Global-in-time solutions for the isothermal Matovich-Pearson equations

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Abstract

In this paper we study the Matovich-Pearson equations describing the process of glass fiber drawing. These equations may be viewed as a 1D-reduction of the incompressible Navier-Stokes equations including free boundary, valid for the drawing of a long and thin glass fiber. We concentrate on the isothermal case without surface tension. Then the

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Matovich-Pearson equations represent a nonlinearly coupled system of an elliptic equation for the axial velocity and a hyperbolic transport equation for the fluid cross-sectional area. We first prove existence of a local solution, and, after constructing appropriate barrier functions, we deduce that the fluid radius is always strictly positive and that the local solution remains in the same regularity class. This estimate leads to the global existence and uniqueness result for this important system of equations.

Keywords: Fiber drawing; Matovich-Pearson equations; incompressible free boundary Navier-Stokes equations; non-local transport equation; iterated comparison; barrier function.

Mathematics Subject Classification MSC2000 (AMS class-code): 35Q30; 35Q35; 35R35; 35L80; 76D05; 76D27

1 Introduction

The drawing of continuous glass fibers is a widely used procedure. Industrial glass fibers are manufactured by a bushing with more than thousand nozzles. Bushings are supplied with a molten glass from a melting furnace. Its temperature ranges from $1300K$ to $1800K$. In order to understand the glass fiber forming process, it is important to study the drawing of a single glass fiber. This is, of course, a significant simplification because we disregard interaction between fibers and between fibers and the surrounding air. For a single glass fiber, the hot glass melt is forced by gravity to flow through a die into air. After leaving the die, the molten glass forms a free liquid jet. It is cooled and attenuated as it proceeds through the air. Finally, the cold fiber is collected on a rotating drum.

The molten glass can be considered as a Newtonian fluid and the process is described by the non-isothermal Navier-Stokes equations for a thermally dilatable but isochoric fluid. Since we deal with a free liquid jet, the problem is posed as a free boundary problem for the Navier-Stokes equations, coupled with the energy equation. We refer to [5] for detailed modeling and analysis of the equations describing the stationary flow inside the die. There are several models proposed to describe the various stages of the flow of a molten glass from the furnace to the winding spool: the slow flow in the die (the "first phase" of the drawing), the jet formation under rapid cooling (the "second phase"), and the terminal fiber profile (the "third phase") (see [6]).

Since we consider long (their length is approximately 10 m) and thin (their diameter varies from 1 mm to 10 μm) fibers, it is reasonable to apply the lubrication approximation to the model equation. This approach yields good results, at least for flows far from the die exit and in the so-called "third phase" of the fiber drawing.

A standard engineering model for the isothermal glass fiber drawing in the "third phase" is represented by the *Matovich-Pearson equations.* For an axially symmetric fiber with a straight central line, they read

$$\partial_t \mathbf{A} + \partial_x(v\mathbf{A}) = 0; \qquad \partial_x(3\mu(T)\mathbf{A}\ \partial_x v) + \partial_x(\sigma(T)\sqrt{\mathbf{A}}) = 0, \tag{1}$$

where $\mathbf{A} = \mathbf{A}(t,x)$ is the cross-area of the fiber section, $v = v(t,x)$ is the effective axial velocity, 3μ is Trouton's viscosity, and σ denotes the surface tension. As the coefficients μ and σ depend on the temperature, it is necessary to take into account an equation for the temperature $T = T(t,x)$.

The original derivation of the system (1) is purely heuristic and obtained under the assumptions that: *(i)* the viscous forces dominate the inertial ones; *(ii)* the effect of the surface tension is balanced with the normal stress at the free boundary; *(iii)* the heat conduction is small compared with the heat convection in the fiber; *(iv)* the fiber is almost straight, and all quantities are axially symmetric. We refer to the classical papers by Kase & Matsuo [18, 19], and Matovich & Pearson [21] for more details concerning the model.

Another derivation of the model based on a lubrication type asymptotic expansion can be found in the work by Schultz *et al.* [10, 23], Dewynne *et al.* [3, 4], and Hagen [13], with more emphasis on the mathematical aspects of the problem. The (formal) asymptotic expansion is developed with respect to a small parameter ε, proportional to the ratio of the characteristic thickness R_E in the radial direction and the characteristic axial length of the fiber L. The fact that the viscosity changes over several orders of magnitude is surprisingly ignored in these studies. As a matter of fact, the viscosity coefficient depends effectively on the temperature, with values varying from 10 to 10^{12} Pa sec, while in the above mentioned asymptotic expansions it is considered simply of order one. A correct formal derivation was given in [1], and it is in full agreement with the model announced in [13]. Finally, a full non-stationary model of a thermally dilatable molten glass, with density depending on the temperature, was derived in [6].

A mathematical analysis of generalized *stationary* Matovich-Pearson equations is performed in [1] (see also [2]). The non-stationary case, without surface tension and with advection equation for the temperature, is studied by Hagen & Renardy [11]. They prove a local-in-time existence result in the class of smooth solutions. Their approach is based on a precise analysis of the dependence of the solution of the mass conservation equation on the velocity. This method requires controlling higher order Sobolev norms in the construction of solutions by means of an iterative procedure and works only for short time intervals. Hagen *et al.* [12, 14, 15] have also undertaken a detailed study of the linearized equations of forced elongation. Despite this considerable effort, global-in-time solvability of the Matovich-Pearson equations was left open. A heuristic argument explaining why Newtonian fluid filaments do not exhibit ductile failure without surface tension is in [17].

The ratio σ/μ is small, and, furthermore, the inertia and gravity effects are negligible in most applications. Accordingly, we consider the Matovich-Pearson equations (1) with $\sigma = 0$, meaning, the isothermal drawing with constant positive viscosity and in the absence of surface tension. For a prescribed velocity at the fiber end points, the important parameter is the *draw ratio*, being the ratio between the outlet and inlet fluid velocities. It is well known that the instability known as a draw resonance occurs at draw ratios in excess of about 20.2. Linear stability analysis was rigorously undertaken by Renardy [22]. Moreover, in [26], it was established that the cross section, given by the Matovich-Pearson equations with $\sigma = 0$, may vary chaotically at a draw ratio higher than 30, under the condition of periodic variations of the input cross section. There are also numerous articles devoted to numerical simulations confirming such a conclusion. Fairly complete simulations can be found in the papers by Gregory Forest & Zhou [9, 25]. Their simulations predict various aspects of the physical process, like a linearized stability principle, bounds on the domain of convergence for linearly stable solutions, and transition to instability. Their analysis completes that of [8]. We also mention somewhat related results by Fontelos in [7] on one-dimensional model for the evolution of thin jets of viscous fluid with a free boundary, and their break-up.

