Analysis of Differential Equations Modelling the Reactive Flow through a Deformable System of Cells

Willi Jäger
Institut für Angewandte Mathematik, Universität Heidelberg
Im Neuenheimer Feld 294, 69121 Heidelberg, Germany
E-mail: jaeger@iwr.uni-heidelberg.de

Andro Mikelić *
Université de Lyon, F-69003 France
Université Lyon 1, Institut Camille Jordan
Site de Gerland, Bât. A, 50, avenue Tony Garnier, 69367 Lyon Cedex 07, France
E-mail: Andro.Mikelic@univ-lyon1.fr

Maria Neuss-Radu
Institut für Angewandte Mathematik, Universität Heidelberg
Im Neuenheimer Feld 294, 69121 Heidelberg, Germany
E-mail: maria.neuss-Radu@iwr.uni-heidelberg.de

Abstract

A system of model equations coupling fluid flow, deformation of solid structure and chemical reactions is formulated starting from processes in biological tissue. The main aim of this paper is to analyse this non-standard system, where the elasticity modules are functionals of a concentration and the diffusion coefficients of the chemical substances are functions of their concentrations. A new approach and new methods are required adapted to these nonlinearities and the transmission conditions on the interface solid-fluid. Strong solutions for the initial and boundary value problem are constructed under suitable regularity assumptions on the data, and stability estimates of the solutions with respect to the initial and boundary values are proved. These estimates imply uniqueness directly.

*This work was done when A.M. was on the sabbatical leave at IWR and Institut für Angewandte Mathematik, Universität Heidelberg, October 1, 2005 - March 31, 2006.

The approach of the paper can be used in more general problems modeling reactive flow and transport and its interaction with elastic cell structures. In a forthcoming paper the approach of this paper is used for getting the upscaled system modeling reactive flow through biological tissue on the macroscopic scale, starting from a system on the cell level.

1 Introduction

In this paper we are studying model equations for processes in a porous elastic structure of cells. Experimental research on the physiology of living cells and tissues is providing more and more detailed information on the nano- and micro-scale. There is an urgent demand for mathematical modeling of reactive flow and transport and its interaction with elastic cell structures. Here we are formulating model equations on the fine scale with ε as scale parameter, which are already a mesoscopic and simplified description of the real processes occurring in the cells, the intercellular space and in the membranes. We are including

1. Fluid flow in the extracellular space, diffusion, transport and reactions of substances in the fluid.
2. Exchange of fluid and substances at the membranes.
3. Diffusion and transport of fluid and substances, chemical reactions inside the cells.
4. Changes of the structures and their mechanical properties, small deformation of the structures.

Due to technical reasons, let us assume that the cells are connected with each other. The final aim is to derive a system of macroscopic equations passing to the limit $\varepsilon \to 0$ and to provide methods to compute the solutions of macroscopic equations using the information coming from processes on the microscopic scale. However, the model equation for the considered multiphysics problem has not been mathematically solved. Before passing to the limit, one has to prove existence and uniqueness of the solutions for positive ε, what is the main aim of this paper. The asymptotic analysis will be done in an independent paper.

In formulating the model equations we start from the underlying real problem and set up a system of partial differential equations modeling the processes in a dimensionless form. This system is studied analytically.

To study of the coupling between the motion of the solid structure and fluid flow, a detailed description of the solid/fluid interfaces may lead to a very complex mathematical and numerical problem. The nonlinearity of the underlying fluid-structure interaction is so severe that even supposing linearly elastic solid structure leads to important complications. To devise a reasonable mathematical model, it is necessary to introduce simplifying model assumptions capturing only the most important physics of the problem.

A common simplification is to suppose "small" displacements and "small" deformation gradients leading to the hypothesis of linear elasticity for the structure. Even for such coupling, the existing mathematical theory does not give the global existence. Namely, in the paper [2], a structure being an elastic plate was considered and existence of at least one weak solution, as long as different parts of the solid structure do not meet, was proven. The corresponding results for the coupling between the Navier-Stokes equations and the linear equations of elastodynamics is more recent and due to Coutand and Shkoller in [4]. They have proven the short time existence for arbitrary data. The same authors extended their results to quasilinear elastodynamic structures in articles [5] and [3].

We are interested in the problem where the biophysical data imply that all assumptions of the linear elasticity are satisfied. Moreover, with our biophysical parameters, the solid structure displacement is very small, the flow is slow and thus, we are allowed to linearize the conditions at the fluid/structure interface. In fact the linearization of the fluid/structure interface introduces the error of the same order as neglecting nonlinear terms in the structure equations. We will motivate the linearization by the dimensional analysis in Section 3. Under similar assumptions, the interaction of fluid with solid structures has been studied in the literature in several papers and passing to the homogenization limit the macroscopic law known as Biot law could be derived, see [6], [8], [9], [20].

The modeling novelty in our paper is dependence of the Young modules on the concentration. Consequently, the cell chemistry causes the deformation. The global existence is then consequence of the energy inequality, resulting from the conservation of energy for the linear elasticity and for the Stokes flow. Nevertheless, adding diffusion, transport and reactions of chemical substances and their interaction with mechanics leads to new obstacles requiring new ideas and methods.

There are two chemicals, which play a role in our model. A first one is present only inside the cells and its cumulated content may change the mechanical properties of the structure. The second chemical, present inside the cells and in the intercellular space, influences the diffusion of the first chemical. These effects are quantitatively demonstrated in experiments, see e.g. [19].

The dependence of the elasticity coefficients on the chemical substance is nonlinear and nonlocal. We assume that the elasticity parameters depend on a Volterra functional of the concentration of the relevant substance. This leads to difficulties for the analysis. To overcome these obstacles we cut off the concentration in the coefficients and first prove existence and uniqueness for the cut-off problem. For the solutions of these problems we then prove lower and upper bounds for the concentrations independent of the cut off. Then, we conclude that the solution of the cut off problem is also a solution of the original problem. However, proving $L^\infty-$estimates requires structural conditions on the nonlinear reaction terms.

Next, we derive higher regularity of the solutions, whereby not only the transmission conditions on the interface between solid and fluid part are causing difficulties, but also the dependence of the elasticity moduli on the concentration of one of the chemical and the dependence of the diffusion coefficient of one of the chemicals on the concentration of the other. The regularity results are crucial for proving uniqueness and more general the dependence of solutions of the system on initial and boundary data.

This paper is organized as follows: In section 2 the model system is formulated, including a set of system parameters and their order of magnitude in the experimental situation taken as a test case. However, getting even the order of magnitude, based on good experiments, is a problem by itself. In section 3, a dimensional analysis is performed leading after some reductions to a dimensionless formulation of the model system. This system represents a larger class of problems coupling fluid flow, solid structure and chemical reaction for slow flow velocity and small deformations, which are typical for biological tissues. The authors are not aware of mathematical results for systems of this type. In section 4 this system is summarized and the assumptions on the data are formulated. Using the Galerkin method, in section 5 the existence of weak solutions with bounded concentrations for the dimensionless problem is proved. Higher regularity of the solutions is derived in section 5. These results are decisive for the analysis of the linearization of the system studied in section 6, where the dependence on the initial and boundary values is estimated yielding also uniqueness of the solutions.

In the forthcoming paper [12] the limit $\varepsilon \to 0$ is analyzed, using similar techniques, however, controlling the dependence on the scale parameter ε, which for simplicity we did not undertake in this paper.

Recently, biophysical and biochemical processes including cell layers or tissue have attracted more attention of mathematical modelling and numerical simulation. Here we mention as examples [1], [18], [21] dealing with model equations formulated on the macroscopic scale. In [18] the Navier Stokes equations for incompressible flow in a vessel coupled with advection-diffusion equation for the solute concentration and its interactions with the wall are treated. In [21] model equations for thrombosis describing flow, transport and reactions in a vessel and at its walls are analyzed and simulated. Flow and transport through interfaces are investigated in [1], assuming a coupling through the Neumann data on the interface. A derivation of effective transmission conditions on a membrane based on microscopic information is presented in [16]. In general, it is appropriate to use concepts which were successful for describing processes in porous media, also for processes in cell layers and tissue, see e.g. [10].

2 Setting of the model

Let us consider the domain $\Omega = (0,1)^3$ consisting of a tissue part formed by elastic cells and a fluid part representing the intercellular space. Initially, (i.e. at $t = 0$) the tissue part is denoted by Ω_s, the fuid part by Ω_f, and the fluid-solid interface by $\Gamma = \partial\Omega_f \cap \partial\Omega_s$. The boundary of the domain Ω consists of three parts

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

where $\Gamma_1 = \{x_1 = 0\} \times (0,1)^2$, $\Gamma_2 = \{x_1 = 1\} \times (0,1)^2$ and $\Gamma_3 = \cup_{j=2,3}(\{x_j = 0\} \cup \{x_j = 1\}) \times (0,1)^2$. We suppose that the solid and fluid parts are smooth and connected. The outer unit normal to $\partial\Omega$ is denoted by ν. On the interface Γ, we denote by ν the outer unit normal to the fluid part Ω_f.

Let $[0,T]$ denote a time interval, with $T > 0$. For simplicity of notation, we define

$$Q_t := \Omega \times (0,t), \quad Q_t^s := \Omega_s \times (0,t), \quad Q_t^f := \Omega_f \times (0,t), \tag{2.1}$$

for all $t \in [0,T]$.

We suppose small deformations of the cells structure. It means that in the solid part Ω_s the equations of linear elasticity hold:

$$\rho_s \frac{\partial^2 w}{\partial t^2} - \nabla\cdot(\sigma(w)) = 0 \ \text{ in } \Omega_s \times (0,T), \tag{2.2}$$

where w is the displacement in the solid part, $D(w)$ is the strain tensor defined by

$$(D(w))_{i,j} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right), \quad i,j = 1,2,3$$

and $\sigma(w)$ is the stress tensor

$$\sigma(w) = A(\mathcal{F}(c_1))D(w). \tag{2.3}$$

In the case when the cells are homogeneous and isotropic bodies, the elasticity coefficients A are given with the help of Lamé's coefficients [1] λ and μ and the stress tensor has the form:

$$\sigma(w) = \lambda(\mathcal{F}(c_1))\nabla\cdot(wI) + 2\mu(\mathcal{F}(c_1))D(w) \tag{2.4}$$

The dependence of the elasticity coefficients A on the concentration c_1 is nonlinear and nonlocal; the coefficients change as a function of cumulated quantity of chemical substance. To describe this dependence, we introduce the operator $\mathcal{F}$ acting on the concentration, and given by

$$\mathcal{F} : L^2(\Omega_s \times [0,T]) \to L^2(\Omega_s \times [0,T]) \tag{2.5}$$

$$\mathcal{F}(c_1)(x,t) = (\mathcal{K} \star_t F(c_1))(x,t) = \int_0^t \mathcal{K}(t-\tau)F(c_1(x,\tau))\, d\tau, \tag{2.6}$$

where $F \in C^2(\mathbb{R})$ is Lipschitz, and the kernel $\mathcal{K}$ has the following properties

$$\mathcal{K} \in C^3[0,T], \quad \mathcal{K}(0) = \mathcal{K}'(0) = \mathcal{K}''(0) = 0. \tag{2.7}$$

In the fluid part, we consider the Navier-Stokes system for a viscous and incompressible fluid

$$\rho_f\left(\frac{\partial v}{\partial t} + (v\nabla)v\right) + \nabla p - \mu_f \Delta v = 0, \text{ in } \Omega_f(t)\times(0,T) \tag{2.8}$$

$$\nabla\cdot v = 0, \text{ in } \Omega_f(t)\times(0,T) \tag{2.9}$$

where $\Omega_f(t)$ is the fluid configuration at time t, $\Omega_f(0) = \Omega_f$, v is the fluid velocity and p is the fluid pressure.

We note that the Lagrangian coordinates are used for the structure and Eulerian for the fluid. Hence, Ω_s is the reference domain and the interface

[1] Other possibility is to use Young's modulus E and Poisson's coefficient ν. They relate to Lamé's coefficients through $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$.

between the two media evolves with the evolution of the structure. The kinematic interface condition is the continuity of the normal velocity and, due to different formulations for our media, it reads

$$v(x + w(x,t),t) = \frac{\partial w}{\partial t}(x,t), \quad \text{on } \Gamma \times (0,T). \tag{2.10}$$

The 3rd Newton's law implies continuity of the contact forces. Expressing continuity of the the contact forces at the interfaces requires introducing the fluid Lagrangian configuration u^f, defined on the initial fluid configuration Ω_f and with values in $\Omega_f(t)$. It is defined through the differential equation $\dfrac{\partial u^f}{\partial t} = v(u^f(x,t),t)$. Then the continuity of the the normal stresses reads

$$(-pI+2\mu_f D(v))(x+w(x,t),t)\cdot(\nabla u^f)^{-1}\nu = \sigma(w)\cdot\nu, \quad \text{on } \Gamma\times(0,T). \tag{2.11}$$

In the simple situation when the solid structure is an elastic curved membrane the condition can be written more explicitly (see e.g. [7]).

