/t]$

p	Pressure $[F/L2]$
V	Characteristic velocity $[L/t]$
μ	Viscosity $[F/L2-t]$
ρ	Density $[M/L3]$
L	Characteristic length M
$\mathcal{P}$	Characteristic pressure $[F/L2]$
ΔP	Pressure drop
Re	Reynolds number
Fr	Froude's number
Ω	Reservoir
Y	Unit cell
Y_s	Solid part of the unit cell
Y_F	Pore
Ω_ε	Pore space (the fluid part of Ω)
ε	Ratio between the pore size ℓ and the reservoir size L
φ	Porosity
$\mathbf{F}$	Dimensionless forcing term
$\mathbf{v}_\varepsilon$	Dimensionless physical velocity
p_ε	Dimensionless pressure
K	Dimensionless permeability

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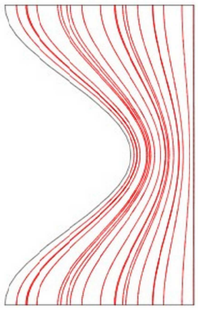

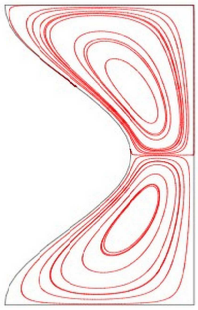

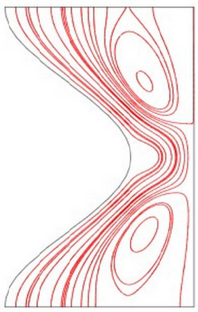

Figure 2: *The streamlines for the Stokes auxiliary problem.* Fig. a *displays streamlines corresponding to the forcing function* $\mathbf{e}_1$ *and so on.*

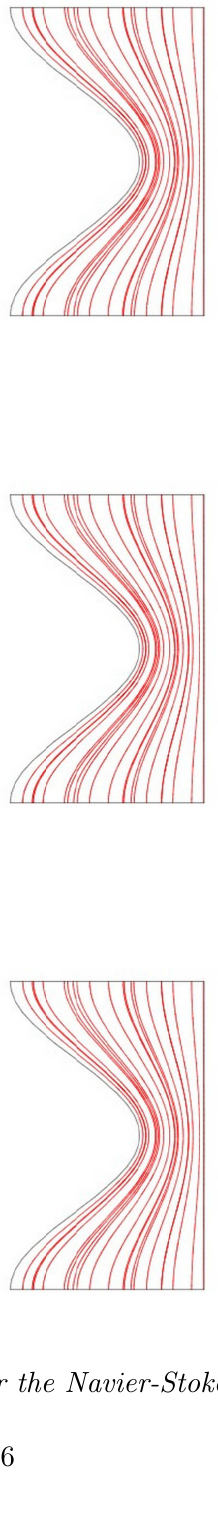

Figure 3: *The streamlines for the Navier-Stokes equations*

$V = \int_{Y_F} v_1 dx$	$\frac{\Delta P}{L}$	$\Delta P^{calc} = f_{F5}(V)$	$\frac{\lvert\Delta P^{calc} - \Delta P\rvert}{\lvert\Delta P\rvert}$	$\Delta P^{D,calc} = f_{F5}(V)$	$\frac{\lvert\Delta P^{D,calc} - \Delta P\rvert}{\lvert\Delta P\rvert}$
3,41E-04	1	1,00	6,39E-07	1,00	6,39E-07
3,41E-01	1000	1000,00	6,88E-07	999,96	3,75E-05
4,29E-01	1258,925412	1258,92	7,17E-07	1258,85	5,90E-05
5,41E-01	1584,893192	1584,89	7,63E-07	1584,75	9,32E-05
6,81E-01	1995,262315	1995,26	8,35E-07	1994,97	1,47E-04
8,57E-01	2511,886432	2511,88	9,49E-07	2511,30	2,33E-04
1,08E+00	3162,27766	3162,27	1,13E-06	3161,11	3,69E-04
1,36E+00	3981,071706	3981,07	1,41E-06	3978,75	5,84E-04
1,71E+00	5011,872336	5011,86	1,83E-06	5007,24	9,24E-04
2,15E+00	6309,573445	6309,56	2,48E-06	6300,35	1,46E-03
2,70E+00	7943,282347	7943,26	3,40E-06	7924,93	2,31E-03
3,40E+00	10000	9999,95	4,58E-06	9963,53	3,65E-03
4,27E+00	12589,2541	12589,19	5,4501E-06	12517,0	0,00574308
5,36E+00	15848,932	15848,87	3,9787E-06	15706,1	0,00901274
6,71E+00	19952,623	19952,79	8,2538E-06	19672,0	0,01406378
8,38E+00	25118,864	25120,39	6,086E-05	24572,5	0,02175123
1,04E+01	31622,777	31630,78	0,00025309	30573,5	0,03318218
1,16E+01	35481,339	35498,46	0,00048243	34037,6	0,04068967
1,29E+01	39810,717	39846,28	0,00089323	37836,6	0,04958874
1,35E+01	41686,938	41734,23	0,0011345	39454,0	0,05356496
1,43E+01	44668,359	44740,35	0,00161169	41988,6	0,059993
1,52E+01	47863,009	47971,60	0,00226881	44656,8	0,0669867

Table 2: *Comparison between the starting pressure drops and the pressure drops recalculated from the 5th order nonlinear filtration law, for velocity obtained by direct solving the stationary Navier-Stokes system*

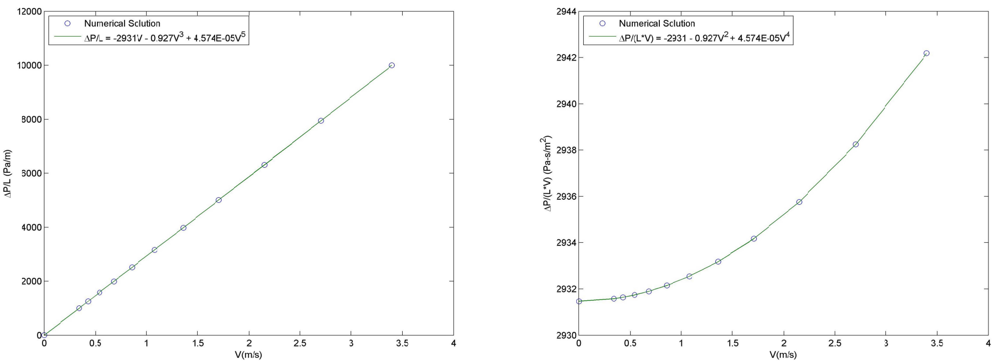

Figure 4: Fig. a: *Darcy's filtration law as the fifth order filtration law in the range of small Reynolds' numbers* $0 \leq \mathbf{Re} \leq 4$*;* Fig. b*: The cubic filtration law as the fifth order filtration law in the range of more important Reynolds' numbers*