The above mentioned simplification of system (1) is briefly discussed in [16], however, without rigorous proofs. Our idea is to use the particular structure of the system with $\sigma = 0$, and to prove short-time existence of smooth solutions satisfying good uniform estimates.

It is in analogy with the known results from [11] and [16], but we suppose less compatibility and (i) regularize the continuity equation for small time and (ii) obtain explicit upper and lower bounds for the cross-section. These bounds, obtained in Lemma 1, are independent of the time interval. Nevertheless, they are obtained using smallness of the time interval.

Then, performing a qualitative analysis of the solutions and constructing appropriate barrier functions in (78) and in (81), we show that the cross-section area remains bounded below away from zero. This observation allows us to deduce existence as well as uniqueness of global-in-time solutions.

2 Isothermal fiber drawing without surface tension

We study the system of equations

$$\partial_t A + \partial_x(vA) = 0 \quad \text{in} \quad Q_T = (0,T) \times (0,L), \tag{2}$$

$$\partial_x(3\mu A\ \partial_x v) = 0 \quad \text{in} \quad Q_T = (0,T) \times (0,L), \tag{3}$$

supplemented with the boundary and initial conditions

$$A(t,0) = S_0(t) \ \text{ in } \ (0,T), \quad A(0,x) = S_1(x) \quad \text{in } \ (0,L), \tag{4}$$

$$v(t,0) = v_{in}(t) \ \text{ in } \ (0,T), \quad v(t,L) = v_L(t) \quad \text{in } \ (0,T). \tag{5}$$

Here v is the axial velocity and A denotes the cross section, L, T are given positive numbers, and $3\mu > 0$ denotes Trouton's viscosity assumed to be constant.

The data satisfy

$$\left\{ \begin{array}{l} 0 < v_m \leq v_{in}(t) < v_L(t) \leq V_M \text{ for any } t \in (0,T), \\ \\ 0 < S_m \leq S_0(t), S_1(x) \leq S_M \text{ for all } (t,x) \in Q_T,\ S_0(0) = S_1(0). \end{array} \right. \tag{6}$$

Moreover, the functions v_{in}, v_L, S_0, S_1 belong to certain regularity classes specified below.

2.1 A priori bounds

Our construction of global-in-time solutions is based on certain *a priori* estimates that hold, formally, for any smooth solution of problem (2) - (5), with the cross-section area $A > 0$. The crucial observation is that, as a direct consequence of (3),

$$A(t,x)\ \partial_x v(t,x) = \chi(t) \text{ for any } t \in (0,T), \tag{7}$$

where χ is a function of the time variable only. Moreover, as A is positive and the axial velocity satisfies the boundary conditions (6), we deduce that

$$\chi(t) > 0 \text{ for any } t \in (0,T), \tag{8}$$

which in turn implies

$$\partial_x v(t,x) > 0 \text{ for all } (t,x) \in Q_T. \tag{9}$$

Accordingly,

$$v_{in}(t) < v(t,x) < v_L(t) \text{ for all } (t,x) \in Q_T.$$

Next, we rewrite equation (2) in the form

$$\partial_t A + v\partial_x A = -\chi \leq 0$$

yielding

$$A(t,x) \leq S_M \text{ for all } (t,x) \in Q_T. \tag{10}$$

Integrating (7) over $(0,L)$ and using (5) and (10) give rise to the uniform bound

$$0 < \chi(t) < \frac{v_L(t) - v_{in}(t)}{L} S_M \text{ for all } t \in (0,T). \tag{11}$$

In order to deduce a lower bound for the cross section area A, we first observe that A and $\partial_x A$ satisfy the same transport equation, namely,

$$\partial_t A + \partial_x(vA) = 0, \tag{12}$$

$$\partial_t \left(\partial_x A\right) + \partial_x \left(v \left(\partial_x A\right)\right) = 0. \tag{13}$$

In particular,

$$\partial_t \left(\partial_x \log(A)\right) + v\partial_x \left(\partial_x \log(A)\right) = 0. \tag{14}$$

Evaluating the boundary values of $\partial_x \log(A)$ with (4), gives

$$\partial_x \log(A)(t,0) = \frac{\partial_x A(t,0)}{S_0(t)},\ \partial_x \log(A)(0,x) = \frac{\partial_x S_1(x)}{S_1(x)},$$

where, in accordance with (2), (4), (5), (7), and (11)

$$\begin{aligned} \partial_x A(t,0) &= -\frac{1}{v_{in}(t)} \left(\chi(t) + \frac{dS_0}{dt}(t)\right) \\ &\geq -\frac{1}{v_{in}(t)} \left(\frac{v_L(t) - v_{in}(t)}{L} S_M + \frac{dS_0}{dt}(t)\right). \end{aligned}$$

We deduce easily the desired lower bound on A

$$A(t,x) \geq A_m > 0 \text{ for all } (t,x) \in Q_T, \tag{15}$$

where the constant A_m is determined solely in terms of v_m, V_M, S_m, S_M, and the first derivatives of S_0, S_1.

The *a priori* bounds derived in (7) - (15) form a suitable platform for the existence theory developed in the remaining part of this paper.

3 Short time existence of regularized strong solutions

In addition to (6), we shall assume that

$$S_0 \in W^{2,\infty}(0,T), \quad S_1 \in H^2(0,L), \quad v_{in}, v_L \in C^1[0,T]. \tag{16}$$

where the symbol $W^{k,p}$ denotes the standard Sobolev space of functions having $k-$derivatives L^p-integrable, and $H^2 \equiv W^{2,2}$. Let $f : X \to \mathbb{R}$ be a real-valued function defined on a measure space $(X, \Sigma, \bar{m})$ and with real values. Then the essential supremum of f, denoted by $\operatorname{ess\,sup} f$, is defined by

$$\operatorname*{ess\,sup}_{\mathbb{R}} f = \inf\{a \in \mathbb{R} : \ \bar{m}(x : \ f(x) > a) = 0\}$$

if the set $\{a \in \mathbb{R} : \ \bar{m}(x : \ f(x) > a) = 0\}$ of essential upper bounds is non-empty, and $\operatorname{ess\,sup} f = +\infty$ otherwise.