At the exterior boundary, for every $t \in (0,T)$, we suppose:

$$(-pI + 2\mu_f D(v)) \cdot (\nabla u^f)^{-1}\, e_1 = 0, \quad \text{on} \quad \Gamma_1 \cap \bar{\Omega}_f \tag{2.12}$$

$$A(\mathcal{F})D(w) \cdot e_1 = 0, \quad \text{on} \quad \Gamma_1 \cap \bar{\Omega}_s \tag{2.13}$$

$$(-pI + 2\mu_f D(v)) \cdot (\nabla u^f)^{-1} e_1 = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3), \quad \text{on} \quad \Gamma_2 \cap \bar{\Omega}_f \tag{2.14}$$

$$A(\mathcal{F})D(w) \cdot e_1 = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3), \quad \text{on} \quad \Gamma_2 \cap \bar{\Omega}_s \tag{2.15}$$

$$v = 0 \ \text{ and } \ w = 0, \quad \text{on} \quad \Gamma_3. \tag{2.16}$$

For simplicity, we suppose initial conditions equal to zero, i.e.

$$\begin{cases} v(x,0) = 0 \ \text{ in } \ \Omega_f, \\ w(x,0) = \dfrac{\partial w}{\partial t}(x,0) = 0 \ \text{ in } \ \Omega_s. \end{cases} \tag{2.17}$$

Next, we write the equations describing the transport of chemical substances. There is a chemical diffusing only inside cells and its cumulated content influences the elastic cells structure:

$$\frac{\partial c_1}{\partial t} - \nabla \cdot \big(D_1(c_2)\nabla c_1\big) = g_1(c_1, c_2) \qquad \text{in } \Omega_s \times (0,T) \tag{2.18}$$

$$D_1(c_2)\nabla c_1 \cdot \nu = 0 \qquad \text{on } \partial\Omega_s \times (0,T) \tag{2.19}$$

$$c_1(0) = c_{10} \qquad \text{in } \Omega_s \tag{2.20}$$

Then, there is a second chemical, which is present in the cells and in the intercellular space, and influencing diffusion and reactive change of the first chemical substance.

$$\frac{\partial c_2}{\partial t} + v \cdot \nabla c_2 - D_2 \Delta c_2 = g_2(c_2) \qquad \text{in} \quad \Omega_f(t) \times (0,T) \tag{2.21}$$

$$\frac{\partial c_2}{\partial t} - D_2 \Delta c_2 = g_3(c_1, c_2) \qquad \text{in} \quad \Omega_s \times (0,T) \tag{2.22}$$

$$(vc_2 - D_2 \nabla c_2)\chi_{\Omega_f} (\nabla u^f)^{-1} \nu = -D_2 \nabla c_2 \chi_{\Omega_s} \cdot \nu \qquad \text{on} \ \Gamma \times (0,T) \tag{2.23}$$

$$c_2 \chi_{\Omega_f}(x + w(x,t)) = K c_2 \chi_{\Omega_s} \qquad \text{on} \ \Gamma \times (0,T) \tag{2.24}$$

$$(\chi_{\Omega_f} + K\chi_{\Omega_s}) c_2 = c_{2D} \qquad \text{on} \ \Gamma_1 \times (0,T) \tag{2.25}$$

$$\nabla c_2 \chi_{\Omega_f} \cdot (\nabla u^f)^{-1} e_1 + \nabla c_2 \chi_{\Omega_s} \cdot e_1 = 0 \qquad \text{on} \ \Gamma_2 \times (0,T). \tag{2.26}$$

$$(vc_2 - D_2 \nabla c_2)\chi_{\Omega_f} \cdot (\nabla u^f)^{-1} \nu = 0 \qquad \text{on} \ \Gamma_3 \times (0,T) \tag{2.27}$$

$$D_2 \nabla c_2 \chi_{\Omega_s} \cdot \nu = 0 \qquad \text{on} \ \Gamma_3 \times (0,T) \tag{2.28}$$

$$(\chi_{\Omega_f} + K\chi_{\Omega_s}) c_2(0) = c_{20} \qquad \text{in} \quad \Omega \tag{2.29}$$

Concerning the transmission conditions at the interface between the cells and the intercellular space, we remark that beside the continuity of the normal fluxes given by (2.23), we have the jump condition (2.24) where $K > 0$ is the so called calibration constant. In order to work with the usual Sobolev spaces in the weak formulation, we redefine c_2 in Ω_s by setting

$$\hat{c}_2 := K c_2 \quad \text{in} \quad \Omega_s \times (0,T).$$

Then problem (2.21) - (2.29) transforms to

$$\frac{\partial c_2}{\partial t} + v \cdot \nabla c_2 - D_2 \Delta c_2 = g_2(c_2) \qquad \text{in} \ \Omega_f(t) \times (0,T) \tag{2.30}$$

$$\frac{1}{K}\frac{\partial \hat{c}_2}{\partial t} - \frac{D_2}{K} \Delta \hat{c}_2 = \hat{g}_3(c_1, \hat{c}_2) \qquad \text{in} \quad \Omega_s \times (0,T) \tag{2.31}$$

$$(vc_2 - D_2 \nabla c_2)\chi_{\Omega_f} (\nabla u^f)^{-1} \nu = -\frac{D_2}{K} \nabla \hat{c}_2 \chi_{\Omega_s} \cdot \nu \qquad \text{on} \ \Gamma \times (0,T) \tag{2.32}$$

$$c_2 \chi_{\Omega_f}(x + w(x,t)) = \hat{c}_2 \chi_{\Omega_s} \qquad \text{on} \ \Gamma \times (0,T) \tag{2.33}$$

$$c_2 \chi_{\Omega_f} + \hat{c}_2 \chi_{\Omega_s} = c_{2D} \qquad \text{on} \ \Gamma_1 \times (0,T) \tag{2.34}$$

$$\nabla c_2 \chi_{\Omega_f} \cdot (\nabla u^f)^{-1} e_1 + \nabla \hat{c}_2 \chi_{\Omega_s} \cdot e_1 = 0 \qquad \text{on} \quad \Gamma_2 \times (0,T) \tag{2.35}$$

$$(vc_2 - D_2 \nabla c_2)\chi_{\Omega_f} (\nabla u^f)^{-1} \nu - \frac{D_2}{K} \nabla \hat{c}_2 \chi_{\Omega_s} \nu = 0 \qquad \text{on} \ \Gamma_3 \times (0,T) \tag{2.36}$$

$$(c_2 \chi_{\Omega_f} + \hat{c}_2 \chi_{\Omega_s})(0) = c_{20} \qquad \text{in} \ \ \Omega \tag{2.37}$$

For simplicity of notation in the following we drop the hat.

3 Dimensional analysis

As already explained in the introduction, the system (2.2)-(2.4), (2.8)-(2.20), (2.30)-(2.37) is very complicated and we will simplify the model, keeping only the most important physics of the problem.

The natural way of analyzing the problem (2.2)-(2.29) is to introduce dimensionless coordinates which are defined in terms of characteristic values of the physical parameters. A detailed analysis with respect to these parameters will be exposed in the forthcoming paper [12], where the homogenization results strongly depend on the relationship between the non-dimensional numbers and the typical size of the non-homogeneities. The goal of the section is to explain why it is reasonable to drop the inertia effects and why the fluid-solid interface could be linearized.

Typical values of characteristic parameters are: $T = 20$ seconds is the characteristic flow time, characteristic domain size is $L = 10^{-2}$ meters, characteristic size of elastic moduli is $\Lambda = 10^4$ pascals, dynamic viscosity is $\mu_f = 1.003 \times 10^{-3}$ kg / (m sec) and densities are $\rho_f \approx \rho_s = 1000$ kg/m^3. Characteristic cell size is $\ell = 1.5 \cdot 10^{-5}$ meters and the cell displacement should not be bigger than 10^{-6} meters. As references for these parameters we quote [19] and personal communication by M. Weiss (BIOMS, Universität Heidelberg). The characteristic size of the heterogeneities is then given by $\varepsilon = \ell/L = 1.5\ 10^{-3}$.

From the data we see that global Reynolds' number is $\mathbf{Re} = VL\rho_f/\mu_f = O(\varepsilon)$ and the local Reynolds' number (defined as $Vl\rho_f/\mu_f$) is of order $O(\varepsilon)^2$. Therefore, the inertia effets are small and we can simply neglect them.

We proceed by setting

$$(3.1) \qquad x^* = \frac{x}{L},\ t^* = \frac{t}{T_D},\ v^* = \frac{v}{V},\ A^* = \frac{A}{\Lambda},\ p^* = \frac{p}{P},\ V \approx \frac{\ell}{T} = \frac{L}{T_D},$$

where $T_D = T/\varepsilon$ is the the characteristic diffusion time. Next we set $\dfrac{\partial w^*}{\partial t^*} = \dfrac{T}{\ell}\dfrac{\partial w}{\partial t}$, but this does not give the correct information about the size of w. To get this information, we remark that in the case of Young moduli independent of the concentration and for negligible inertia effects at the external boundaries, we have the energy equality

$$(3.2) \qquad \rho_f \int_{\Omega_f(t)} |v|^2\ dx + \rho_s \int_{\Omega_s} |\frac{\partial w}{\partial t}|^2\ dx + \int_{\Omega_s} AD(w) : D(w)\ dx +$$
$$2\mu_f \int_0^t \int_{\Omega_f(\tau)} |\nabla v|^2\ dxd\tau = 2\int_0^t \int_\Gamma \mathcal{S}(\chi_{\bar\Omega_f(\tau)} v + \chi_{\bar\Omega_s} \frac{\partial w}{\partial \tau})\ d\Gamma_2 d\tau$$

Now, the right hand side in (3.2) can be further estimated. For this we make use of the fact that the H^1-norm on Ω_s is equivalent to

$$\Big(||w||^2_{L^2(\Omega_s)} + ||D(w)||^2_{L^2(\Omega_s)}\Big)^{\frac{1}{2}}$$

with a constant independent of ε, see [17], of the estimate

$$||w||^2_{L^2(\Omega_s)} \leq \int_0^t ||\partial_\tau w||_{L^2(\Omega_s)} d\tau$$

and of $w(0,x) = 0$. Finally, applying Gronwall's inequality, we obtain

$$||D(w)||_{L^\infty(L^2)} \approx \frac{C}{\sqrt{\Lambda}}\mathcal{A}, \quad ||\partial_t w||_{L^\infty(L^2)} \approx \frac{C}{\sqrt{\rho_s}}\mathcal{A} \tag{3.3}$$

$$||v||_{L^\infty(L^2)} \approx \frac{C}{\sqrt{\rho_f}}\mathcal{A}, \quad ||\nabla v||_{L^2(L^2)} \approx \frac{C}{\sqrt{\mu_f}}\mathcal{A} \tag{3.4}$$

where

$$\mathcal{A} = \frac{C}{\sqrt{\rho_f \mu_f}} \int_0^t ||\mathcal{S}||^2_{L^2(\Gamma^2 \cap \bar{\Omega}_f)^3} d\tau +$$
$$\left(\frac{1}{\sqrt{\rho_s \Lambda}} + \frac{1}{\rho_s}\right)\left(\int_0^t ||\partial_t \mathcal{S}||^2_{L^2(\Gamma^2 \cap \bar{\Omega}_f)^3} d\tau\right) + ||\mathcal{S}||^2_{L^\infty(0,T;L^2(\Gamma^2\cap\bar{\Omega}_f)^3)}$$

As a consequence we find out that (a) the fluid velocity and the deformation velocity of the structure are of the same order with respect to ε; (b) the strain tensor of the structure is even smaller, but the fluid velocity gradient could be of higher order with respect to the structure displacement. Thus, for data at the exterior boundary being "compatible" with linearization and small Reynolds' number, we see that terms $\nabla_x v(x,t) \cdot w(x,t)$, and the higher order ones are negligible and it is justified to linearize the kinematic condition (2.10) and the coefficients in the dynamic conditions (2.11).

