An existence result for the equations describing a gas-liquid two-phase flow

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Abstract

We consider the immiscible two phase mixture of water and hydrogen in a porous medium. The water phase is incompressible and the hydrogen phase is compressible. The hydrogen dissolves in the water. The flow is described by the system of non-linear evolution equations for the water saturation and the hydrogen pressure. Under non-degeneracy and slow oscillation assumptions on the diagonal coefficients and with small data for the hydrogen, we establish the existence of a weak solution. *To cite this article: A. Name1, A. Name2, C. R. Mecanique 333 (2005).*

Résumé

Un résultat d'existence pour les équations décrivant un écoulement diphasique gaz −liquide.
Nous considérons le mélange diphasique immiscible de l'eau et de l'hydrogène dans un milieu poreux. L'eau est incompressible et l'hydrogène est compressible. L'hydrogène se dissout dans l'eau. L'écoulement est décrit par le système des équations non linéaires d'évolution pour la saturation de l'eau et la pression d'hydrogène. Sous les conditions de la non-dégénérescence et des petites oscillations des coefficients diagonaux et avec de petites données pour l'hydrogène, nous établissons l'existence d'une solution faible. *Pour citer cet article : A. Name1, A. Name2, C. R. Mecanique 333 (2005).*

Key words: M12 porous media ; two-phase flow ; compressible gaz phase ; existence of a solution

Mots-clés : M12 - Milieux poreux ; écoulement diphasique ; phase gazeuse compressible ; existence d'une solution

1. Introduction to the model

In the context of a deep geological radioactive waste repository, it is expected that significant quantities of hydrogen will be generated mostly by the corrosion of metal components.

The impact of gas transfers on the evolution of the repository is a major concern for the French national radioactive waste management agency **Andra** and it launched the Couplex-Gaz exercise as a benchmark to simulate hydrogen transfers in porous media.

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This important problem renews the mathematical interest in the equations describing multiphase/multicomponent flows through porous media. It turns out that there is a satisfactory mathematical theory for the two-phase incompressible immiscible flows and for more details we refer to the books [1], [2] and [3], articles [4] and [5] and to subsequent publications. In the case of two-phase flows with one (or more) compressible phases, there are practically no mathematical results. Namely, in the case of two-phase incompressible flows equations could be reduced to a parabolic-elliptic system using "global pressure" introduced by G. Chavent. The system is "weakly" coupled and the sophisticated theory, developed for the scalar degenerate parabolic equations could be applied. If one of the phases is compressible, this transformation does not help any more. Consequently one has to deal with a degenerate parabolic system. We note here that if the thermodynamical variables are supposed to depend not on the physical pressure, but on Chavent's "global pressure", then it is possible to extend the known results to the compressible case. For more details we refer to [6]. It is difficult to justify such approach from physical point of view and we prefer keep the modeling from fundamental references.

For general modeling of multiphase/multicomponent flows through porous media we refer to the book [7] and to the article [8]. Here we deal with 2 phases: water (liquid phase) and hydrogen (gas phase). We suppose that both phases satisfy Darcy's law:

$$v^\ell = -\frac{\mathbf{K} k_{r\ell}(S_\ell)}{\mu_\ell}\big(\nabla P_\ell - \rho^\ell g \nabla x_2\big) \quad \text{and} \quad v^g = -\frac{\mathbf{K} k_{rg}(S_g)}{\mu_g}\big(\nabla P_g - \rho^g g \nabla x_2\big); \quad S_\ell + S_g = 1. \tag{1}$$

The indices ℓ and g relate to the liquid and to the gas phase, respectively. $S_i, i = \ell, g$ stands for the saturation, P_i for the pressure, K for the absolute permeability, k_{ri} for the relative permeability, μ_i for the viscosity and ρ^i for the density. g is the gravity acceleration and for simplicity we suppose a 2D situation.