For further use, we introduce the notation

$$Q^{0,0} = -\frac{dS_0}{dt}(0) - v_{in}(0)\frac{dS_1}{dx}(0). \tag{17}$$

Let us begin with a list of definitions:

Definition 1. *Let t_0 be a positive number. A pair (A, v), defined on $Q_{t_0} = (0, t_0) \times (0, L)$, is a strong solution of (2)-(5) if*

$$A \in W^{1,\infty}((0, t_0) \times (0, L)), \tag{18}$$

$$v, \ \partial_t v, \ \partial_x v, \ \partial^2_{tx} v, \ \partial^2_x v \in L^\infty((0, t_0) \times (0, L)), \tag{19}$$

$$(A, v) \quad \textit{satisfy equations } (2) - (3) \ \textit{a. e. in } (0, t_0) \times (0, L), \tag{20}$$

$$A > 0 \quad \textit{on } Q_{t_0} \ \textit{ and } A \ \textit{ satisfies } (4) \ \textit{ pointwise }, \tag{21}$$

$$v \quad \textit{satisfies } (5) \textit{ pointwise.} \tag{22}$$

Definition 2. *Let t_0 be a constant, $0 < t_0 \le T$. For $h \in L^\infty(0, t_0; H^1(0, L))$ with $\partial_x h \in L^\infty(0, L; L^2(0, t_0))$, we define the energy functional $\mathcal{E}$ as*

$$\mathcal{E}(h)^2 = \operatorname*{ess\ sup}_{0<t<t_0} ||h(t, \cdot)||^2_{H^1(0,L)} + v_m \operatorname*{ess\ sup}_{0<x<L} ||\partial_x h(\cdot, x)||^2_{L^2(0,t_0)}. \tag{23}$$

Similarly, the radius R is defined by

$$\frac{1}{8}R^2 = \int_0^L \left\{ S_1^2 + \left(\frac{dS_1}{dx}\right)^2 \right\}(x) \ dx + \int_0^T v_{in}(t) \left\{ S_0^2 + \frac{2}{v_{in}^2} \left|\frac{dS_0}{dt}\right|^2 \right\}(t) \ dt. \tag{24}$$

Definition 3. *For R given by (24) and $0 < t_0 \le T$, we denote by $\mathcal{S}(t_0, R)$ the convex set of nonnegative functions h defined on Q_{t_0} such that*

$$h \in W^{1,\infty}(0, t_0; L^2(0, L)) \cap L^\infty(0, t_0; H^1(0, L)) \cap \\ \cap W^{1,\infty}(0, L; L^2(0, t_0)) \cap L^\infty(0, L; H^1(0, t_0)), \tag{25}$$

$$\mathcal{E}(h) \le R, \quad \textit{and}$$

$$\operatorname{ess}\sup_{0<t<t_0} ||\partial_t h(t, \cdot)||_{L^2(0,L)} + \sup_{x\in[0,L]} ||\partial_t h(\cdot, x)||_{L^2(0,t_0)} \le \\ RV_M \left(2 + \frac{1}{L} + \frac{\sqrt{T}}{L^{3/2}}\right) + |Q^{0,0}| \left(\sqrt{T} + \sqrt{L}\right) \tag{26}$$

$$h(0, x) = S_1(x) \quad \textit{and} \quad h(t, 0) = S_0(t). \tag{27}$$

Definition 4. *For R given by (24),*

$$\alpha = \frac{4}{v_m}|Q^{0,0}|^2 + 4S_M L|Q^{0,0}|, \quad \beta = \frac{4V_M^2 R^2}{v_m L^3} + 4S_M^2 V_M + \frac{4S_M V_M R}{\sqrt{L}},$$

we denote $t^* \in (0, T]$ a positive time satisfying

$$t^* \le \frac{S_m L^{3/2}}{4RV_M} \quad \textit{and} \quad t^* \le \frac{R^2}{8(\alpha + \beta)}. \tag{28}$$

In this section, for $\delta > 0$ small enough, we construct a family of approximate solutions (A^δ, v^δ) solving the following initial-boundary value problem:

$$\partial_t A^\delta + \partial_x(v^\delta A^\delta) = 0 \quad \text{in} \quad Q^\delta_{t_0} = (\delta, t_0) \times (0, L), \tag{29}$$

$$\partial_t A^\delta + v^\delta \partial_x A^\delta + Q^{0,0}\left(1 - \frac{t}{\delta}\right) + \\ \frac{v_L(\delta) - v_{in}(\delta)}{\displaystyle\int_0^L \frac{d\xi}{A^\delta(\delta, \xi)}} \frac{t}{\delta} = 0 \quad \text{in} \quad (0, \delta] \times (0, L), \tag{30}$$

$$\partial_x(3\mu A^\delta \, \partial_x v^\delta) = 0 \quad \text{in} \quad Q_{t_0} = (0, t_0) \times (0, L), \tag{31}$$

$$A^\delta(t, 0) = S_0(t) \;\; \text{in} \;\; (0, T), \quad A^\delta(0, x) = S_1(x) \quad \text{in} \;\; (0, L), \tag{32}$$

$$v^\delta(t, 0) = v_{in}(t) \;\; \text{in} \;\; (0, T), \quad v^\delta(t, L) = v_L(t) \quad \text{in} \;\; (0, T). \tag{33}$$

Specifically, we prove the following result.

Theorem 1. *Let v_L, v_{in}, S_0 and S_1 satisfy (16). Consider $t^* > 0$ given by (28), $t_0 \in (0, t^*)$, and let $\delta \in (0, \min\{1, t_0\})$ be a small number satisfying*

$$\frac{S_m}{12} - \delta|Q^{0,0}| > 0. \tag{34}$$

Then the initial-boundary value problem (29) - (33) possesses a unique solution (A^δ, v^δ) *in the class*

$$A^\delta \in C^1([0,t_0]; H^1(0,L)) \cap C([0,t_0]; H^2(0,L)), \tag{35}$$

$$v^\delta \in C^1([0,t_0]; H^2(0,L)) \cap C([0,t_0]; H^3(0,L)). \tag{36}$$

Note that for a strictly positive $h \in \mathcal{S}(t_0, R)$, the corresponding velocity field v solving $\partial_x(h\partial_x v) = 0$ with boundary conditions (5) reads

$$v(t,x) = v_{in}(t) + \frac{v_L(t) - v_{in}(t)}{\displaystyle\int_0^L \frac{d\xi}{h(t,\xi)}} \int_0^x \frac{d\xi}{h(t,\xi)}, \tag{37}$$

$$\partial_x v(t,x) = \frac{v_L(t) - v_{in}(t)}{\displaystyle\int_0^L \frac{d\xi}{h(t,\xi)}} \frac{1}{h(t,x)}. \tag{38}$$

Remark 1. *Under the compatibility condition*

$$Q^{0,0} = -\frac{dS_0}{dt}(0) - v_{in}(0)\frac{dS_1}{dx}(0) = \frac{v_L(0) - v_{in}(0)}{\displaystyle\int_0^L \frac{dx}{S_1(x)}}, \tag{39}$$

we could set $\delta = 0$. *However, imposing (39) is not physically clear. Note that this condition is systematically used in [11] as well as [16].*

The rest of this section is devoted to the proof of Theorem 1. The basic and rather standard idea is to construct a sequence approaching a fixed point of a suitable nonlinear mapping. The proof is carried over by means of several steps. We fix $t_0 \in (0, t^*)$ and $\delta \in (0, \min\{1, t_0\})$ such that (34) is satisfied.