Now, after dropping the stars, we obtain the dimensionless equations for the fluid-structure interaction:

$$\mathbf{Sh\ Re}\ \frac{\partial v}{\partial t} + \frac{PL}{\mu_f V}\nabla p = \Delta v \qquad \text{in } \Omega_f \times (0,T) \tag{3.5}$$

$$\nabla \cdot v = 0 \qquad \text{in } \Omega_f \times (0,T) \tag{3.6}$$

$$\frac{\rho_s \ell L}{\mu_f T}\frac{\partial^2 w}{\partial t^2} = \frac{\Lambda L}{\mu_f V}\operatorname{div}(AD(w)) \qquad \text{in } \Omega_s \times (0,T) \tag{3.7}$$

$$v(x,t) = \frac{\partial w}{\partial t}(x,t) \qquad \text{on } \Gamma \times (0,T) \tag{3.8}$$

$$(-\frac{PL}{\mu_f V}pI + 2D(v)) \cdot \nu = \frac{\Lambda L}{\mu_f V}AD(w)) \cdot \nu \qquad \text{on } \Gamma \times (0,T), \tag{3.9}$$

where the product of Reynolds' and Strouhal's number **Sh Re** is equal to $L^2\varepsilon\rho_f/(T\mu_f)$. Using the above reference values, we obtain that $\frac{\Lambda L}{\mu_f V}$ is very large. Thus, in the nondimensional equation for the flow, we will consider a viscosity coefficient of order $O(1)$, and in the nondimensional equation for the structure we consider elasticity coefficients of the form $\Lambda_0 = \frac{\Lambda L}{\mu_f V}$.

The reference pressure P is chosen such that $\frac{PL}{\mu_f V} = O(1)$. Due to the continuity of the velocities (3.8) at the fixed reference interface Γ and assuming the initial displacements in the fluid to be zero, it is natural to introduce a displacement function $u : \Omega \times (0, T) \to \mathbb{R}^3$ such that

$$v = \frac{\partial u}{\partial t} \qquad \text{on } \Omega_f \times (0, T) \tag{3.10}$$

$$w = u \qquad \text{on } \Omega_s \times (0, T). \tag{3.11}$$

We note that the same equations are obtained by taking the characteristic time T, instead of T_D. Only difference is that $\Lambda_0 = \frac{\Lambda \ell}{\mu_f V}$, **Sh Re** is equal to $L^2\rho_f/(T\mu_f)$ and the coefficients in (3.7) change on the corresponding way.

Finally, we write the equations for the concentrations in dimensionless form. Let $D_R = L^2\varepsilon/T$. Then we set

$$c_1^* = \frac{c_1}{C_1},\ c_2^* = \frac{c_2}{C_2},\ D_j^* = \frac{D_j}{D_R},\ g_1^* = \frac{g_1}{G_1^R},\ g_2^* = \frac{g_2}{G_2^R},\ g_3^* = \frac{g_3}{G_3^R}. \tag{3.12}$$

After dropping the stars, we obtain

$$\frac{\partial c_1}{\partial t} - \operatorname{div}\big(D_1(c_2)\nabla c_1\big) = \frac{G_1^R T_D}{C_1} g_1(c_1, c_2) \qquad \text{in } \Omega_s^\varepsilon \times (0, T) \tag{3.13}$$

$$\frac{\partial c_2}{\partial t} + \frac{\partial u}{\partial t}\cdot\nabla c_2 - D_2\Delta c_2 = \frac{G_2^R T_D}{C_2} g_2(c_2) \qquad \text{in } \Omega_f \times (0, T) \tag{3.14}$$

$$\frac{1}{K}\frac{\partial c_2}{\partial t} - \frac{1}{K} D_2\Delta c_2 = \frac{G_3^R T_D}{C_2} g_3(c_1, c_2) \qquad \text{in } \Omega_s \times (0, T) \tag{3.15}$$

Thus denoting

$$G_1 = \frac{G_1^R T_D}{C_1}, \quad G_2 = \frac{G_2^R T_D}{C_2}, \quad G_3 = \frac{G_3^R T_D}{C_2}$$

we finally can write down the full set of dimensionless equations coupling fluid flow, deformation of the solid structure and chemical reactions.

4 Statement of the equations and assumption on the data

Let us first make some remarks on the function spaces we are using: For a given smooth bounded domain $G \subset \mathbb{R}^3$, we use the usual Sobolev spaces $W^{m,q}(G)$ of functions from $L^q(G)$ having derivatives of order m in $L^q(G)$. For $q = 2$, these spaces are denoted by $H^m(G)$. We also use the spaces of functions depending on space and time $W_q^{2l,l}(G \times (0,T))$, $l > 0$, consisting of functions having derivatives with respect to space up to order $2l$ and with respect to time up to order l in L^q. For the precise definition of these spaces see [14].

In this paper, we look for solutions (u, c_1, c_2), with

$$u \in W^{3,\infty}(0,T; L^2(\Omega)) \cap W^{2,\infty}(0,T; H^1(\Omega)) \cap H^3(0,T; H^1(\Omega_f)),$$

$$c_1 \in W_2^{2,1}(Q_T^s), \quad \text{and} \quad c_2 \in W_2^{1,1}(Q) \cap W_2^{2,1}(Q_T^s \cup \Omega_T^f),$$

satisfying the problem for the fluid/structure interaction:

$$\mathbf{Sh}\ \mathbf{Re}\ \frac{\partial^2 u}{\partial t^2} + \nabla p = \Delta\left(\frac{\partial u}{\partial t}\right) \qquad \text{in } \Omega_f \times (0,T) \tag{4.1}$$

$$\nabla \cdot \left(\frac{\partial u}{\partial t}\right) = 0 \qquad \text{in } \Omega_f \times (0,T) \tag{4.2}$$

$$\frac{\rho_s \ell L}{\mu_f T}\frac{\partial^2 u}{\partial t^2} = \Lambda_0 \nabla \cdot (AD(u)) \qquad \text{in } \Omega_s \times (0,T) \tag{4.3}$$

$$u\chi_{\Omega_f} = u\chi_{\Omega_s} \qquad \text{on } \Gamma \times (0,T) \tag{4.4}$$

$$\left(-pI + 2D\left(\frac{\partial u}{\partial t}\right)\right) \cdot \nu = \Lambda_0 AD(u) \cdot \nu \qquad \text{on } \Gamma \times (0,T) \tag{4.5}$$

$$\left(-pI + 2D\left(\frac{\partial u}{\partial t}\right)\right) \cdot\ e_1 = 0, \qquad \text{on } \Gamma_1 \cap \bar{\Omega}_f \times (0,T) \tag{4.6}$$

$$\Lambda_0 AD(u) \cdot e_1 = 0, \qquad \text{on } \Gamma_1 \cap \bar{\Omega}_s \times (0,T) \tag{4.7}$$

$$\left(-pI + 2D\left(\frac{\partial u}{\partial t}\right)\right) \cdot e_1 = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3), \qquad \text{on } \Gamma_2 \cap \bar{\Omega}_f \times (0,T) \tag{4.8}$$

$$\Lambda_0 AD(u) \cdot e_1 = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3), \qquad \text{on } \Gamma_2 \cap \bar{\Omega}_s \times (0,T) \tag{4.9}$$

$$\frac{\partial u}{\partial t}\chi_{\Omega_f} + u\chi_{\Omega_s} = 0 \qquad \text{on } \Gamma_3 \times (0,T) \tag{4.10}$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \qquad \text{in } \Omega \tag{4.11}$$

together with the problem for the evolution of the concentrations:

$$\frac{\partial c_1}{\partial t} - \nabla \cdot \big(D_1(c_2)\nabla c_1\big) = G_1 g_1(c_1, c_2) \quad \text{in} \;\; \Omega_s \times (0,T) \tag{4.12}$$

$$D_1(c_2)\nabla c_1 \cdot \nu \; = 0 \;\; \text{on} \quad \Omega_s \times (0,T) \tag{4.13}$$

$$c_1(0) \; = c_{10} \;\; \text{in} \;\; \Omega_s \tag{4.14}$$

$$\frac{\partial c_2}{\partial t} + \frac{\partial u}{\partial t} \cdot \nabla c_2 - D_2 \Delta c_2 \; = G_2 g_2(c_2) \quad \text{in} \quad \Omega_f \times (0,T) \tag{4.15}$$

$$\frac{1}{K}\frac{\partial c_2}{\partial t} - \frac{1}{K} D_2 \Delta c_2 \; = G_3 g_3(c_1, c_2) \quad \text{in} \quad \Omega_s \times (0,T) \tag{4.16}$$

$$\left(\frac{\partial u}{\partial t} c_2 - D_2 \nabla c_2\right) \chi_{\Omega_f} \cdot \nu = -\frac{D_2}{K} \nabla c_2 \chi_{\Omega_s} \cdot \nu \quad \text{on} \;\; \Gamma \times (0,T) \tag{4.17}$$

$$c_2 \chi_{\Omega_f} = c_2 \chi_{\Omega_s} \;\; \text{on} \;\; \Gamma \times (0,T) \tag{4.18}$$

$$c_2 \chi_{\Omega_f} + c_2 \chi_{\Omega_s} \; = c_{2D} \;\; \text{on} \;\; \Gamma_1 \times (0,T) \tag{4.19}$$

$$(\nabla c_2 \chi_{\Omega_f} + \nabla c_2 \chi_{\Omega_s}) \cdot e_1 = 0 \;\; \text{on} \;\; \Gamma_2 \times (0,T) \tag{4.20}$$

$$\left(\frac{\partial u}{\partial t} c_2 - D_2 \nabla c_2\right) \chi_{\Omega_f} \cdot \nu - \frac{D_2}{K} \nabla \hat{c}_2 \chi_{\Omega_s} \cdot \nu = 0 \qquad \text{on} \;\; \Gamma_3 \times (0,T) \tag{4.21}$$

$$(c_2 \chi_{\Omega_f} + c_2 \chi_{\Omega_s})(0) = c_{20} \qquad \text{in} \;\; \Omega. \tag{4.22}$$

Before starting with the analysis of the problem (4.1)-(4.22), we give the precise assumptions on the data.

4.1 Assumptions on the data

We assume that the components of the symmetric fourth order elasticity tensor A belong to $W^{3,\infty}(\mathbb{R})$ as function of $\mathcal{F}$, and that there exists $\lambda_0 \in \mathbb{R}, \lambda_0 > 0$ such that

$$\lambda_0 ||M||^2 \leq A(\cdot)MM \leq \frac{1}{\lambda_0}||M||^2, \tag{4.23}$$

for all symmetric matrices M, a.e. on $\mathbb{R}$. Further, we suppose that

$$(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) \in H^3(0,T; L^2(\Gamma_2))^3, \tag{4.24}$$

$$(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)(0) = \partial_t(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)(0) = 0. \tag{4.25}$$

Concerning the reaction terms we suppose that $G_1 > 0, G_2 > 0, G_3 > 0$. We also assume that g_1, g_2, g_3 are Lipschitz continuous with respect to their arguments. This implies that there exist positive constants c_1, c_2 and c_3 such that

$$|g_1(y,z)| \leq c_1(1+|(y,z)|) \quad \text{for all } (y,z) \in \mathbb{R}^2 \tag{4.26}$$
$$|g_2(z)| \leq c_2(1+|z|) \quad \text{for all } z \in \mathbb{R} \tag{4.27}$$
$$|g_3(y,z)| \leq c_3(1+|(y,z)|) \quad \text{for all } (y,z) \in \mathbb{R}^2 \tag{4.28}$$

Additionally we have to impose on g_1, g_2, g_3 structural conditions which guarantee positivity of the solutions and for c_1 also a uniform upper bound. A possible choice of such conditions is given in the following.

$$x^- g_1(x^-, y)G_1 + y^- g_3(x, y^-)G_3 \leq C((x^-)^2 + (y^-)^2) \tag{4.29}$$
$$y^- g_2(y^-)G_2 \leq C(y^-)^2 \tag{4.30}$$

for all $x, y \in \mathbb{R}$, where $x^- = \min\{x, 0\}$. We also require that there exist constants $A_1, M_1 \in \mathbb{R}, A_1 \geq 0, M_1 > 0$, such that

$$g_1(x,y) \leq A_1 x, \quad \text{for } x \geq M_1,\ y \in \mathbb{R}. \tag{4.31}$$

For the initial and boundary concentrations we assume that

$$c_{10} \in C^2(\bar{\Omega}_s) \text{ with } \nabla c_{10} \cdot n = 0 \text{ on } \partial\Omega_s, \text{ and } 0 \leq c_{10} \leq M_1, \tag{4.32}$$

where M_1 is the constant in the assumption (4.31). We also assume that there exists $\beta > 0$ and $M_2 > 0$, such that

$$c_{20} \in H^1(\Omega) \cap C^\beta(\bar{\Omega}) \cap C^{2+\beta}(\bar{\Omega}_s) \cap C^{2+\beta}(\bar{\Omega}_f) \tag{4.33}$$

and

$$c_{20}|_{\Gamma_1} = c_{2D}|_{t=0}, \quad \nabla c_{20} \cdot n = 0 \text{ on } \Gamma_2 \cup \Gamma_3, \text{ and } 0 \leq c_{20} \leq M_2. \tag{4.34}$$