In our particular simplified model, we suppose that water vapor is not present in the gas phase. Then the continuity equation for the gas phase reads

$$\partial_t(\phi S_\ell \rho^\ell X^\ell_{H_2} + \phi S_g \rho^g) + \text{ div } \big(\rho^\ell X^\ell_{H_2} v^\ell + \rho^g v^g\big) - \text{ div } \big(\rho^\ell D^\ell_{H_2} \nabla X^\ell_{H_2}\big) = r_g, \tag{2}$$

where ϕ is the porosity, $X^\ell_{H_2}$ is the mass fraction of the hydrogen in the gas phase and $D^\ell_{H_2}$ is the corresponding diffusion coefficient.

The continuity equation for the liquid phase is

$$\partial_t(\phi S_\ell \rho^\ell X^\ell_{H_2O}) + \text{ div } \big(\rho^\ell X^\ell_{H_2O} v^\ell\big) = r_\ell; \qquad X^\ell_{H_2O} + X^\ell_{H_2} = 1. \tag{3}$$

System (1)-(3) is not complet and we add (i) the capillary pressure relation, (ii) the constitutive law for the gas, and (iii) Henry's law:

$$P_g - P_\ell = P_{c\ell g}(S_\ell); \quad \rho^g = R^* P_g; \quad \rho^l_{H_2} = \frac{X^\ell_{H_2}\rho_w}{1 - X^\ell_{H_2}} = K^*_H P_g, \tag{4}$$

where ρ_w is the standard pure water density, $P_{c\ell g}(S_\ell)$ is the capillary pressure, being a monotone decreasing function of the saturation S_ℓ, and K^*_H and R^* are (small) positive constants. For the detailed presentation of the model we refer to the presentation of the benchmark Couplex-Gaz by J. Talandier and the corresponding Web pages in [9]. For more general modeling of the two-phase two-component flows through porous media one could consult the recent article [10] by A. Bourgeat and M. Jurak.

In this short note our goal is to establish an existence result in a simple case when (i) the evaporation is neglected; (ii) degeneracy is avoided , (iii) the permeability $\mathbf{K}$ is supposed to be a scalar and (iv) the boundary conditions are simplified. We choose as unknowns the saturation of the liquid phase $S = S_\ell$ and the rescaled hydrogen mass density $U = m(S)\rho^l_{H_2}$ in the liquid phase. We have

$$a=\frac{R^*}{K_H^*}\approx 50;\; m(S)=S+a(1-S);\; \frac{dm}{dS}=1-a<0;\; b(S)=\frac{k_{r\ell}(S)}{\mu_\ell}+a\frac{k_{rg}(S)}{\mu_g};\; \rho^g=a\rho^\ell_{H_2}. \quad (5)$$

Henry's law from (4) is generalized to large values of P_g by setting $\rho^\ell_{H_2}=1-e^{-M_{H_2}K_HP_g}$. Since we are going to establish the existence for small hydrogen density, the fact that we know the constitutive laws only for small values of it does not pose problems.

With notations

$$A_{11}(S,U)=\mathbf{K}\frac{dP_g}{d\rho^\ell_{H_2}}\rho^\ell_{H_2}\frac{b(S)}{m(S)}+\frac{D^\ell_{H_2}}{m(S)}\frac{\rho_w}{\rho_w+\frac{U}{m(S)}};\quad \varphi_1(S,U)=\mathbf{K}\rho^\ell_{H_2}b(S)g\nabla x_2, \quad (6)$$

$$A_{12}(S,U)=\frac{\mathbf{K}U}{m(S)}\Big\{\Big(\frac{dP_g}{d\rho^\ell_{H_2}}\rho^\ell_{H_2}b(S)+\frac{D^\ell_{H_2}\rho_w}{\rho_w+\frac{U}{m(S)}}\Big)\frac{(a-1)}{m(S)}-P_c'(S)\frac{k_{rw}(S)}{\mu_w}\Big\} \quad (7)$$

$$A_{21}(S,U)=\frac{k_{rw}(S)}{\mu_w m(S)}\mathbf{K}\frac{dP_g}{d\rho^\ell_{H_2}}=\frac{1}{M_{H_2}K_H}\frac{\mathbf{K}}{U+m(S)}\frac{k_{rw}(S)}{\mu_w} \quad (8)$$