STEP 1

We take an arbitrary $A^0 \in \mathcal{S}(t_0, R)$. Then we substitute $h = A^0$ into (37) and calculate $v^0 = v$. We note that $A^0 \in \mathcal{S}(t_0, R)$ implies

$$v^0 \in W^{1,\infty}([0,t_0]; H^1(0,L)) \cap L^\infty([0,t_0]; H^2(0,L)), \tag{40}$$

$$t \mapsto \partial_x v^0(t,0) = \frac{v_L(t) - v_{in}(t)}{\displaystyle\int_0^L \frac{d\xi}{A^0(t,\xi)}} \frac{1}{S_0(t)} \in H^1(0,t_0), \tag{41}$$

$$\partial_x v^0(t,x) > 0 \quad \text{and} \quad v_{in}(t) \le v^0(t,x) \le v_L(t) \ \text{ in } \ [0,t_0] \times [0,L]. \tag{42}$$

Next, we introduce functions Q^0 and $Q^{0,\delta}$ defined on $[0,t_0]$ by

$$Q^0(t) = \frac{v_L(t) - v_{in}(t)}{\displaystyle\int_0^L \frac{d\xi}{A^0(t,\xi)}} \tag{43}$$

$$Q^{0,\delta}(t) = \begin{cases} Q^0(t), & \text{for } \delta \le t \le t_0, \\ \\ Q^{0,0} + (Q^0(\delta) - Q^{0,0})\dfrac{t}{\delta}, & \text{for } 0 \le t < \delta. \end{cases} \tag{44}$$

Obviously, $Q^{0,\delta} \in W^{1,\infty}(0,t_0)$, and, using Jensen's inequality and (6), we get

$$\begin{aligned} 0 \le Q^0(t) &\le \frac{v_L(t)}{\displaystyle\int_0^L \frac{dx}{A^0(t,x)}} \le \frac{v_L(t)}{L^2}\int_0^L A^0(t,x)\,dx \\ &\le \frac{V_M}{L^{3/2}}||A^0(t,\cdot)||_{L^2(0,L)} \le \frac{V_M R}{L^{3/2}}, \end{aligned} \tag{45}$$

since $A^0 \in \mathcal{S}(t_0,R)$, as well as

$$\left|Q^{0,\delta}(t)\right| \le |Q^{0,0}| \left(1 - \frac{t}{\delta}\right)_+ + \frac{V_M R}{L^{3/2}} \tag{46}$$

Now we are in a position to define the solution operator:

For $A^0 \in \mathcal{S}(t_0,R)$ and v^0 given by (37) with A^0 instead of h, we solve the initial-boundary value problem

$$\partial_t u(t,x) = -v^0(t,x)\partial_x u(t,x) - Q^{0,\delta}(t), \quad (t,x) \in Q_{t_0}, \tag{47}$$

$$u(0,x) = S_1(x),\ x \in (0,L), \qquad u(t,0) = S_0(t),\ t \in (0,t_0). \tag{48}$$

Because of (40)-(42) and regularity and compatibility of the data, we may apply a result of Hagen & Renardy [11] (see Theorem 4 in Appendix) to problem (47)-(48) to obtain a unique solution $u \in C^1([0,t_0];H^1(0,L)) \cap C([0,t_0];H^2(0,L))$. We set

$$A^1(t,x) = u(t,x), \quad (t,x) \in Q_{t_0}. \tag{49}$$

Relation (49) defines a nonlinear operator, assigning to a given $A^0 \in \mathcal{S}(t_0,R)$ the unique function A^1 in the class $C^1([0,t_0];H^1(0,L)) \cap C([0,t_0];H^2(0,L))$. By (47), $\partial_t^2 u + \partial_t Q^{0,\delta} \in L^\infty(Q_{t_0})$. Using a compactness lemma of Aubin type (see e.g. [24]) we conclude that our nonlinear operator, defined on $\mathcal{S}(t_0, R)$, is compact.

At this stage we follow the ideas of [11] and [16] to show that this nonlinear operator has a fixed point for all $t_0 < t^*$. They study the nonisothermal fiber spinning and their system of equations is different. Hence we are obliged to give an independent proof of short time existence, but the result remains close to their considerations.

The natural approach is to apply Schauder's fixed point theorem. To use it we have to prove that $\mathcal{S}(t_0, R)$ is a relatively compact convex set invariant under our nonlinear mapping. Finally, we should establish the continuity of the mapping (49).

STEP 2

We establish uniform $L^\infty-$ bounds for the function u. We have

Lemma 1. *Let R be given by Definition 3 and let $t_0 \leq t^*$. Then the solution u of problem (47)-(48) satisfies the estimate*

$$\frac{S_m}{4} \leq u(t,x) \leq S_M + \frac{2S_m}{3} \leq 2S_M \quad \textit{in} \ [0,t_0] \times [0,L]. \tag{50}$$

Remark 2. *It should be noted that the bounds (50) are independent of the time interval. Nevertheless, they are obtained using the "smallness" of t^*.*

Proof. Let $s_m \in W^{2,\infty}(0,t_0)$ be the solution to the Cauchy problem

$$\frac{ds_m(t)}{dt} = -Q^{0,\delta}(t) \text{ in } \ (0,t_0), \ s_m(0) = \frac{2S_m}{3}. \tag{51}$$