Finally, for the boundary concentration c_{2D} we require

$$c_{2D} \in C^{\beta, \frac{\beta}{2}}(\Gamma_1 \times [0,T]) \cap H^2(\Gamma_1 \times (0,T)), \tag{4.35}$$
$$c_{2D} \in C^{2+\beta, 1+\frac{\beta}{2}}((\bar{\Omega}_s \cap \Gamma_1) \times [0,T]) \cap C^{2+\beta, 1+\frac{\beta}{2}}((\bar{\Omega}_f \cap \Gamma_1) \times [0,T]), \tag{4.36}$$

and

$$0 \leq c_{2D} \leq M_2. \tag{4.37}$$

5 Existence of weak solutions

We start with the analysis of the problem (4.1)-(4.22) by proving existence of weak solutions. Since in this paper we are not interested how the constants in the estimates depend explicitly on the coefficients, we replace

$$A(\mathcal{F}(c_1^N) := \Lambda_0 A(\mathcal{F}(c_1^N), \quad g_1 := G_1 g_1, \quad g_2 := G_2 g_2, \quad g_3 := G_3 g_3.$$

and we set the coefficients **Sh Re** , $\dfrac{\rho_s \ell L}{\mu_f T}$ to 1. For the proof of the existence of week solutions, we will use the Galerkin method. Thus. let us write down the variational formulation of the problem (4.1)-(4.22):

Find (u, c_1, c_2) with

$$u \in W^{2,\infty}(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H^1(\Omega)) \cap H^2(0,T;H^1(\Omega_f))$$
$$0 \le c_1 \le M_1 e^{A_1 t},\ c_1 \in L^2(0,T;H^1(\Omega_s)) \cap W_2^{1,1/2}((0,T)\times\Omega_s),$$
$$0 \le c_2 \le C,\ c_2 \in L^2(0,T;H^1(\Omega)) \cap W_2^{1,1/2}((0,T)\times\Omega),$$

and $c_2 - c_{2D}(1-x_1) \in \{\phi \in L^2(0,T;H^1(\Omega)); \phi = 0 \,\mathrm{on}\, \Gamma_1\}$, satisfying for a.e. $t \in (0,T)$

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}(t)\varphi dx + 2\int_{\Omega_f} D(\frac{\partial u}{\partial t}(t)) : D(\varphi)\,dx + \int_{\Omega_s} A(\mathcal{F}(c_1))D(u(t)) : D(\varphi) = \int_{\Gamma_2} (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)\varphi\,dS, \tag{5.1}$$

$$\nabla \cdot \frac{\partial u}{\partial t} = 0, \qquad \text{in } \Omega_f \times (0,T), \tag{5.2}$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \qquad \text{in } \Omega, \tag{5.3}$$

$$\left\langle \frac{\partial c_1}{\partial t}(t), \psi \right\rangle + \int_{\Omega_s} D_1(c_2)\nabla c_1(t)\nabla\psi\,dx = \int_{\Omega_s} g_1(c_1,c_2)\psi dx, \tag{5.4}$$

$$\left\langle \{\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s}\}\frac{\partial c_2}{\partial t}(t), \zeta \right\rangle + \int_\Omega D_2\{\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s}\}\nabla c_2(t)\nabla\zeta\,dx - \int_{\Omega_f}\frac{\partial u}{\partial t}(t)c_2(t)\nabla\zeta dx + \int_{\Gamma_2\cap\bar\Omega_f}\frac{\partial u}{\partial t}(t)\cdot e_1 c_2(t)\zeta dS = \int_\Omega \{g_2(c_2)\chi_{\Omega_f} + g_3(c_1,c_2)\chi_{\Omega_s}\}\zeta dx, \tag{5.5}$$

$$c_1(0) = c_{10} \ \text{ in } \ \Omega_s, \quad c_2(0) = c_{20} \ \text{ in } \ \Omega, \tag{5.6}$$

for all $\varphi \in V$, $\psi \in H^1(\Omega_s)$, and $\zeta \in H^1(\Omega)$ with $\zeta = 0$ on Γ_1. Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing of $H^1(\Omega)^*$ and $H^1(\Omega)$. The space V is defined as follows:

$$V = \{\varphi \in H^1(\Omega)^3; \quad \nabla \cdot \varphi = 0 \text{ in } \Omega_f, \quad \varphi = 0 \text{ on } \Gamma_3\}.$$

Theorem 1 *Under the assumptions on the data from section 4.1 the problem (5.1)-(5.6) has at least a weak solution* (u, c_1, c_2).

Proof: The existence proof consists of the following steps.

1. Local existence for the discretized cut-off problem
2. Global existence for the discretized cut-off problem
3. Compactness estimates for the discretized cut-off problem
4. Convergence of the approximates for the cut-off problem
5. Passing to the limit in the approximate cut-off problem
6. Non-negativity of the concentrations
7. Uniform upper bounds for the concentrations

1. Step: Local existence for the discretized cut-off problem

In a first step we construct Galerkin-approximations for our unknowns u, c_1, c_2. Thus, let $\{\alpha_j\}_{j\in\mathbb{N}}$ be a smooth basis for V, $\{\beta_j\}_{j\in\mathbb{N}}$ be a smooth basis for $H^1(\Omega_s)$ and $\{\gamma_j\}_{j\in\mathbb{N}}$ be a smooth basis for $W = \{\phi \in H^1(\Omega), \phi = 0 \,\text{on}\, \Gamma_1\}$. We are looking for an approximate solution in the form

$$u_N(t) = \sum_{j=1}^N \delta_j(t)\alpha_j, \quad c_1^N(t) = \sum_{j=1}^N \xi_j(t)\beta_j, \quad c_2^N(t) = \sum_{j=1}^N \zeta_j(t)\gamma_j + c_{2D}(1-x_1)$$

satisfying the approximate cut-off problem

$$\int_\Omega \frac{\partial^2 u_N}{\partial t^2}(t)\alpha_k \, dx + 2\int_{\Omega_f} D(\frac{\partial u_N}{\partial t})(t) : D(\alpha_k)\, dx + \int_{\Omega_s} A(\mathcal{F}(\tilde{c}_1^N))D(u_N(t)) : D(\alpha_k) \;=\; \int_{\Gamma_2} (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)\alpha_k \, dS, \tag{5.7}$$

for all $k = 1, \ldots, N$, a.e. in $(0, T)$, and

$$u_N(0) = 0, \quad \frac{du_N}{dt}(0) = 0 \quad \text{in } \Omega, \tag{5.8}$$

$$(5.9)\quad \int_{\Omega_s} \frac{\partial c_1^N}{\partial t}(t)\beta_k\,dx + \int_{\Omega_s} D_1(c_2^N)\nabla c_1^N(t)\nabla\beta_k\,dx = $$
$$= \int_{\Omega_s} g_1(c_1^N, c_2^N)\beta_k dx,$$

$$(5.10)\int_\Omega \{\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s}\}\frac{\partial c_2^N}{\partial t}(t)\gamma_k\,dx + \int_\Omega D_2\{\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s}\}\nabla c_2^N(t)\nabla\gamma_k\,dx$$
$$- \int_{\Omega_f} \frac{\partial u_N}{\partial t}(t)c_2^N(t)\nabla\gamma_k dx + \int_{\Gamma_2\cap\bar{\Omega}_f} \frac{\partial u_N}{\partial t}(t)\cdot e_1 c_2^N(t)\gamma_k dS$$
$$= \int_\Omega \{g_2(c_2)\chi_{\Omega_f} + g_3(c_1, c_2)\chi_{\Omega_s}\}\gamma_k dx,$$

for all $k = 1, \ldots, N$, a.e. in $(0, T)$ and

$$(5.11)\qquad c_1^N(0) = c_{10}^N = \sum_{j=1}^N \xi_j(0)\beta_j, \quad c_2^N(0) = c_{20}^N = \sum_{j=1}^N \zeta_j(0)\gamma_j.$$

We remark that we have to cut of the concentration c_1^N in the coefficients $A(\mathcal{F}(c_1^N))$ and the cut-off is given by

$$(5.12)\qquad \tilde{c}_1^N = \inf\{\sup\{c_1^N, 0\}, M\} + \sup\{\inf\{c_1^N, 0\}, -M\}.$$

Since g_1, g_2 and g_3 are Lipschitz continuous, the Cauchy problem (5.7)-(5.11) has a unique solution $\{u_N, c_1^N, c_2^N - c_{2D}(1 - x_1)\} \in C^2([0, T_N]; V) \times C^1([0, T_N]; H^1(\Omega_s)) \times C^1([0, T_N]; W)$, for some $T_N > 0$.

2. Step: Global existence for the discretized cut-off problem In this step we prove that $T_N = T$ by obtaining the corresponding apriori estimates. Starting from here, we use for the partial derivative with respect to time the notation ∂_t.

First, we test (5.7) by $\partial_t u_N$ and get

$$(5.13)\qquad \frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_t u_N(t)|^2 dx + 2\int_{\Omega_f} |D(\partial_t u_N)(t)|^2 dx +$$
$$\int_{\Omega_s} A(\mathcal{F}(\tilde{c}_1^N))D(u_N)(t) : D(\partial_t u_N)(t) =$$
$$\int_{\Gamma_2} (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)\partial_t u_N(t) dS = \frac{d}{dt}\int_{\Gamma_2} (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) u_N(t) dS$$
$$- \int_{\Gamma_2} \partial_t(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) u_N(t) dS$$

Let us transform the elastic energy term:

$$\int_{\Omega_s} A(\mathcal{F}(\tilde{c}_1^N))D(u_N)(t) : D(\partial_t u_N)(t)dx =$$
$$\frac{d}{dt}\int_{\Omega_s} A(\mathcal{K} \star_t F(\tilde{c}_1^N))D(u_N)(t) : D(u_N)(t)dx -$$
$$-\int_{\Omega_s} A(\mathcal{K} \star_t F(\tilde{c}_1^N))D(\partial_t u_N)(t) : D(u_N)(t)dx -$$
$$-\int_{\Omega_s} \frac{dA}{d\mathcal{F}} D(u_N)(t) : D(u_N)(t)\big(\frac{d\mathcal{K}}{dt} \star_t F(\tilde{c}_1^N)\big)(t)dx$$

Due to the symmetry of A we get

$$\int_{\Omega_s} A(\mathcal{F}(\tilde{c}_1^N))D(u_N)(t) : D(\partial_t u_N)(t)dx = \tag{5.14}$$
$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_s} A(\mathcal{K} \star_t F(\tilde{c}_1^N))D(u_N)(t) : D(u_N)(t)dx -$$
$$-\frac{1}{2}\int_{\Omega_s} \frac{dA}{d\mathcal{F}} D(u_N)(t) : D(u_N)(t)\big(\frac{d\mathcal{K}}{dt} \star_t F(\tilde{c}_1^N)\big)(t)dx$$

After inserting (5.14) in (5.13), integrating with respect to time and using Korn's inequality, the following energy inequality is obtained

$$\int_{\Omega} |\partial_t u_N(t)|^2 dx + \int_0^t \int_{\Omega_f} |D(\partial_t u_N)|^2 dx d\tau + \tag{5.15}$$
$$\int_{\Omega_s} A(\mathcal{K} \star_t F(\tilde{c}_1^N))D(u_N)(t) : D(u_N)(t)dx \leq$$
$$C||\frac{dA}{d\mathcal{F}}||_{L^\infty(\mathbb{R})^9}(1 + ||\tilde{c}_1^N||_{L^\infty}) \int_0^t \int_{\Omega_s} A(\mathcal{F}(\tilde{c}_1^N))D(u_N) : D(u_N)dx d\tau$$
$$C\left(||D(u_N)(t)||_{L^2(\Omega)^9} + \int_0^t ||D(u_N)||_{L^2(\Omega)^9} d\tau\right)$$

In order to estimate $||D(u_N)(t)||_{L^2(\Omega_f)^9}$, we remark that since $D(u_N)(0) = 0$, we have

$$||D(u_N)(t)||_{L^2(\Omega_f)^9} = ||\int_0^t D(\partial_\tau u_N)d\tau||_{L^2(\Omega_f)^9} \tag{5.16}$$
$$\leq \int_0^t ||D(\partial_\tau u_N)||_{L^2(\Omega_f)^9} d\tau \leq \sqrt{t}||D(\partial_t u_N)||_{L^2(0,T_N;L^2(\Omega_f))^9}$$

Now using Gronwall's lemma and the fact that the 4th order tensor A is elliptic, uniformly with respect to its argument, the estimates (5.14) and (5.16) imply

$$||\partial_t u_N||_{L^\infty(0,T_N;L^2(\Omega))^3} \leq C(M) \tag{5.17}$$
$$||D(\partial_t u_N)||_{L^2(0,T_N;L^2(\Omega_f))^9} \leq C(M) \tag{5.18}$$
$$||D(u_N)||_{L^\infty(0,T_N;L^2(\Omega_s))^9} \leq C(M) \tag{5.19}$$

where M is the cut-off constant in the definition of $\tilde{c}_1^N$, see (5.12).