$$A_{22}(S,U)=\mathbf{K}\frac{k_{rw}(S)}{\mu_w}\Big\{\frac{dP_g}{d\rho^\ell_{H_2}}\frac{(a-1)U}{m(S)^2}-P_c'(S)\Big\};\quad \varphi_2(S,U)=\mathbf{K}\frac{k_{rw}(S)}{\mu_w}(\rho_w+U/m(S))g\nabla x_2, \quad (9)$$

the system (1)-(3), (4)-(5) becomes

$$\partial_t(\phi U)-\text{ div }\Big\{A_{11}(S,U)\nabla U+A_{12}(S,U)\nabla S\Big\}+\text{div }\big(\varphi_1(S,U)\big)=r_g \quad (10)$$

$$\partial_t(\phi S)-\text{ div }\Big(A_{21}(S,U)\nabla U+A_{22}(S,U)\nabla S\Big)+\text{ div }\Big(\varphi_2(S,U)\Big)=\frac{r_w}{\rho_w} \quad (11)$$

Through the paper we suppose the following 2 hypothesis
(H1) Let $\lim_{S\to 0,1}P_c'(S)k_{rw}(S)$ exist and let they be different from zero.
(H2) Let there be a constant $\beta>0$ such that $\beta k_{rw}(S)<-P_c'(S)$ on $[0,1]$.
Now for the coefficients we have
Lemma 1.1 *A_{11}, A_{21} and A_{22} are C^∞ functions of S and U on $[0,1]\times[0,+\infty[$, taking values in the intervals $(a_{11,m},a_{11,M})$, $(0,a_{21,M})$ and $(a_{22,m},a_{22,M})$, respectively. $A_{12}=Ua_{12}(S,U)$, where a_{12} is a C^∞ function of S and U on $[0,1]\times[0,+\infty[$, taking values between 2 positive constants, $a_{12,m}$ and $a_{12,M}$. $\varphi_1(S,U)=\chi(U)\Psi_1(S)g\nabla x_2$, where Ψ_1 is a bounded C^∞ function and χ is a C^∞ bounded function of U on $[0,+\infty[$, behaving as U for small values of the variable. $\varphi_2(S,U)$ is a bounded C^∞ function.*

2. Existence of a solution

We start by defining the coefficients for S outside the interval $[0,1]$: $A_{ij}(S,U)=A_{ij}(1,U)$ for $S>1$ and $A_{ij}(S,U)=A_{ij}(S_+,U)$ for $S<0$. Next, for $U<0$, we set $A_{ij}(S,U)=A_{ij}(S,U_+)$. We make the following assumptions on the data:

(i) $\Omega\subset\mathbb{R}^2$ is open, bounded and connected with smooth boundary. Let $V=L^2(0,T;H^1(\Omega))$, $0<T<+\infty$ and $Q_T=\Omega\times(0,T)$.

(ii) For the data we assume $S_0\in W^{1,3}(\Omega)$, $U_0\in W^{1,3}(\Omega)$, $r_w,\ r_g\in L^2(Q_T)$ and $r_w,r_g\geq 0$. ϕ is a positive constant.

Définition 2.1 *We call $\{S,U\}$ a weak solution of the system (10)-(11) if the following properties are fulfilled: $\{S,U\}\in V^2\cap L^\infty(Q_T)^2$, $\partial_t\{S,U\}\in(V^*)^2$ and for all $\{w_1,w_2\}\in V^2$ we have*

$$\int_0^T < \partial_t(\phi U), w_1 > dt + \int_0^T \int_\Omega \Big\{ A_{11}\nabla U + A_{12}\nabla S - \varphi_1(S,U) \Big\} \nabla w_1 dxdt = \int_0^T \int_\Omega r_g w_1 dxdt \tag{12}$$

$$\int_0^T < \partial_t(\phi S), w_2 > dt + \int_0^T \int_\Omega \Big\{ A_{21}(\nabla U + A_{22}\nabla S - \varphi_2(S,U) \Big\} \nabla w_2 dxdt = \int_0^T \int_\Omega \frac{r_w}{\rho_w} w_2 dxdt \tag{13}$$

and for all $w_i \in V \cap W^{1,1}(0,T; L^\infty(\Omega))$, $i = 1,2$, *with* $w_i(T) = 0$,

$$\int_0^T < \partial_t U, w_1 > dt + \int_0^T \int_\Omega (U - U_0)\partial_t w_1 dxdt = 0; \quad \int_0^T < \partial_t S, w_2 > dt + \int_0^T \int_\Omega (S - S_0)\partial_t w_2 dxdt = 0. \tag{14}$$