Then s_m clearly solves (47) and it follows from (6) and (48) that

$$u(0,x) = S_1(x) \geq S_m \geq s_m(0), \quad x \in (0,L),$$

while (6), (34), the nonnegativity of Q^0, (48), and (51) ensure that

$$\begin{aligned} s_m(t) &= \frac{2S_m}{3} - \int_0^t Q^{0,\delta}(s)ds \leq \frac{2S_m}{3} - \int_0^{\min\{\delta,t\}} Q^{0,\delta}(s)ds \\ &\leq \frac{2S_m}{3} - \int_0^{\min\{\delta,t\}} Q^{0,0}\left(1 - \frac{s}{\delta}\right) ds \leq \frac{2S_m}{3} + |Q^{0,0}|\frac{\delta}{2} \\ &\leq S_m \leq S_0(t) = u(t,0). \end{aligned}$$

The comparison principle then entails that $u(t,x) \geq s_m(t)$ for $(t,x) \in Q_{t_0}$, whence, by (28), (34), and (46),

$$\begin{aligned} u(t,x) &\geq s_m(t) \geq \frac{2S_m}{3} - \int_0^t |Q^{0,\delta}(s)| ds \\ &\geq \frac{2S_m}{3} - |Q^{0,0}|\frac{\delta}{2} - \frac{V_M R}{L^{3/2}}\, t^* \geq \frac{S_m}{4}. \end{aligned}$$

This proves the lower bound. Proving the upper bound is analogous. The comparison function is now given by the solution to the Cauchy problem

$$\frac{ds_M(t)}{dt} = -Q^{0,\delta}(t) \text{ in } (0,t_0),\ s_M(0) = S_M + \frac{S_m}{2}. \tag{52}$$

Indeed, s_M clearly solves (47) with $s_M(0) \geq S_M \geq S_1(x) = u(0,x)$ for $x \in (0,L)$ and it follows from (6), (28), (34), (46), (48), and (52) that

$$s_M(t) \geq s_M(0) - |Q^{0,0}|\frac{\delta}{2} - \frac{V_M R}{L^{3/2}}\, t^* \geq S_M \geq u(t,0).$$

Applying again the comparison principle gives $u(t,x) \leq s_M(t)$ for $(t,x) \in Q_{t_0}$, which completes the proof since (34) guarantees that

$$s_M(t) \leq S_M + \frac{S_m}{2} + |Q^{0,0}|\frac{\delta}{2} \leq S_M + \frac{2S_m}{3}.$$

This proves the Lemma. □

Now we use equation (47) to calculate $G(t) = \partial_x u(t,0) \in C([0,t_0])$ getting:

$$G(t) = \partial_x u(t,0) = -\frac{1}{v_{in}(t)}\left(\frac{dS_0(t)}{dt} + Q^{0,\delta}(t)\right). \tag{53}$$

Due to the assumptions on the data, $G \in W^{1,\infty}(0,t_0)$ and we have

$$|G(t)| \leq \frac{1}{v_{in}(t)}\left(|\frac{dS_0(t)}{dt}| + |Q^{0,\delta}(t)|\right). \tag{54}$$

Next, we take the derivative of equation (47) with respect to the x variable. This yields that $S = \partial_x u$ solves

$$\partial_t S = -v^0 \partial_x S - \partial_x v^0 S, \quad (t,x) \in Q_{t_0}, \tag{55}$$

$$S(0,x) = \frac{dS_1}{dx}(x),\ x \in (0,L); \qquad S(t,0) = G(t),\ t \in (0,t_0), \tag{56}$$

the function v^0 being still given by (37) with A^0 instead of h.

Multiplying equation (47) by u, equation (55) by S, integrating both equations on $(0,t)\times(0,x)$, $(t,x)\in Q_{t_0}$, and adding the obtained identities, we deduce:

$$\frac{1}{2}\int_0^x \{u^2+S^2\}(t,\xi)\ d\xi+\frac{1}{2}\int_0^t v^0(\tau,x)\{u^2+S^2\}(\tau,x)\ d\tau+$$
$$\frac{1}{2}\int_0^t\int_0^x \partial_x v^0(\tau,\xi)\{S^2-u^2\}(\tau,\xi)\ d\xi d\tau+\int_0^t\int_0^x Q^{0,\delta}(\tau)u(\tau,\xi)\ d\xi d\tau$$
$$=\frac{1}{2}\int_0^x \left\{S_1^2+\left(\frac{dS_1}{dx}\right)^2\right\}(\xi)\ d\xi+\frac{1}{2}\int_0^t v_{in}(\tau)\left\{S_0^2+G^2\right\}(\tau)\ d\tau. \quad (57)$$

Now we use (6), (46) with (54) to get

$$\int_0^t v_{in}(\tau)G^2(\tau)\ d\tau\le\int_0^t \frac{2}{v_{in}(\tau)}\left\{\left|\frac{dS_0}{d\tau}\right|^2+|Q^{0,\delta}(\tau)|^2\right\}\ d\tau\le$$
$$\int_0^t \frac{2}{v_{in}(\tau)}\left|\frac{dS_0}{d\tau}\right|^2\ d\tau+\frac{4\min\{t,\delta\}}{v_m}|Q^{0,0}|^2+\frac{4V_M^2R^2}{v_mL^3}t. \quad (58)$$

The third term on the left hand side of (57) is estimated with the help of (43)-(44) and (50) as

$$\int_0^t\int_0^x \partial_x v^0(\tau,\xi)(u^2-S^2)(\tau,\xi)\ d\xi d\tau\le\int_0^t\int_0^x \frac{Q^0(\tau)}{A^0(\tau,\xi)}u^2(\tau,\xi)\ d\xi d\tau$$
$$\le 4S_M^2\int_0^t Q^0(\tau)\int_0^x \frac{d\xi}{A^0(\tau,\xi)}\ d\tau\le 4S_M^2\int_0^t \left(v^0(\tau,x)-v_{in}(\tau)\right)\ d\tau$$
$$\le 4S_M^2V_Mt, \quad (59)$$

while the fourth term obeys

$$\left|\int_0^t\int_0^x Q^{0,\delta}(\tau)u(\tau,\xi)\ d\xi d\tau\right|\le 2S_ML\int_0^t |Q^{0,\delta}(\tau)|\ d\tau$$
$$\le 2S_ML\left(|Q^{0,0}|+\frac{V_MR}{L^{3/2}}\right)t, \quad (60)$$

thanks to (46) and (50).