Next let us differentiate (5.7) with respect to t. It yields for all $k = 1, \dots, N$

$$\begin{aligned}
&\int_\Omega \partial_{ttt} u_N(t)\alpha_k\, dx + 2\int_{\Omega_f} D(\partial_{tt}u_N)(t) : D(\alpha_k)\, dx + \\
&\int_{\Omega_s} A(\mathcal{F}(\tilde{c}_1^N))D(\partial_t u_N)(t) : D(\alpha_k)dx + \\
&\int_{\Omega_s} \frac{dA}{d\mathcal{F}}(\mathcal{F}(\tilde{c}_1^N))D(u_N)(t) : D(\alpha_k)\big(\frac{d\mathcal{K}}{dt} \star_t F(\tilde{c}_1^N)\big)(t)dx = \\
&\int_{\Gamma_2} \partial_t(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)\alpha_k\, dS
\end{aligned} \tag{5.20}$$

We now test equation (5.20) by $\partial_{tt}u_N$ and get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_{tt}u_N(t)|^2 dx + 2\int_{\Omega_f} |D(\partial_{tt}u_N)(t)|^2 dx + \\
&\int_{\Omega_s} A(\mathcal{F}(\tilde{c}_1^N))D(\partial_t u_N)(t) : D(\partial_{tt}u_N)dx + \\
&\int_{\Omega_s} \frac{dA}{d\mathcal{F}}(\mathcal{F}(\tilde{c}_1^N))D(u_N)(t) : D(\partial_{tt}u_N)\big(\frac{d\mathcal{K}}{dt} \star_t F(\tilde{c}_1^N)\big)(t)dx \\
&= \int_{\Gamma_2} \partial_t(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)\partial_{tt}u_N(t)\, dS
\end{aligned} \tag{5.21}$$

In (5.21) it is necessary to transform several terms:

$$\begin{aligned}
&\int_{\Omega_s} A(\cdot)D(\partial_t u_N) : D(\partial_{tt}u_N)dx = \\
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_s} A(\cdot)D(\partial_t u_N) : D(\partial_t u_N)dx - \\
&-\frac{1}{2}\int_{\Omega_s} \frac{dA}{d\mathcal{F}}(\cdot)D(\partial_t u_N)(t) : D(\partial_t u_N)(t)\big(\frac{d\mathcal{K}}{dt} \star_t F(\tilde{c}_1^N)\big)(t)dx
\end{aligned} \tag{5.22}$$

$$(5.23)\qquad \int_{\Omega_s} \frac{dA}{d\mathcal{F}}(\cdot)D(u_N) : D(\partial_{tt}u_N)\big(\frac{d\mathcal{K}}{dt}\star_t F(\tilde{c}_1^N)\big)(t)dx =$$

$$\frac{d}{dt}\int_{\Omega_s} \frac{dA}{d\mathcal{F}}(\cdot)D(u_N) : D(\partial_t u_N)\big(\frac{d\mathcal{K}}{dt}\star_t F(\tilde{c}_1^N)\big)(t)dx -$$

$$-\int_{\Omega_s} \frac{dA}{d\mathcal{F}}(\cdot)D(\partial_t u_N) : D(\partial_t u_N)\big(\frac{d\mathcal{K}}{dt}\star_t F(\tilde{c}_1^N)\big)(t)dx -$$

$$-\int_{\Omega_s} \frac{dA}{d\mathcal{F}}(\cdot)D(u_N) : D(\partial_t u_N)\big(\frac{d^2\mathcal{K}}{dt^2}\star_t F(\tilde{c}_1^N)\big)(t)dx -$$

$$-\int_{\Omega_s} \frac{d^2A}{d\mathcal{F}^2}(\cdot)D(u_N) : D(\partial_t u_N)\big(\frac{d\mathcal{K}}{dt}\star_t F(\tilde{c}_1^N)\big)^2(t)dx$$

$$(5.24)\qquad \int_{\Gamma_2} \partial_t(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)\partial_{tt}u_N(t)\,dS = \frac{d}{dt}\int_{\Gamma_2} \partial_t(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)\partial_t u_N(t)\,dS$$

$$-\int_{\Gamma_2} \partial_{tt}(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)\partial_t u_N(t)\,dS$$

The terms involving lower derivatives in time are estimated as follows:

$$\left|\int_{\Omega_s} \frac{dA}{d\mathcal{F}}(\cdot)D(\partial_t u_N)(t) : D(\partial_t u_N)(t)\big(\frac{d\mathcal{K}}{dt}\star_t F(\tilde{c}_1^N)\big)(t)dx\right| \le$$

$$C\left|\left|\frac{dA}{d\mathcal{F}}\right|\right|_{L^\infty(\mathbb{R})^9}(1+||\tilde{c}_1^N||_{L^\infty((0,T_N)\times\Omega_s)})||D(\partial_t u_N)(t)||^2_{L^2(\Omega_s)^9}$$

$$\left|\int_{\Omega_s} \frac{dA}{d\mathcal{F}}(\cdot)D(u_N) : D(\partial_t u_N)\big(\frac{d^2\mathcal{K}}{dt^2}\star_t F(\tilde{c}_1^N)\big)(t)dx\right| \le$$

$$C\left|\left|\frac{dA}{d\mathcal{F}}\right|\right|_{L^\infty(\mathbb{R})^9}(1+||\tilde{c}_1^N||_{L^\infty((0,T_N)\times\Omega_s)})||D(u_N)(t)||_{L^2(\Omega_s)^9}||D(\partial_t u_N)(t)||_{L^2(\Omega_s)^9}$$

$$\left|\int_{\Omega_s} \frac{d^2A}{d\mathcal{F}^2}(\cdot)D(u_N) : D(\partial_t u_N)\big(\frac{d\mathcal{K}}{dt}\star_t F(\tilde{c}_1^N)\big)^2(t)dx\right| \le$$

$$C\left|\left|\frac{d^2A}{d\mathcal{F}^2}\right|\right|_{L^\infty(\mathbb{R})^9}(1+||\tilde{c}_1^N||_{L^\infty((0,T_N)\times\Omega_s)})||D(u_N)(t)||_{L^2(\Omega_s)^9}||D(\partial_t u_N)(t)||_{L^2(\Omega_s)^9}$$

$$|\int_{\Gamma_2} \partial_{tt}(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)(t)\partial_t u_N(t)\,dS| \le$$

$$||\partial_{tt}(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)(t)||_{L^2(\Gamma_2)^3}||\partial_t u_N(t)||_{L^2(\Gamma_2)^3} \le$$

$$||\partial_{tt}(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)(t))||_{L^2(\Gamma_2)^3}||D(\partial_t u_N)(t)||_{L^2(\Omega)^9}$$

It remains now to calculate and estimate $\partial_{tt}u_N$ at $t = 0$. Thus let us evaluate equation (5.7) at $t = 0$.

$$\int_\Omega \partial_{tt}u_N(0)\alpha_k dx + 2\int_{\Omega_f} D(\partial_t u_N(0)) : D(\alpha_k)\, dx +$$
$$\int_{\Omega_s} A(\mathcal{F}(\tilde{c}_1^N))D(u_N(0)) : D(\alpha_k) \;=\; \int_{\Gamma_2} (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)(0)\alpha_k\, dS$$

Taking into accout the initial conditions (5.8) and the assumption $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)(0) = 0$, we have that $\partial_{tt}u_N(0) = 0$.

Now integrating with respect to time in (5.21), using the regularity assumptions from Section 4.1 on A and $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ as well as the estimates above, we obtain

$$\frac{1}{2}\int_\Omega |\partial_{tt}u_N(t)|^2 dx + 2\int_0^t \int_{\Omega_f} |D(\partial_{tt}u_N)|^2 dx d\tau +$$
$$\frac{1}{2}\int_{\Omega_s} A(\mathcal{F}(\tilde{c}_1^N))D(\partial_t u_N)(t) : D(\partial_t u_N)(t) dx \leq$$
$$C(M)\int_0^t ||D(\partial_t u_N)||^2_{L^2(\Omega_s)^9} d\tau + C(M)||D(\partial_t u_N)(t)||_{L^2(\Omega_s)^9} +$$
$$C(M)\left(||D(\partial_t u_N)(t)||^2_{L^2(\Omega_f)^9} + \int_0^t ||D(\partial_t u_N)||^2_{L^2(\Omega_f)^9}\right) +$$
$$||(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)(t))||^2_{H^2(0,T;L^2(\Gamma_2))^3}$$

In order to estimate $||D(\partial_t u_N)||^2_{L^2(\Omega_f)^9}$, we proceed like in (5.16) and, since $D(\partial_t u_N)(0) = 0$, we obtain:

$$||D(\partial_t u_N)(t)||_{L^2(\Omega_f)^9} \leq \sqrt{t}||D(\partial_{tt}u_N)||_{L^2(0,T_N;L^2(\Omega_f))^9} \tag{5.25}$$

Thus, Gronwall's inequality implies the following estimates:

$$||\partial_{tt}u_N||_{L^\infty(0,T_N;L^2(\Omega))^3} \;\leq\; C(M) \tag{5.26}$$
$$||D(\partial_{tt}u_N)||_{L^2(0,T_N;L^2(\Omega_f))^9} \;\leq\; C(M) \tag{5.27}$$
$$||D(\partial_t u_N)||_{L^\infty(0,T_N;L^2(\Omega_s))^9} \;\leq\; C(M) \tag{5.28}$$

Next, we test (5.9) by c_1^N and obtain

(5.29)
$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_s} |c_1^N(t)|^2_{L^2(\Omega_s)} + \int_{\Omega_s} D_1(c_2^N)|\nabla c_1^N(t)|^2 dx = \int_{\Omega_s} g_1(c_1^N, c_2^N)c_1^N dx$$

Finally, we use $\hat{c}_2^N = c_2^N - c_{2D}(1-x_1)$ as a test function for (5.10). It yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s})|\hat{c}_2^N(t)|^2 dx + D_2 \int_\Omega (\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s})|\nabla \hat{c}_2^N(t)|^2 dx -$$
$$- \int_{\Omega_f} \partial_t u_N \hat{c}_2^N \nabla \hat{c}_2^N dx + \int_{\Gamma_2 \cap \bar{\Omega}_f} \partial_t u_N \cdot e_1 \hat{c}_2^N \hat{c}_2^N dS =$$
$$\int_\Omega \{g_2(c_2)\chi_{\Omega_f} + g_3(c_1,c_2)\chi_{\Omega_s}\}\hat{c}_2^N dx - \int_\Omega (\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s})\partial_t c_{2D}(1-x_1)\hat{c}_2^N dx$$
$$-D_2 \int_\Omega (\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s})\nabla(c_{2D}(1-x_1))\nabla \hat{c}_2^N dx$$
$$+ \int_{\Omega_f} \partial_t u_N (c_{2D}(1-x_1))\nabla \hat{c}_2^N dx - \int_{\Gamma_2 \cap \bar{\Omega}_f} \partial_t u_N \cdot e_1 (c_{2D}(1-x_1))\hat{c}_2^N dS$$

Now we have to estimate several terms. First, since $\partial_t u_N$ is bounded in $L^\infty(0,T_N;L^6(\Omega_f))$ we can conclude, using a Gagliardo-Nirenberg-type inequality, see e. g. inequality (2.9) on pag. 62 in [14], and Young's inequality, that

$$(5.30)\quad |\int_0^t \int_{\Omega_f} \partial_t u_N \hat{c}_2^N \nabla \hat{c}_2^N dx d\tau| \leq$$
$$||\partial_t u_N||_{L^\infty(0,T_N;L^6(\Omega_f))} ||\hat{c}_2^N||_{L^2(0,T_N;L^3(\Omega_f))} ||\nabla \hat{c}_2^N||_{L^2(0,T_N;L^2(\Omega_f))} \leq$$
$$C(M)||\nabla \hat{c}_2^N||^{3/2}_{L^2((0,T_N)\times\Omega_f)} ||\hat{c}_2^N||^{1/2}_{L^2((0,T_N)\times\Omega_f)} \leq$$
$$\delta ||\nabla \hat{c}_2^N||^2_{L^2((0,T_N)\times\Omega_f)} + C(M,\delta)||\hat{c}_2^N||^2_{L^2((0,T_N)\times\Omega_f)}$$