The system (10)-(11) could loose parabolicity for large values of U. Let $A^R_{ij}(S,U) = A_{ij}(S, \sup\{R, U_+\})$.
Theorem 2.1 *There exist* $\mathcal{R} > 0$ *such that for* $0 < R \le \mathcal{R}$, *the system (12)-(14), with the matrix* $[A_{ij}]$ *replaced by* $[A^R_{ij}]$, *has at least one weak solution* $\{S^R, U^R\}$.
Proof: We use the general theory of quasilinear elliptic-parabolic differential equations from the article [11]. It is enough to check the ellipticity. It is equivalent to the positive definiteness of the quadratic form

$$(\xi_1, \xi_2) \to A^R_{11}(S,U)\xi_1^2 + (A^R_{12}(S,U) + A^R_{21}(S,U))\xi_1\xi_2 + A^R_{22}(S,U)\xi_2^2 \tag{15}$$

For $U = 0$ the sufficient and necessary conditions for the quadratic form (15) to be positive definite are (i) $A^R_{11}(S,0) > 0$, $A^R_{22}(S,0) > 0$ and (ii)$(A^R_{12}(S,0) + A^R_{21}(S,0))^2 < 4A^R_{11}(S,0)A^R_{22}(S,0)$. The former conditions are consequence of the hypothesis that $\lim_{S\to 0,1} P'_c(S)k_{rw}(S)$ exist and is different from zero. The later condition reads

$$(\mathbf{K}k_{rw}(S))/((M_{H_2}K_H)^2\mu_w) < (D^\ell_{H_2}(-P'_c(S))/(m(S)) \qquad \text{on} \quad [0,1]. \tag{16}$$

Under hypothesis **(H2)** , (16) is achieved by multiplying the 2nd equation by $(M_{H_2}K_H)^2 D^\ell_{H_2}\mu_w\beta$ divided by $\mathbf{K}\max m(S)$. By continuity, there is a constant $\mathcal{R} > 0$ such that the quadratic form (15) remains positive definite for $|U| < \mathcal{R}$. This proves the theorem. □
Lemma 2.2 *Let* $r_g, r_w \ge 0$, $k_{rw}(0) = 0$, *let the initial values be non-negatives and let the assumptions of theorem 2.1 be fulfilled. Then solutions are non-negatives.*
Proof: We note that $k_{rw}(0) = 0$ implies $A_{12}(0,U) = 0$. Furthermore, $A_{21}(S,0) = 0$. Now we test the system (10)-(11) by the negative parts of a solution and obtain the result. □

It remains to prove that for small data the L^∞-norm of U is small.

Let $b > 0$ be a given constant. Then for the problem

$$\frac{1}{b}\partial_t u - \Delta u = f + \text{ div } F \text{ in } Q_T; \quad \frac{\partial u}{\partial n} + F\cdot n = 0 \text{ on } \partial\Omega \times (0,T); \quad u = u_0 \text{ on } \Omega. \tag{17}$$

the theory of parabolic potential (see e.g. [12], pages 271-276) gives

$$\|\nabla u\|_{L^q(Q_T)} \le \mathcal{A}(b,q)\Big\{ C_0(b)\|u_0\|_{W^{1,q}(\Omega)} + \|F\|_{L^q(Q_T)^2} \Big\} + \bar{\mathcal{A}}(b,q)(\|f\|_{L^q(Q_T)} + \|u_0\|_{W^{1,q}(\Omega)}), \tag{18}$$

where $\mathcal{A}(b,q)$ is a continuous function of q, equal to 1 for $q = 2$.
Proposition 2.3 *Let* $q \in (2, +\infty)$ *be such that*

$$\ell(q) = \max\{\mathcal{A}(a_{22,M}, q) \sup_{x\in[0,1], y>0} \Big| \frac{A_{22}(x,y)}{a_{22,M}} - 1 \Big|, \ \mathcal{A}(a_{11,M}, q) \sup_{x\in[0,1], y>0} \Big| \frac{A_{11}(x,y)}{a_{11,M}} - 1 \Big| \} < 1. \tag{19}$$