Now, setting

$$y(t)=\sup_{x\in[0,L]}\left\{\int_0^x \left(u^2+S^2\right)(t,\xi)\ d\xi+\int_0^t v^0(\tau,x)\{u^2+S^2\}(\tau,x)\ d\tau\right\},$$

we may insert (58)-(60) into (57) and use (24) and (28) to get the following estimate:

$$
\begin{aligned}
y(t) &\leq \int_0^L \left\{ S_1^2 + \left(\frac{dS_1}{d\xi}\right)^2 \right\}(\xi)\ d\xi \\
&+ \int_0^t v_{in}(\tau) \left\{ S_0^2 + \frac{2}{v_{in}^2(\tau)} \left|\frac{dS_0}{d\tau}\right|^2 \right\}(\tau)\ d\tau + (\alpha+\beta)t \\
&\leq \frac{R^2}{8} + \frac{R^2}{8}\frac{t}{t^*} \leq \frac{R^2}{4}.
\end{aligned}
$$

Recalling (6) and (42), we thus conclude that

$$\mathcal{E}(u) < R. \tag{61}$$

Next, we use equation (47) and estimate (61) to control $\partial_t u$, obtaining

$$
\begin{aligned}
&\sup_{t\in[0,t_0]} ||\partial_t u(t)||_{L^2(0,L)} + \sup_{x\in[0,L]} ||\partial_t u(\cdot,x)||_{L^2(0,t_0)} < \\
&\qquad RV_M\big(2 + \frac{1}{L} + \frac{\sqrt{T}}{L^{3/2}}\big) + |Q^{0,0}|(\sqrt{T}+\sqrt{L}).
\end{aligned} \tag{62}
$$

Therefore the image of a nonnegative function from $\mathcal{S}(t_0,R)$ remains in $\mathcal{S}(t_0,R)$ and satisfies the $L^\infty-$ bound (50). Therefore our nonlinear map (49) maps the convex set

$$\mathcal{S}_0(t_0,R) = \{f \in \mathcal{S}(t_0,R) \,|\, f \text{ satisfies (50) a.e.}\} \tag{63}$$

into itself.

STEP 3

Let X be the intersection of the Banach spaces $W^{1,\infty}([0,t_0];L^2(0,L))$, $L^\infty([0,t_0];H^1(0,L))$, $W^{1,\infty}(0,L;L^2(0,t_0))$ and $L^\infty(0,L;H^1(0,t_0))$. Clearly, $\mathcal{S}_0(t_0,R)$ is a convex, bounded and closed subset of the Banach space X. We apply the Schauder fixed point theorem to prove that the mapping (49) admits a fixed point in $\mathcal{S}_0(t_0,R)$. After Step 2, it remains only to prove the sequential continuity of the map (49). Hence let $\{A^k\}_{k\in\mathbb{N}}$, be a sequence converging in $\mathcal{S}_0(t_0,R)$ to A. Then we have

$$A^k \to A \quad \text{strongly in } C([0,t_0];L^2(0,L)) \cap L^2(0,t_0;C([0,L])), \tag{64}$$

$$Q_k^0 = \frac{v_L - v_{in}}{\displaystyle\int_0^L \frac{dx}{A^k(t,x)}} \to Q^0 = \frac{v_L - v_{in}}{\displaystyle\int_0^L \frac{dx}{A(t,x)}} \quad \text{uniformly in } C([0,t_0]), \tag{65}$$

$$v^k = v_{in} + Q_k^0 \int_0^x \frac{d\xi}{A^k(t,\xi)} \to v = v_{in} + Q^0 \int_0^x \frac{d\xi}{A(t,\xi)}$$
$$\text{strongly in } C([0,t_0];H^1(0,L)). \tag{66}$$

Let u^k be the solution to (47)-(48), corresponding to A^k and u the solution corresponding to A. Then by analogous calculations to those performed in Step 2, we get $u^k \to u$ in $C([0,t_0];H^1(0,L)) \cap L^2(0,t_0;C^1([0,L]))$. Using the equation (47), we find out that one also has $\partial_t u^k \to \partial_t u$ in $C([0,t_0];L^2(0,L)) \cap L^2(0,t_0;C([0,L]))$. Therefore the mapping (49) is continuous and compact and by Schauder's fixed point theorem there is at least one fixed point $A^\delta \in \mathcal{S}_0(t_0,R)$.

Denoting the corresponding velocity field by v^δ, we have

$$v^\delta \in W^{1,\infty}(0,t_0;H^1(0,L)) \cap L^\infty(0,t_0;H^2(0,L))$$

and since $t \mapsto \partial_x v^\delta(t,0) \in W^{1,\infty}(0,t_0)$, we may apply Theorem 4 in Appendix to conclude that, in fact,

$$A^\delta \in C^1([0,t_0];H^1(0,L)) \cap C([0,t_0];H^2(0,L)).$$

It remains to prove uniqueness.

STEP 4

With the obtained smoothness, uniqueness is easy to establish. It suffices to notice that $L^2(0,t_0;H^1(0,L))$ perturbation of v^δ is controlled by the L^2-perturbation of A^δ in x and t. Then we use this observation, regularity of A^δ, and Gronwall's lemma to obtain uniqueness.

The proof of Theorem 1 is now complete.

4 Global existence of regularized strong solutions

Now we suppose that the regularity of the solution or/and the strict positivity of A^δ, stated in (50), breaks down at the time t_p. Our goal is to prove that $t_p = T$.

Theorem 2. *Under the hypotheses of Theorem 1, the initial-boundary value problem (29)-(33) has a unique solution*

$$A^\delta \in C^1([0,T]; H^1(0,L)) \cap C([0,T]; H^2(0,L)),$$

$$v^\delta \in C^1([0,T]; H^2(0,L)) \cap C([0,T]; H^3(0,L)).$$

The remaining part of this section is devoted to the proof of Theorem 2.

STEP 1

First we recall that, owing to (31) and (33), v^δ is given by

$$v^\delta(t,x) = v_{in}(t) + \frac{v_L(t) - v_{in}(t)}{\displaystyle\int_0^L \frac{d\xi}{A^\delta(t,\xi)}} \int_0^x \frac{d\xi}{A^\delta(t,\xi)}. \tag{67}$$

Next, by (6)

$$Q^\delta(t) = \frac{v_L(t) - v_{in}(t)}{\displaystyle\int_0^L \frac{dx}{A^\delta(t,x)}} > 0 \quad \text{in } \overline{Q_{t_0}}. \tag{68}$$

and we put

$$Q^{cut,\delta}(t) = \begin{cases} Q^\delta(t), & \text{for } \delta \le t \le t_0, \\ \\ Q^{0,0} + (Q^\delta(\delta) - Q^{0,0})\dfrac{t}{\delta}, & \text{for } 0 \le t < \delta. \end{cases} \tag{69}$$