To estimate the next term we have to use the embedding of $H^1(\Omega_f)$ into the space of traces $L^4(\Gamma_2 \cap \bar{\Omega}_f))$, see [14], Theorem 2.1, Chapter 2, to obtain

$$(5.31)\quad |\int_0^t \int_{\Gamma_2 \cap \bar{\Omega}_f} \partial_t u_N \cdot e_1 \hat{c}_2^N \hat{c}_2^N dS d\tau| \leq$$
$$\int_0^t ||\partial_t u_N||_{L^4(\Gamma_2 \cap \bar{\Omega}_f)} ||\hat{c}_2^N||^2_{L^{8/3}(\Gamma_2 \cap \bar{\Omega}_f)} d\tau \leq$$
$$||\partial_t u_N||_{L^\infty(0,T_N;H^1(\Omega_f))} \int_0^t ||\hat{c}_2^N||^2_{L^{8/3}(\Gamma_2 \cap \bar{\Omega}_f)} d\tau$$

Using now the interpolation inequality for $L^{8/3}$ between L^2 and L^4, and the trace estimate (2.21) from [14] pag. 69, we calculate

$$||\hat{c}_2^N||_{L^{8/3}(\Gamma_2 \cap \bar{\Omega}_f)} \leq ||\hat{c}_2^N||^{1/2}_{L^2(\Gamma_2 \cap \bar{\Omega}_f)} ||\hat{c}_2^N||^{1/2}_{L^4(\Gamma_2 \cap \bar{\Omega}_f)} \leq$$
$$C||\hat{c}_2^N||^{1/4}_{L^2(\Omega_f)} ||\hat{c}_2^N||^{3/4}_{H^1(\Omega_f)}$$

Now using Young's inequality with $p = 4, q = \frac{4}{3}$ we obtain

$$(5.32)\quad \int_0^t ||\hat{c}_2^N||^2_{L^{8/3}(\Gamma_2\cap\bar{\Omega}_f)} d\tau \le C\int_0^t ||\hat{c}_2^N||^{1/2}_{L^2(\Omega_f)} ||\hat{c}_2^N||^{3/2}_{H^1(\Omega_f)} d\tau$$
$$\le \delta \int_0^t ||\hat{c}_2^N||^2_{H^1(\Omega_f)} d\tau + C(\delta)\int_0^t ||\hat{c}_2^N||^2_{L^2(\Omega_f)} d\tau$$
$$\le \delta \int_0^t ||\nabla\hat{c}_2^N||^2_{L^2(\Omega_f)^3} d\tau + \bar{C}(\delta)\int_0^t ||\hat{c}_2^N||^2_{L^2(\Omega_f)} d\tau$$

Inserting now (5.32) in (5.31) we have estimated the boundary term as follows

$$(5.33)\, |\int_0^t \int_{\Gamma_2\cap\bar{\Omega}_f} \partial_t u_N \cdot e_1 \hat{c}_2^N \hat{c}_2^N dS d\tau| \le$$
$$C(M)\left(\delta \int_0^t ||\nabla\hat{c}_2^N||^2_{L^2(\Omega_f)^3} d\tau + \bar{C}(\delta)\int_0^t ||\hat{c}_2^N||^2_{L^2(\Omega_f)} d\tau\right)$$

To estimate the remaining terms, we use Hölder inequality and the assumptions on the data, to obtain

$$(5.34)\quad |\int_0^t \int_\Omega (\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s})\partial_t c_{2D}(1-x_1)\hat{c}_2^N dx d\tau| \le$$
$$C||\partial_t c_{2D}||_{L^2((0,t)\times\Omega)} ||\hat{c}_2^N||_{L^2((0,t)\times\Omega)}$$

$$(5.35)\quad |\int_0^t \int_\Omega (\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s})\nabla(c_{2D}(1-x_1))\nabla\hat{c}_2^N dx d\tau| \le$$
$$C||c_{2D}(1-x_1)||_{L^2(0,t;H^1(\Omega))} ||\nabla\hat{c}_2^N||_{L^2((0,t)\times\Omega)^3}$$

Then using again the boundedness of $\partial_t u_N$ in $L^\infty(0,T_N;L^6(\Omega_f))$ we have

$$(5.36)\, |\int_0^t \int_{\Omega_f} \partial_t u_N (c_{2D}(1-x_1))\nabla\hat{c}_2^N dx d\tau| \le$$
$$C||c_{2D}(1-x_1)||_{L^2(0,t;H^1(\Omega_f))} ||\partial_t u_N||_{L^\infty(0,t;L^6(\Omega_f))^3} ||\nabla\hat{c}_2^N||_{L^2((0,t)\times\Omega_f)^3},$$

The last boundary term can be estimated by the same techniques as for (5.33), and thus

$$(5.37)\, |\int_0^t \int_{\Gamma_2\cap\bar{\Omega}_f} \partial_t u_N \cdot e_1 (c_{2D}(1-x_1))\hat{c}_2^N dS d\tau| \le$$
$$C(M)\left(\delta \int_0^t ||\nabla(c_{2D}(1-x_1))||^2_{L^2(\Omega_f)^3} d\tau + \bar{C}(\delta)\int_0^t ||\hat{c}_2^N||^2_{L^2(\Omega_f)} d\tau\right)$$

Collecting now the estimates (5.29)-(5.37), we obtain

$$
\begin{aligned}
(5.38) \qquad & \frac{1}{2}\int_{\Omega_s}|c_1(t)|^2dx+\frac{1}{2}\int_{\Omega}(\chi_{\Omega_f}+\frac{1}{K}\chi_{\Omega_s})|\hat{c}_2^N(t)|^2dx \\
& C_0\int_0^t\int_{\Omega_s}|\nabla c_1^N(t)|^2dxd\tau+C_0\int_0^t\int_{\Omega}|\nabla\hat{c}_2^N(t)|^2dxd\tau\le \\
& C(M)\int_0^t\int_{\Omega}|\hat{c}_2^N(t)|^2dxd\tau+C.
\end{aligned}
$$

Now, using Gronwalls lemma, we finally obtain for the concentrations the estimates

$$
(5.39) \qquad ||c_1^N||_{L^\infty(0,T_N;L^2(\Omega_s))}\le C(M), \quad ||\nabla c_1^N||_{L^2((0,T_N)\times\Omega_s)}\le C(M)
$$

$$
(5.40) \qquad ||c_2^N||_{L^\infty(0,T_N;L^2(\Omega))}\le C(M), \quad ||\nabla c_2^N||_{L^2((0,T_N)\times\Omega)}\le C(M)
$$

Since we succeeded to prove apriori estimates with constants independent of T_N, there exists a time $T>0$ such that the solution $\{u_N,c_1^N,c_2^N\}$ to the problem (5.7)-(5.11) is defined on $(0,T)$, for all $N\in\mathbb{N}$.

3. Step: Compactness estimates for the discretized cut-off problem In order to establish strong compactness of the concentrations we try to prove an estimate of the type

$$
\int_0^{T-h}\int_{\Omega}\frac{1}{h}|c_2^N(t+h,x)-c_2^N(t,x)|^2dxdt\le C, \quad h>0.
$$

Clearly, it is enough to obtain the result for the sequence $\{c_2^N\}$. The corresponding estimate for $\{c_1^N\}$ is analogous.

In the analogy with the literature, we integrate equation (5.10) with respect to time between t and $t+h$ and test with $\bar{c}_2^N=c_2^N(t+h)-c_2^N(t)$. We obtain the following inequality

$$
\begin{aligned}
(5.41) \qquad & \int_0^{T-h}\int_{\Omega}\{\chi_{\Omega_f}+\frac{1}{K}\chi_{\Omega_s}\}|\bar{c}_2^N(t)|^2dxdt\le \\
& C\Big\{\int_0^{T-h}\int_{\Omega}\left|\int_t^{t+h}\nabla c_2^N(\tau)d\tau\right||\nabla\bar{c}_2^N(t)|dxdt+ \\
& \int_0^{T-h}\int_{\Omega_f}\left|\int_t^{t+h}\partial_\tau u_N(\tau)c_2^N(\tau)d\tau\right||\nabla\bar{c}_2^N(t)|dxdt+ \\
& \int_0^{T-h}\int_{\Gamma_2\cap\bar{\Omega}_f}\left|\int_t^{t+h}\partial_\tau u_N(\tau)\cdot e_1c_2^N(\tau)d\tau\right||\bar{c}_2^N(t)|dxdt+ \\
& \int_0^{T-h}\int_{\Omega}\left|\int_t^{t+h}\{g_2(c_2^N)\chi_{\Omega_f}+g_3(c_1^N,c_2^N)\chi_{\Omega_s}\}d\tau\right||\bar{c}_2^N(t)|dxdt\Big\}
\end{aligned}
$$

Obviously, it is enough to estimate the transport term, all other terms are much easier to handle. In the following calculations we use several times Hölders inequality and the apriori estimates for u_N and c_N^2.

$$
\begin{aligned}
(5.42)\quad &\int_0^{T-h}\int_{\Omega_f}\left|\int_t^{t+h}\partial_\tau u_N(\tau)c_2^N(\tau)d\tau\right||\nabla\bar{c}_2^N(t)|dxdt \le \\
&\int_0^{T-h}\left(\int_{\Omega_f}\left|\int_t^{t+h}\partial_\tau u_N(\tau)c_2^N(\tau)d\tau\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla\bar{c}_2^N(t)|^2dx\right)^{\frac{1}{2}}dt \le \\
&C\int_0^{T-h}\left(\int_{\Omega_f} h\int_t^{t+h}|\partial_\tau u_N(\tau)c_2^N(\tau)|^2 d\tau\, dx\right)^{\frac{1}{2}}||\bar{c}_2^N(t)||_{H^1(\Omega_f)}\,dt \le \\
&C\left(\int_0^{T-h}||\bar{c}_2^N(t)||^2_{H^1(\Omega_f))}\right)^{\frac{1}{2}}\left(\int_0^{T-h}\int_{\Omega_f} h\int_t^{t+h}|\partial_\tau u_N(\tau)c_2^N(\tau)|^2\right)^{\frac{1}{2}} \le \\
&C\,h^{\frac{1}{2}}\left(\int_0^{T-h}\int_t^{t+h}\left(\int_{\Omega_f}|c_2^N(\tau)|^6\right)^{\frac{1}{3}}\left(\int_{\Omega_f}|\partial_\tau u_N(\tau)|^3\right)^{\frac{2}{3}}d\tau dt\right)^{\frac{1}{2}} \le \\
&C\,h^{\frac{1}{2}}||\partial_t u_N||_{L^\infty(0,T;L^3(\Omega_f))}\left(\int_0^{T-h}\int_t^{t+h}\left(\int_{\Omega_f}|c_2^N(\tau)|^6\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} \le \\
&C(M)\,h^{\frac{1}{2}}\,h^{\frac{1}{2}}||c_2^N||_{L^2(0,T;L^6(\Omega_f))} \le C(M)\,h
\end{aligned}
$$

Therefore we conclude that the following estimates hold true

$$
(5.43)\qquad \int_0^{T-h}\frac{||c_2^N(t+h,\cdot)-c_2^N(t,\cdot)||^2_{L^2(\Omega)}}{h}dt \le C(M)
$$

$$
(5.44)\qquad \int_0^{T-h}\frac{||c_1^N(t+h,\cdot)-c_1^N(t,\cdot)||^2_{L^2(\Omega_s)}}{h}dt \le C(M)
$$

4. Step: Convergence of the approximates for the cut-off problem Now we are able to formulate the compactness properties for the sequence (u_N, c_1^N, c_2^N).