Then we have

$$\|\nabla S^R\|_{L^q(Q_T)} \leq \frac{\mathcal{A}(a_{22,M},q)}{1-\ell(q)}\Big\{ \frac{\|A_{21}\|_\infty}{a_{22,M}}\|\nabla U^R\|_{L^q(Q_T)} + C_{\varphi,2} + C_0\|S_0\|_{W^{1,q}(\Omega)}\Big\} + C_1\|r_w\|_{L^q(Q_T)} \quad (20)$$

$$\|\nabla U^R\|_{L^q(Q_T)} \leq \frac{\mathcal{A}(a_{11,M},q)}{1-\ell(q)}\Big\{ \frac{\|A_{12}\|_\infty}{a_{11,M}}R\|\nabla S^R\|_{L^q(Q_T)} + RC_{\varphi,1} + \|U_0\|_{W^{1,q}(\Omega)}C_3\Big\} + C_2\|r_g\|_{L^q(Q_T)} \quad (21)$$

Proof: Applying (18) to (11) yields

$$\|\nabla S^R\|_{L^q(Q_T)} \leq \mathcal{A}(a_{22,M},q)\Big\{ \sup_{x\in[0,1],y>0}\Big|\frac{A_{22}(x,y)}{a_{22,M}}-1\Big|\|\nabla S^R\|_{L^q(Q_T)} + \frac{\|A_{21}\|_\infty}{a_{22,M}}\|\nabla U^R\|_{L^q(Q_T)} + \|\varphi_2\|_{L^q(Q_T)}\Big\} + C_0\|S_0\|_{W^{1,q}(\Omega)} + C_1\|r_w\|_{L^q(Q_T)}. \quad (22)$$

Next we apply (18) to (10) and get

$$\|\nabla U^R\|_{L^q(Q_T)} \leq \mathcal{A}(a_{11,M},q)\Big\{ \sup_{x\in[0,1],y>0}\Big|\frac{A_{11}(x,y)}{a_{11,M}}-1\Big|\|\nabla U^R\|_{L^q(Q_T)} + \|\varphi_1\|_{L^q(Q_T)} + \frac{\|a_{12}\|_\infty}{a_{11,M}}\|\nabla S^R\|_{L^q(Q_T)}\|U^R\|_{L^\infty(Q_T)}\Big\} + C_0\|U_0\|_{W^{1,q}(\Omega)} + C_1\|r_g\|_{L^q(Q_T)}. \quad (23)$$

After inserting (23) into (22) and using (19) we obtain (20)-(21).□

Corollary 2.4 *Under conditions of Proposition 2.3, there are constants I_S^1 and I_S^2, depending only on $\|S_0\|_{W^{1,q}(\Omega)}$ and on bornes on coefficients, such that*

$$\|\nabla S^R\|_{L^q(Q_T)}\big(1-\frac{\mathcal{A}(a_{11,M},q)\mathcal{A}(a_{22,M},q)}{(1-\ell(q))^2}\frac{a_{21,M}}{a_{22,M}}R\big) \leq \frac{\mathcal{A}(a_{22,M},q)}{1-\ell(q)}\Big(I_S^1 + RI_S^2\Big). \quad (24)$$

Next, we will need the De Giorgi-Nash-Moser L^∞-estimate for the equation (11). Unfortunately the classical result establishes the bound, but without really caring how the constants depend on data. We have to establish that for small data the L^∞-norm is small. We use the result proved in the Appendix A (see also [14]), which applied to the equation (11) reads