We point out that the upper bound in (50) is independent of t^*. Indeed, we prove now that it is valid regardless the length of the time interval. To this end, we introduce the solution $S_M^\delta \in W^{2,\infty}(0,t_0)$ to the Cauchy problem

$$\frac{dS_M^\delta(t)}{dt} = |Q^{0,0}|\chi_{[0,\delta]}(t), \quad t \in (0,t_0), \quad S_M^\delta(0) = S_M. \tag{70}$$

Owing to the positivity (68) of Q^δ, S_M^δ is a supersolution to (29)-(30) with $S_M^\delta(0) = S_M \ge S_1(x) = A^\delta(0,x)$ for $x \in [0,L]$ and $S_M^\delta(t) \ge S_M^\delta(0) \ge S_0(t) = A^\delta(t,0)$ for $t \in (0,t_0)$. The comparison principle then implies

$$A^\delta(t,x) \le S_M^\delta(t) \le S_M + \delta|Q^{0,0}|. \tag{71}$$

This proves the upper bound and, in addition, the estimate is independent of the length of the time interval.

Therefore, by Jensen's inequality, we may infer that

$$
\begin{aligned}
0 < Q^\delta(t) &\le \frac{V_M}{\displaystyle\int_0^L \frac{d\xi}{A^\delta(t,\xi)}} \le \frac{V_M}{L^2}\int_0^L A^\delta(t,\xi)\ d\xi \\
&\le \frac{V_M}{L}(S_M + \delta|Q^{0,0}|), \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
|G^\delta(t)| = |\partial_x A^\delta(t,0)| &= \left| -\frac{1}{v_{in}(t)}\left(\frac{dS_0(t)}{dt} + Q^{cut,\delta}(t)\right)\right| \\
\le \frac{1}{v_m}&\left(\left|\frac{dS_0(t)}{dt}\right| + |Q^{0,0}| + \frac{V_M}{L}(S_M + \delta|Q^{0,0}|)\right), \qquad (73)
\end{aligned}
$$

for any $t < t_p$, where t_p is the critical time at which $A^\delta(t,x)$ attains zero for some x.

STEP 2

Having established that $\partial_x A^\delta(t,0)$ is bounded in $L^\infty(0,t)$ independently of t, we now look for a strictly positive lower bound for A^δ.

Proposition 1. *There are constants C_1 and C_2, independent of the length of the time interval and of δ, such that we have*

$$
C_1 \le \partial_x \log A^\delta(t,x) \le C_2 \qquad \textit{on} \quad \overline{Q_{t_0}}. \qquad (74)
$$

Proof. We notice that $y = \partial_x \log A^\delta$ satisfies the equation

$$
\partial_t y + v^\delta \partial_x y + \frac{y}{A^\delta}(Q^\delta - Q^{cut,\delta}) = 0 \ \text{ in } \ Q_{t_p}, \qquad (75)
$$

and

$$
y(0,x) = \frac{d\log S_1}{dx}(x), \quad x \in (0,L), \qquad y(t,0) = \frac{G^\delta(t)}{S_0(t)}, \quad t \in (0,t_p). \qquad (76)
$$

The function $F(t,x) = (Q^\delta - Q^{cut,\delta})(t)/A^\delta(t,x)$ vanishes for $t \ge \delta$ and, due to (50), satisfies the following bound

$$
\begin{aligned}
\int_0^t ||F(\tau)||_{L^\infty(0,L)}\ d\tau &= \int_0^{\min\{t,\delta\}} ||F(\tau)||_{L^\infty(0,L)}\ d\tau \\
&\le \frac{8\delta}{S_m}\left(|Q^{0,0}| + ||Q^\delta||_{L^\infty(0,\delta)}\right) \qquad (77)
\end{aligned}
$$

for every t since $\delta \le t_0$.

We next introduce the solution y_m to the ordinary differential equation

$$\frac{dy_m}{dt}(t) = ||F(t)||_{L^\infty(0,L)}\ y_m(t) \quad \text{in } \ (0,T), \tag{78}$$

with initial condition

$$y_m(0) = \min\left\{\min_{[0,L]}\left\{\frac{d\log S_1}{dx}\right\}, -G_M\right\} \leq 0, \tag{79}$$

with

$$G_M = \frac{1}{v_m}\left[\left\|\frac{dS_0}{dt}\right\|_{L^\infty(0,T)} + |Q^{0,0}| + \frac{V_M}{L}(S_M + \delta|Q^{0,0}|)\right]. \tag{80}$$

Owing to (73), (76), (78), and (79), y_m is nonpositive and thus a subsolution to (75), and satisfies $y_m(t) \leq y_m(0) \leq y(t,0)$ for $t \in (0,t_p)$ and $y_m(0) \leq y(0,x)$ for $x \in (0,L)$. The comparison principle then entails that

$$y(t,x) \geq y_m(t) = y_m(0)\exp\int_0^t ||F(\tau)||_{L^\infty(0,L)}\ d\tau.$$

Since $y_m(0) \leq 0$, we deduce from (77) that $y(t,x) \geq C_1$ for some non-positive constant C_1, independent of δ and t_0.

Similarly, let Y_M be the solution to the ordinary differential equation

$$\frac{dY_M}{dt}(t) = ||F(t)||_{L^\infty(0,L)}\ Y_M(t) \quad \text{in } \ (0,T), \tag{81}$$

with initial condition

$$Y_M(0) = \max\left\{\max_{[0,L]}\left\{\frac{d\log S_1}{dx}\right\}, G_M\right\} \geq 0, \tag{82}$$

the constant G_M being defined in (80). It follows from (73), (76), (81), and (82) that Y_M is a supersolution to (75) which satisfies $Y_M(t) \geq Y_M(0) \geq y(t,0)$ for $t \in (0,t_p)$ and $Y_M(0) \geq y(0,x)$ for $x \in (0,L)$. Using once more the comparison principle and (77), we conclude that

$$y(t,x) \leq Y_M(t) = Y_M(0)\exp\int_0^t ||F(\tau)||_{L^\infty(0,L)}\ d\tau \leq C_2,$$

the constant C_2 being independent on δ and t. □

Since $\log A^\delta(t,0) = \log S_0(t) \geq \log S_m > -\infty$ by (6) and

$$\log A^\delta(t,x) = \log S_0(t) + \int_0^x \partial_\xi \log A^\delta(t,\xi)\ d\xi,$$

we infer from (74) that

$$||\log A^\delta||_{L^\infty(0,t_0;W^{1,\infty}(0,L))} \leq C. \tag{83}$$

Therefore A^δ is strictly positive on $\overline{Q_t}$ for all $t \leq t_p$, and bounded from below and from above by a constant which is independent of both t and δ.