Proposition 2 *There exist* (u, c_1, c_2), *with*

$$
\begin{aligned}
&u \in W^{2,\infty}(0,T;L^2(\Omega))\cap W^{1,\infty}(0,T;H^1(\Omega))\cap H^2(0,T;H^1(\Omega_f))\\
&c_1 \in L^\infty(0,T;L^2(\Omega_s))\cap L^2(0,T;H^1(\Omega_s))\cap W_2^{1,1/2}((0,T)\times\Omega_s),\\
&c_2 \in L^\infty(0,T;L^2(\Omega_s))\cap L^2(0,T;H^1(\Omega))\cap W_2^{1,1/2}((0,T)\times\Omega),
\end{aligned}
$$

and a subsequence, denoted again (u_N, c_1^N, c_2^N), *such that*

$$\begin{array}{rcll}
\partial_t u_N & \rightharpoonup & \partial_t u & \textit{weak}^\star \textit{ in } \quad L^\infty(0,T;H^1(\Omega)) \\
\partial_t u_N & \to & \partial_t u & \textit{strongly in } \quad C([0,T];L^q(\Omega)),\ q<6 \\
\partial_{tt} u_N & \rightharpoonup & \partial_{tt} u & \textit{weak}^\star \textit{ in } \quad L^\infty(0,T;L^2(\Omega)) \\
\partial_{tt} u_N & \rightharpoonup & \partial_{tt} u & \textit{weakly in } \quad L^2((0,T)\times\Omega_f) \\
c_1^N & \rightharpoonup & c_1 & \textit{weakly in } \quad L^2(0,T;H^1(\Omega_s)) \\
c_1^N & \rightharpoonup & c_1 & \textit{weak}^\star \textit{ in } \quad L^\infty(0,T;L^2(\Omega_s)) \\
c_1^N & \to & c_1 & \textit{strongly in } \quad L^2((0,T)\times\Omega_s) \\
c_2^N & \rightharpoonup & c_2 & \textit{weakly in } \quad L^2(0,T;H^1(\Omega)) \\
c_2^N & \rightharpoonup & c_2 & \textit{weak}^\star \textit{ in } \quad L^\infty(0,T;L^2(\Omega)) \\
c_2^N & \to & c_2 & \textit{strongly in } \quad L^2((0,T)\times\Omega)
\end{array}$$

Proof: The apriori estimates (5.17) - (5.19), (5.26) - (5.28), (5.39) - (5.40), (5.43) - (5.44) together with compactness results from classical parabolic theory (see e.g. [14]) imply the above weak and strong compactness properties of the sequence (u_N, c_1^N, c_2^N).

5. Step: Passing to the limit in the discretized cut-off problem The convergence properties from Proposition 2 allow us any easy passing to the limit in the approximate problem. Therefore any limit functions (u, c_1, c_2) satisfy for a. e. $t \in (0,T)$ the problem

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}(t)\varphi\,dx + 2\int_{\Omega_f} D(\frac{\partial u}{\partial t}(t)) : D(\varphi)\,dx + \int_{\Omega_s} A(\mathcal{F}(\tilde{c}_1))D(u(t)) : D(\varphi) \;=\; \int_{\Gamma_2} (\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)\varphi\,dS, \tag{5.45}$$

for all $\varphi \in V = \{\varphi \in H^1(\Omega)^3;\ \nabla\cdot\varphi = 0 \text{ in } \Omega_f,\ \varphi = 0 \text{ on } \Gamma_3\}$, and

$$\nabla\cdot(\partial_t u) = 0, \qquad \text{in } (0,T)\times\Omega_f \tag{5.46}$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \qquad \text{in } \Omega \tag{5.47}$$

$$u(t,x) = 0 \qquad \text{in } (0,T)\times\Gamma_3 \tag{5.48}$$

(5.49) $$\left\langle \frac{\partial c_1}{\partial t}(t), \psi \right\rangle + \int_{\Omega_s} D_1(c_2)\nabla c_1(t)\nabla\psi\, dx = \int_{\Omega_s} g_1(c_1, c_2)G_1\psi dx,$$

(5.50) $$\left\langle \{\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s}\}\frac{\partial c_2}{\partial t}(t), \zeta \right\rangle + \int_{\Omega} D_2\{\chi_{\Omega_f} + \frac{1}{K}\chi_{\Omega_s}\}\nabla c_2(t)\nabla\zeta\, dx$$
$$- \int_{\Omega_f} \frac{\partial u}{\partial t}(t)c_2(t)\nabla\zeta dx = - \int_{\Gamma_2\cap\bar{\Omega}_f} \frac{\partial u}{\partial t}(t)\cdot e_1 c_2(t)\zeta dS$$
$$+ \int_{\Omega}\{g_2(c_2)\chi_{\Omega_f} + g_3(c_1, c_2)\chi_{\Omega_s}\}\zeta dx,$$

for all $\psi \in H^1(\Omega_s)$ and $\zeta \in W = \{\zeta \in H^1(\Omega),\ \zeta = 0 \text{ on } \Gamma_1\}$, and

(5.51) $$c_1(0) = c_{10} \text{ in } \Omega_s, \quad c_2(0) = c_{20} \text{ in } \Omega$$

(5.52) $$c_2|_{(0,T)\times\Gamma_1} = c_{2D}$$

where the cut-off function $\tilde{c}_1$ is defined by

$$\tilde{c}_1 = \inf\{\sup\{c_1, 0\}, M\} + \sup\{\inf\{c_1, 0\}, -M\}.$$

In order to conclude the proof of Theorem 1, we now show that a solution to the cut-off problem (5.45)-(5.52) is also a solution to our original problem (5.1)-(5.6). For this we find lower and upper bounds for the concentrations c_1, c_2 which depend only on the bounds for the data but not on the cut-off constant M.

6. Step: Non-negativity of concentrations

Thus let us first prove that $c_1, c_2 \geq 0$. We test equation (5.49) by $c_1^- = \inf\{c_1, 0\}$, equation (5.50) by $c_2^- = \inf\{c_2, 0\}$, and add the obtained equalities. We get, with $k_0 = \min\{1, 1/K\}$

$$\frac{1}{2}\int_{\Omega_s}|c_1^-|^2 + \frac{k_0}{2}\int_{\Omega}|c_2^-|^2 + \int_0^t\int_{\Omega_s} D_1(c_2)|\nabla c_1^-|^2 + k_0 D_2\int_0^t\int_{\Omega}|\nabla c_2^-|^2$$
$$\leq |\int_0^t\int_{\Omega_s}\partial_t u\nabla c_2^- c_2^-| + |\int_0^t\int_{\Gamma_2\cap\bar{\Omega}_f}\partial_t u\cdot e_1 c_2^-(t)c_2^- dS| +$$
$$\int_0^t\int_{\Omega}\{c_1^- g_1(c_1^-, c_2) + c_2^- g_3(c_1, c_2^-)\}\chi_{\Omega_s} + c_2^- g_2(c_2^-)\chi_{\Omega_f} dxd\tau$$

The first two terms on the right hand side can be estimated analogously to the similar terms in (5.30) and (5.31) respectively. To estimate the last term on the right hand side we use the assumption (4.29) and (4.30) on g_1, g_2 and

g_3. We obtain

$$\frac{1}{2}\int_{\Omega_s}|c_1^-|^2+\frac{k_0}{2}\int_{\Omega}|c_2^-|^2+\int_0^t\int_{\Omega_s}D_1(c_2)|\nabla c_1^-|^2+k_0D_2\int_0^t\int_{\Omega}|\nabla c_2^-|^2$$
$$\leq C\left(\int_0^T\int_{\Omega_s}|c_1^-|^2+\int_0^T\int_{\Omega}|c_2^-|^2\right)$$

Next, we apply Gronwalls inequality to get

$$c_1^-=c_2^-=0$$

7. Step: Uniform upper bounds for the concentrations

In the last step of our proof we will construct upper bounds for the concentrations c_1, c_2. Let us start with the bound for c_1 and test equation (5.49) with the function

$$\psi(t,x)=e^{-A_1t}\psi_1(t,x)$$

where $\psi_1\in L^2((0,T),H^1(\Omega_s))$ and $\psi_1(0,x)=0$. We obtain

$$\int_{\Omega_s}\partial_t c_1 e^{-A_1t}\psi_1 dx+\int_{\Omega_s}D_1(c_2)e^{-A_1t}\nabla c_1\nabla\psi_1 dx=\int_{\Omega_s}g_1(c_1,c_2)e^{-A_1t}\psi_1 dx$$

Now we want to set

$$\psi_1=(e^{-A_1t}c_1-M_1)_+=\sup\{e^{-A_1t}c_1-M_1,0\},\ \text{a.e. on}\,[0,T]\times\Omega_s. \tag{5.53}$$

Therefore we write the term containing the time derivative as follows

$$\int_{\Omega_s}\partial_t c_1 e^{-A_1t}\psi_1 dx=\int_{\Omega_s}\partial_t(e^{-A_1t}c_1-M_1)\psi_1 dx+\int_{\Omega_s}A_1e^{-A_1t}c_1\psi_1 dx$$

Now taking ψ_1 as in (5.53), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_s}|(e^{-A_1t}c_1-M_1)_+|^2\,dx+\int_{\Omega_s}A_1e^{-A_1t}c_1(e^{-A_1t}c_1-M_1)_+dx\\ \leq\int_{\Omega_s}g_1(c_1,c_2)e^{-A_1t}(e^{-A_1t}c_1-M_1)_+dx\end{aligned} \tag{5.54}$$

Now due to the structural condition (4.31) on g_1, we can estimate the right hand side in the above inequality by

$$\begin{aligned}\int_{\Omega_s}g_1(c_1,c_2)e^{-A_1t}(e^{-A_1t}c_1-M_1)_+dx\leq\\ \leq\int_{\Omega_s}A_1c_1e^{-A_1t}(e^{-A_1t}c_1-M_1)_+dx\end{aligned} \tag{5.55}$$

Then from (5.54) and (5.55) we obtain after integration with respect to time

$$\int_{\Omega_s} |(e^{-A_1 t} c_1 - M_1)_+|^2(t)\, dx \leq 0$$

This implies

$$c_1 \leq M_1 e^{A_1 t} \quad \text{a.e. on } [0,T] \times \Omega_s.$$

Due to the transport term in the equation for c_2, we have to use a different technique to prove the upper bound for c_2. Thus, let us write the equation for c_2 in the form

$$\hat{a}\partial_t c_2 + v\chi_{\Omega_f} \nabla c_2 = \nabla \cdot (\hat{a} D_2 \nabla c_2) + f \qquad \text{in} \quad \Omega \times (0,T) \tag{5.56}$$

$$[vc_2\chi_{\Omega_f} - \hat{a} D_2 \nabla c_2] \cdot \nu = 0 \qquad \text{on} \quad \Gamma \times (0,T) \tag{5.57}$$

$$c_2 \chi_{\Omega_f} = c_2 \chi_{\Omega_s} \qquad \text{on} \quad \Gamma \times (0,T) \tag{5.58}$$

$$c_2 = c_{2D} \qquad \text{on} \quad \Gamma_1 \times (0,T) \tag{5.59}$$

$$\nabla c_2 \cdot e_1 = 0 \qquad \text{on} \quad \Gamma_2 \times (0,T) \tag{5.60}$$

$$(vc_2\chi_{\Omega_f} - \hat{a} D_2 \nabla c_2) \cdot \nu = 0 \qquad \text{on} \quad \Gamma_3 \times (0,T) \tag{5.61}$$

$$c_2(0) = c_{20} \qquad \text{in} \quad \Omega \tag{5.62}$$

where

$$\hat{a} = \begin{cases} 1 & \text{in } \Omega_f \\ \frac{1}{K} & \text{in } \Omega_s \end{cases} \quad \text{and} \quad f = \begin{cases} g_2(c_2) & \text{in } \Omega_f \\ g_3(c_1, c_2) & \text{in } \Omega_s \end{cases} \tag{5.63}$$

and $[vc_2\chi_{\Omega_f} - \hat{a} D_2 \nabla c_2] \cdot \nu$ represents the jump in the normal flux. For this problem, we have that the coefficient $\hat{a}$ is bounded in $L^\infty(\Omega)$ and strictly positive. From the properties of u we get

$$v = \partial_t u \in W^{1,\infty}(0,T; L^2(\Omega_f)) \cap L^\infty(0,T; H^1(\Omega_f)) \cap H^1(0,T; H^1(\Omega_f))$$

and thus

$$v \in L^\infty(0,T; L^6(\Omega_f)).$$

The reaction term satisfies $f \in L^\infty(0,T; L^6(\Omega))$ due to the conditions (4.27) - (4.28). These regularity properties imply that the conditions (7.1)-(7.2), form ([14], page 181) are satisfied with $\chi_1 = \frac{1}{2}$, $r = +\infty$, $q = 3$. Thus, analogously to Theorem 7.1 in the same reference, we can proof the boundedness of the solution c_2, i.e. there exists $C_2 \geq 0$, such that

$$\sup |c_2(x,t)| \leq C_2, \quad \text{a.e. on } \Omega \times [0,T].$$

6 Regularity of weak solutions

In this section, we prove that assuming higher time regularity for the data of the fluid/structure problem, we can prove more time regularity for the displacements. Using this result, we then show higher regularity for the concentrations c_1 and c_2 in time and space. Thus, let us first prove the following theorem.