Lemma 2.5 *Let $q\in(4,+\infty)$ satisfies (19), let $\Lambda\in(0,1)$ and let $\beta = 2\beta_1(\Omega) + (\frac{T}{|\Omega|})^{1/4}$ be the imbedding constant of $V_2^{1,0}(Q_T)$ in $L^4(Q_T)$. Let us suppose that $\beta^2(|\Omega|T)^{(q-4)/(2q)} \leq \min\{1,a_{11,m}\}\Lambda^2/4$. Then we have*

$$\|U^R\|_{C(\bar{Q}_T)} \leq 65\Lambda\Big(2\|U_0\|_{L^\infty(\Omega)} + \|r_g\|_{L^{q/2}(Q_T)} + \frac{4}{\sqrt{a_{11,m}}}(\|A_{12}\nabla S\|_{L^q(Q_T)} + \|\varphi_1\|_{L^q(Q_T)})\Big) \quad (25)$$

The upper bound for R is equal or smaller than $\min\{\frac{1}{2}\frac{(1-\ell(q))^2}{\mathcal{A}(a_{11,M},q)\mathcal{A}(a_{22,M},q)},\mathcal{R}\}$. Consequently, we estimate $\|\nabla S^R\|_{L^q(Q_T)}$ by the inequality (24) and get as the upper bound $I_S = \frac{2\mathcal{A}(a_{22,M},q)}{1-\ell(q)}\Big(I_S^1 + RI_S^2\Big)$,with R replaced by the above value.

Theorem 2.6 *Let us suppose (19) with $q>4$ and the hypothesis of Lemmas 1.1, 2.2 and 2.5 and Theorem 2.1. Let $\{S^R,U^R\}$ be a weak solution for the truncated coefficients A_{ij}, constructed in Theorem 2.1, and let $R = \min\{\frac{1}{2}\frac{(1-\ell(q))^2}{\mathcal{A}(a_{11,M},q)\mathcal{A}(a_{22,M},q)},\mathcal{R}\}$ and $\Lambda_0 = \frac{\sqrt{a_{11,m}}}{4}(a_{12,M}I_S + C_{\varphi,1}|Q_T|^{1/q})^{-1}$. We suppose the following conditions on the data*

$$\beta^2(|\Omega|T)^{(q-4)/(2q)} \leq \min\{1,a_{11,m}\}\Lambda_0^2/4 \quad \text{and} \quad 130\Lambda_0\Big(\frac{1}{2}\|r_g\|_{L^{q/2}(Q_T)} + \|U_0\|_{L^\infty(\Omega)}\Big) < R. \quad (26)$$

Then $\|U^R\|_{L^\infty(Q_T)} \leq R$ *and* $\{S^R, U^R\}$ *is a weak solution for the system (12)-(14).*
Proof: It is a direct consequence of the preceding Lemmas, of the choice of R and Λ_0 and the conditions (26). □
Remark 1 *We note that (19) holds if the oscillation of coefficients* A_{11} *and* A_{22}*, given by (6) and (9), is not large. Physically, under the hypothesis (H1)-(H2) it reduces to the smalness of* $\dfrac{dP_g}{d\rho^\ell_{H_2}}$*. Assumptions of Lemma 2.5 are always fulfied if the length of the time interval is not too large.*

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Appendix A. Adapted De Giorgi-Nash-Moser parabolic estimate

In this appendix we establish a variant of the classical De Giorgi-Nash-Moser parabolic estimate, which gives the precise dependence of the L^∞-norm of the solution on the data. We repeat, after [13], that $V_2^{1,0}(Q_T) = C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$. Let β be the imbedding constant of $V_2^{1,0}(Q_T)$ in $L^4(Q_T)$. After [13], page 77, it is given by

$$\beta = 2\beta_1(\Omega) + (\frac{T}{|\Omega|})^{1/4}. \tag{A.1}$$

Theorem A.1 *Let $B \in (L^\infty(Q_T))^4$ be a such that $B\xi\xi \geq \alpha_0 |\xi|^2, \forall \xi \in \mathbb{R}^2$ and let $g, |\overrightarrow{f}|^2 \in L^q(Q_T)$, $q > 2$ and $u_0 \in C(\overline{\Omega})$. Let $u \in V_2^{1,0}(Q_T)$ be any weak solution for the boundary/initial value problem*

$$\partial_t u - \ div\ (B(x,t)\nabla u) = g + \ div\ \overrightarrow{f} \quad in \quad Q_T \tag{A.2}$$

$$u|_{t=0} = u_0 \ \ in \ \ \Omega \ ; \quad (B\nabla u + \overrightarrow{f}) \cdot \nu = 0 \ on\ \partial\Omega \times (0,T). \tag{A.3}$$