STEP 3

Having established (83) we easily obtain the following estimates:

$$||\partial_t A^\delta||_{L^\infty(Q_t)} + ||\partial_x A^\delta||_{L^\infty(Q_t)} \leq C, \tag{84}$$

$$||v^\delta||_{L^\infty(Q_t)} + ||\partial_x v^\delta||_{L^\infty(Q_t)} + ||\partial_x^2 v^\delta||_{L^\infty(Q_t)} + ||\partial_{xt}^2 v^\delta||_{L^\infty(Q_t)} \leq C, \tag{85}$$

where C is again independent of t and δ.

The estimate (85) guarantees that the coefficients in equations (29)-(30) remain regular, whence Theorem 4 is applicable. Consequently, A^δ remains bounded in $C^1([0,t];H^1(0,L)) \cap C([0,t];H^2(0,L))$.

We conclude that, for all t, A^δ is bounded from below by a positive constant, independent of t, and the norm of A^δ in $C^1([0,t];H^1(0,L)) \cap C([0,t];H^2(0,L))$ remains bounded by a constant, also independent of t, that may, however, depend on δ. The maximal solution therefore extends to $[0,T]$, and, in fact, we have established the existence of a unique strictly positive solution A^δ on $(0,T)\times(0,L)$. The corresponding velocity v^δ is given by (37), with $h = A^\delta$.

This completes the proof of Theorem 2 .

5 Existence of a unique strong solution

At this stage, we are ready to establish the main result of the paper.

Theorem 3. *Under the hypotheses (6), (16), the initial-boundary value problem (2)-(5) possesses a unique (strong) solution A, v on $\bar{Q}_T$, belonging to the class*

$$A,\ \partial_t A,\ \partial_x A \in L^\infty((0,L)\times(0,T)),$$
$$v,\ \partial_t v,\ \partial_{t,x}^2 v,\ \partial_{x,x}^2 v \in L^\infty((0,L)\times(0,T)).$$

Proof. We just recall the estimates obtained in the proof of Theorem 2, valid independently of δ:

$$0 < C_1 \leq A^\delta(t,x) \leq C \qquad \text{in} \quad \overline{Q_T}, \tag{86}$$

$$||\partial_x A^\delta||_{L^\infty(Q_T)} + ||\partial_t A^\delta||_{L^\infty(Q_T)} \leq C. \tag{87}$$

Since $L^\infty(Q_T)$ is the dual space of the separable Banach space $L^1(Q_T)$, Alaoglu's weak* compactness theorem gives weak* sequential compactness. Therefore there exist A and v such that

$$A^\delta \to A \qquad \text{uniformly in} \quad \overline{Q_T}, \ \text{ as } \ \delta \to 0, \tag{88}$$

$$\partial_x A^\delta \rightharpoonup \partial_x A \qquad \text{weakly}^* \ \text{ in } \quad L^\infty(Q_T), \ \text{ as } \ \delta \to 0, \tag{89}$$

$$\partial_t A^\delta \rightharpoonup \partial_t A \qquad \text{weakly}^* \ \text{ in } \quad L^\infty(Q_T), \ \text{ as } \ \delta \to 0, \tag{90}$$

$$v^\delta \to v = v_{in}(t) + \frac{v_L(t) - v_{in}(t)}{\displaystyle\int_0^L \frac{d\xi}{A(t,\xi)}} \int_0^x \frac{d\xi}{A(t,\xi)} \text{ uniformly in } \overline{Q_T}, \ \text{ as } \ \delta \to 0. \tag{91}$$

$$\partial_x v^\delta \to \partial_x v = \frac{v_L(t) - v_{in}(t)}{\int_0^L \frac{d\xi}{A(t,\xi)}} \frac{1}{A(t,x)} \text{ uniformly in } \overline{Q_T}, \ \text{ as } \ \delta \to 0, \tag{92}$$

at least for suitable subsequences. Obviously, (A, v) solves the system (2)-(5). Moreover, by virtue of the interior regularity, the equations (2)-(3) are satisfied pointwise. Finally, according to the smoothness of A and v, the proof of uniqueness is straightforward. □

Remark 3. *The standard way of proving uniqueness relies on Gronwall's inequality. For any bounded time interval, small L^2-perturbations of the data v_L, v_{in}, S_0 in the L^2- norm result in the corresponding variation of the solution in the same norm, that may depend exponentially on the length of the time interval. Better estimates would require refined analytical arguments.*

6 Appendix

Here we recall the result from [11], which is used in this paper:

Theorem 4. *Let f and p be given continuous functions defined on $[0, t_0] \times [0, b]$, $b > 0$. Let $u^0 \in C([0, b])$ and $u^b \in C([0, t_0])$ and let us suppose*[1] *that*

$$
\begin{aligned}
& p, f \in W^{1,\infty}([0, t_0]; H^1(0, b)) \cap L^\infty([0, t_0]; H^2(0, b)), \\
& \text{are such that} \quad \partial_x p(\cdot, 0) \text{ and } \partial_x f(\cdot, 0) \in H^1(0, t_0); \qquad (93) \\
& p < 0 \quad \text{on} \quad [0, t_0] \times [0, b]; \qquad (94) \\
& u^0 \in H^2(0, b), \quad u^b \in H^2(0, t_0); \qquad (95) \\
& u^0(0) = u^b(0); \quad \partial_t u^b(0) = p(0, 0)\partial_x u^0(0) + f(0, 0). \qquad (96)
\end{aligned}
$$

Then the boundary-initial value problem

$$\partial_t u = p(t, x)\partial_x u + f(t, x), \quad (x, t) \in (0, t_0) \times (0, b), \tag{97}$$

$$u(0, x) = u^0(x), \ x \in (0, b); \qquad u(t, 0) = u^b(t), \ t \in (0, t_0). \tag{98}$$

has a solution

$$u \in C^1([0, t_0]; H^1(0, b)) \cap C([0, t_0]; H^2(0, b)), \tag{99}$$

which is unique in $W^{1,\infty}(0, t_0; L^2(0, b)) \cap L^\infty(0, t_0; H^1(0, b))$.

7 Note added in revision

After this article was submitted for publication, an independent global existence proof by Hagen and Renardy, in [27], came to our attention. Their proof is different and relies on using a new unknown (the Lagrange variable) and properties of the linear transport equation.

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