Let $\Lambda \in (0,1)$ be an arbitrary constant. Furthermore, let us suppose that

$$\beta^2 (|\Omega| T)^{(q-2)/(2q)} \leq \frac{\Lambda^2}{4} \min\{1, \alpha_0\}. \tag{A.4}$$

Then $u \in C(\overline{Q}_T)$ and we have

$$\max_{(x,t)\in \overline{Q}_T} u(x,t) \leq 2\Lambda (\frac{2}{\sqrt{\alpha_0}} \| |\overrightarrow{f}|^2 \|^{1/2}_{L^q(Q_T)} + \frac{1}{2} \|g\|_{L^q(Q_T)} + \|u_0\|_{L^\infty(\Omega)}) \Big(1 + 2^{2q(q-1)/(q-2)^2} \Big). \tag{A.5}$$

Proof: We follow the proof of the corresponding result from the book [13], pages 181-186. It is the Theorem 7.1. As there, we test the weak form of the problem (A.2)-(A.3) by $u^{(k)} = \sup\{u-k, 0\}$, $k \geq \|u_0\|_{L^\infty(\Omega)}$. Exactly, as in [13], pages 183-184, after straightforward calculations we get

$$\frac{1}{2} \int_\Omega (u^k(x,t))^2 \ dx + \alpha_0 \int_0^t \int_{A_k(\tau)} |\nabla u|^2 \ d\tau dx \leq \int_0^t \int_{A_k(\tau)} \Big(\frac{\alpha_0}{2} |\nabla u|^2 + \frac{2}{\alpha_0} |\overrightarrow{f}|^2 + |g|(u-k) \Big) \ dx d\tau, \tag{A.6}$$

where $A_k(\tau) = \{x \in \Omega \,|\, u(x,t) > k\}$. The inequality (A.6) implies

$$\frac{1}{2} \min\{1, \alpha_0\} \|u^{(k)}\|^2_{V_2^{1,0}} \leq \int_0^t \int_{A_k(\tau)} \Big(\frac{4}{\alpha_0} \frac{|\overrightarrow{f}|^2}{\delta^2} + \frac{1}{2} \frac{|g|}{\delta} \Big) \Big((u-k)^2 + k^2 \Big) \ dx d\tau, \tag{A.7}$$

for every $\delta > 0$ and $k \geq \max\{\|u_0\|_{L^\infty(\Omega)}, \delta\}$. We choose $\delta = \Lambda(\frac{2}{\sqrt{\alpha_0}} \| |\overrightarrow{f}|^2 \|^{1/2}_{L^q(Q_T)} + \frac{1}{2} \|g\|_{L^q(Q_T)} + \|u_0\|_{L^\infty(\Omega)})$. Then using interpolation and embedding inequalities, exactly as in [13], page 185, we obtain

$$\frac{\Lambda}{2} \min\{1, \alpha_0\} \|u^{(k)}\|^2_{V_2^{1,0}} \leq \beta^2 \mu^{(q-2)/(2q)}(k) \|u^{(k)}\|^2_{V_2^{1,0}} + k^2 \mu^{(q-1)/q}(k), \quad \forall k \geq \delta, \tag{A.8}$$

where $\mu(k) = \int_0^t |A_k(\tau)| \ d\tau$ and β is the embedding constant of $V_2^{1,0}(Q_T)$ in $L^4(Q_T)$, given by (A.1). Next, the assumption (A.4) implies that (A.8) reduces to

$$\frac{\Lambda}{4} \min\{1, \alpha_0\} \|u^{(k)}\|^2_{V_2^{1,0}} \leq k^2 \mu^{(q-1)/q}(k), \quad \forall k \geq \delta. \tag{A.9}$$

Finally we use Theorem 6.1., pages 102-103 form [13], to conclude that (A.9) implies (A.5).□