Theorem 3 *Let* $A \in (W^{3,\infty}(\mathbb{R}))^9$, $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) \in H^3(0,T; L^2(\Gamma_2))^3$ *and* $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)(0) = \partial_t(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)(0) = 0$. *Then, we have*

$$||\partial_{ttt} u||_{L^\infty(0,T;L^2(\Omega))^3} \leq C \tag{6.1}$$

$$||D(\partial_{ttt} u)||_{L^2(0,T;L^2(\Omega_f))^9} \leq C \tag{6.2}$$

$$||D(\partial_{tt} u)||_{L^\infty(0,T;L^2(\Omega_s))^9} \leq C \tag{6.3}$$

Proof: In order to simplify the notation, we denote the partial derivative of order j with respect time by ∂_t^j. Let us start by differentiating equation (5.20) with respect to t. It yields

$$\begin{aligned}(6.4)\quad & \int_\Omega \partial_t^4 u_N(t)\alpha_k \, dx + 2\int_{\Omega_f} D(\partial_t^3 u_N)(t) : D(\alpha_k)\, dx + \\ & \int_{\Omega_s} AD(\partial_t^2 u_N)(t) : D(\alpha_k)dx + 2\int_{\Omega_s} \frac{dA}{d\mathcal{F}}\frac{d\mathcal{F}}{dt} D(\partial_t u_N)(t) : D(\alpha_k)dx + \\ & \int_{\Omega_s} \left\{ \frac{d^2A}{d\mathcal{F}^2}\left(\frac{d\mathcal{F}}{dt}\right)^2 + \frac{dA}{d\mathcal{F}}\frac{d^2\mathcal{F}}{dt^2} \right\} D(u_N)(t) : D(\alpha_k)dx = \\ & \int_{\Gamma_2} \partial_t^2(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)\alpha_k \, dS\end{aligned}$$

We now test equation (6.4) by $\partial_t^3 u_N$ and get

$$\begin{aligned}(6.5)\quad & \frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_t^3 u_N(t)|^2 dx + 2\int_{\Omega_f} |D(\partial_t^3 u_N)(t)|^2 dx + \\ & \int_{\Omega_s} AD(\partial_t^2 u_N)(t) : D(\partial_t^3 u_N)(t)dx + \\ & 2\int_{\Omega_s} \frac{dA}{d\mathcal{F}}\frac{d\mathcal{F}}{dt} D(\partial_t u_N)(t) : D(\partial_t^3 u_N)(t)dx + \\ & \int_{\Omega_s} \left\{ \frac{d^2A}{d\mathcal{F}^2}\left(\frac{d\mathcal{F}}{dt}\right)^2 + \frac{dA}{d\mathcal{F}}\frac{d^2\mathcal{F}}{dt^2} \right\} D(u_N)(t) : D(\partial_t^3 u_N)(t)dx = \\ & \int_{\Gamma_2} \partial_t^2(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)\partial_t^3 u_N(t)\, dS\end{aligned}$$

In (6.10) it is necessary to transform serveral terms:

$$\int_{\Omega_s} AD(\partial_t^2 u_N) : D(\partial_t^3 u_N)dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega_s} AD(\partial_t^2 u_N) : D(\partial_t^2 u_N)dx \tag{6.6}$$
$$-\frac{1}{2}\int_{\Omega_s} \frac{dA}{d\mathcal{F}}\frac{d\mathcal{F}}{dt} D(\partial_t^2 u_N) : D(\partial_t^2 u_N)dx$$

$$\int_{\Omega_s} \frac{dA}{d\mathcal{F}}\frac{d\mathcal{F}}{dt} D(\partial_t u_N) : D(\partial_t^3 u_N)dx = \tag{6.7}$$
$$\frac{d}{dt}\int_{\Omega_s} \frac{dA}{d\mathcal{F}}\frac{d\mathcal{F}}{dt} D(\partial_t u_N) : D(\partial_t^2 u_N)dx$$
$$-\int_{\Omega_s} \frac{dA}{d\mathcal{F}}\frac{d\mathcal{F}}{dt} D(\partial_t^2 u_N) : D(\partial_t^2 u_N)dx$$
$$-\int_{\Omega_s} \left\{ \frac{d^2A}{d\mathcal{F}^2}\left(\frac{d\mathcal{F}}{dt}\right)^2 + \frac{dA}{d\mathcal{F}}\frac{d^2\mathcal{F}}{dt^2} \right\} D(\partial_t u_N)(t) : D(\partial_t^2 u_N)dx$$

$$\int_{\Omega_s} \left\{ \frac{d^2A}{d\mathcal{F}^2}\left(\frac{d\mathcal{F}}{dt}\right)^2 + \frac{dA}{d\mathcal{F}}\frac{d^2\mathcal{F}}{dt^2} \right\} D(u_N) : D(\partial_t^3 u_N)dx = \tag{6.8}$$
$$\frac{d}{dt}\int_{\Omega_s} \left\{ \frac{d^2A}{d\mathcal{F}^2}\left(\frac{d\mathcal{F}}{dt}\right)^2 + \frac{dA}{d\mathcal{F}}\frac{d^2\mathcal{F}}{dt^2} \right\} D(u_N) : D(\partial_t^2 u_N)dx$$
$$-\int_{\Omega_s} \left\{ \frac{d^2A}{d\mathcal{F}^2}\left(\frac{d\mathcal{F}}{dt}\right)^2 + \frac{dA}{d\mathcal{F}}\frac{d^2\mathcal{F}}{dt^2} \right\} D(\partial_t u_N) : D(\partial_t^2 u_N)dx$$
$$-\int_{\Omega_s} \left\{ \frac{d^3A}{d\mathcal{F}^3}\left(\frac{d\mathcal{F}}{dt}\right)^3 + 3\frac{d^2A}{d\mathcal{F}^2}\frac{d\mathcal{F}}{dt}\frac{d^2\mathcal{F}}{dt^2} + \frac{dA}{d\mathcal{F}}\frac{d^3\mathcal{F}}{dt^3} \right\} \times$$
$$\times\, D(u_N) : D(\partial_t^2 u_N)dx$$

$$\int_{\Gamma_2} \partial_t^2(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)\partial_t^3 u_N\, dS = \frac{d}{dt}\int_{\Gamma_2} \partial_t^2(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)\partial_t^2 u_N\, dS \tag{6.9}$$
$$-\int_{\Gamma_2} \partial_t^3(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)\partial_t^2 u_N\, dS$$

As before, $(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)(0) = \partial_t(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)(0) = 0$ implies $\partial_t^3 u_N = 0$. Thus, integrating (6.10) with respect to time and using the regularity of the coefficients as well as the boundedness of c_1 in $L^\infty(\Omega_s)$, we obtain:

$$
\begin{aligned}
(6.10)\qquad & \frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_t^3 u_N(t)|^2 dx + 2\int_{\Omega_f} |D(\partial_t^3 u_N)(t)|^2 dx + \\
& \frac{1}{2}\int_{\Omega_s} AD(\partial_t^2 u_N)(t) : D(\partial_t^2 u_N)(t) dx \\
& \leq C\int_0^t ||D(\partial_t^2 u_N)(\tau)||^2_{L^2(\Omega_s)^9} d\tau + \\
& C||D(\partial_t u_N)(t)||_{L^2(\Omega_s)^9} ||D(\partial_t^2 u_N)(t)||_{L^2(\Omega_s)^9} + \\
& C\int_0^t ||D(\partial_t u_N)(\tau)||_{L^2(\Omega_s)^9} ||D(\partial_t^2 u_N)(\tau)||_{L^2(\Omega_s)^9} + \\
& C||D(u_N)(t)||_{L^2(\Omega_s)^9} ||D(\partial_t^2 u_N)(t)||_{L^2(\Omega_s)^9} + \\
& C\int_0^t ||D(u_N)(\tau)||_{L^2(\Omega_s)^9} ||D(\partial_t^2 u_N)(\tau)||_{L^2(\Omega_s)^9} + \\
& C\left(||D(\partial_t^2 u_N)(t)||^2_{L^2(\Omega_f)^9} + \int_0^t ||D(\partial_t^2 u_N)(\tau)||^2_{L^2(\Omega_f)^9} \right) + \\
& ||(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)(t))||^2_{H^3(0,T;L^2(\Gamma_2))^3}
\end{aligned}
$$

In order to estimate $||D(\partial_t^2 u_N)||^2_{L^2(\Omega_f)^9}$, we proceed like in (5.16) and, since $D(\partial_t^3 u_N)(0) = 0$, we obtain:

$$
(6.11)\qquad ||D(\partial_t^2 u_N)(t)||_{L^2(\Omega_f)^9} \leq \sqrt{t}||D(\partial_t^3 u_N)||_{L^2(0,T_N;L^2(\Omega_f))^9}
$$

Thus, using the estimates (5.17)- (5.19) and (5.26)-(5.28) from Step 2 in the proof of Theorem 1, Gronwall's inequality leads to:

$$
\begin{aligned}
(6.12)\qquad ||\partial_t^3 u_N||_{L^\infty(0,T_N;L^2(\Omega))^3} &\leq C \\
(6.13)\qquad ||D(\partial_t^3 u_N)||_{L^2(0,T_N;L^2(\Omega_f))^9} &\leq C \\
(6.14)\qquad ||D(\partial_t^2 u_N)||_{L^\infty(0,T_N;L^2(\Omega_s))^9} &\leq C
\end{aligned}
$$

Finally, after passing to the limit for $N \to 0$, we obtain the assertion of the theorem.

Let us now show more regularity for the concentrations c_1, c_2. First, by direct generalization of the proof of Theorem 10.1 in ([14], page 204-206), we have:

LEMMA 4 *Let $c_{2D} \in C^{\beta,\frac{\beta}{2}}(\Gamma_1 \times [0,T])$ and $c_{20} \in H^\beta(\bar{\Omega})$, $c_{20}|_{\Gamma_1} = c_{2D}|_{t=0}$, for some $\beta > 0$. Then every weak solution c_2 to the problem* $(5.56) - (5.62)$, *constructed in Theorem 1, is Hölder-continuous on $\bar{\Omega} \times [0,T]$.*

Next, we note that the problem $(5.56)-(5.62)$ is a transmission problem. Using the classical results on difraction problems from ([14], page 224-232), we obtain higher regularity for c_2 in Ω_f and Ω_s separately.

LEMMA 5 *Let c_{20} and c_{2D} satisfy the assumptions from section 4.1. Then, we have $c_2 \in C^{2+\beta,1+\frac{\beta}{2}}(\Omega_f \times [0,T])$ and $c_2 \in C^{2+\beta,1+\frac{\beta}{2}}(\Omega_s \times [0,T])$.*

Additionally to this interior regularity results, we now prove a global higher integrability for the derivatives.

Theorem 6 *Let $c_{20} \in H^1(\Omega)$ and $c_{2D} \in H^2(\Gamma_1 \times (0,T))$. Then, for any weak solution c_2 of the problem $(5.56)-(5.62)$, we have $\nabla c_2 \in L^\infty(0,T;L^2(\Omega))$, $\Delta c_2 \in L^2(0,T;L^2(\Omega_f \cup \Omega_s))$ and $\partial_t c_2 \in L^2((0,T)\times\Omega)$.*

Proof: We multiply equation (5.56) in Ω_s by $D_2\Delta c_2$ and in Ω_f by $\nabla\cdot(D_2\nabla c_2 - vc_2)$, and integrate with respect to x. We obtain

$$\frac{D_2}{K}\int_{\Omega_s}\partial_t c_2\Delta c_2 - \frac{D_2^2}{K}\int_{\Omega_s}|\Delta c_2|^2 = D_2\int_{\Omega_s} g_3\Delta c_2 \tag{6.15}$$

$$\begin{aligned}&\int_{\Omega_f}\partial_t c_2\nabla\cdot(D_2\nabla c_2 - vc_2) - \int_{\Omega_f}|\nabla\cdot(D_2\nabla c_2 - vc_2)|^2 \\ &= \int_{\Omega_f} g_2\nabla\cdot(D_2\nabla c_2 - vc_2)\end{aligned} \tag{6.16}$$

In (6.15), (6.16) we integrate by parts in the terms containing the time derivative and add the two equations to get

$$\begin{aligned}&-\frac{D_2}{K}\int_{\Omega_s}\partial_t\nabla c_2\nabla c_2 - \int_{\Omega_f}\partial_t\nabla c_2(D_2\nabla c_2 - vc_2) - \frac{D_2^2}{K}\int_{\Omega_s}|\Delta c_2|^2 \\ &- \int_{\Omega_f}|\nabla\cdot(D_2\nabla c_2 - vc_2)|^2 + \frac{D_2}{K}\int_{\partial\Omega_s}\partial_t c_2\nabla c_2\cdot n \\ &+ \int_{\partial\Omega_f}\partial_t c_2(D_2\nabla c_2 - vc_2)\cdot n \\ &= D_2\int_{\Omega_s} g_3\Delta c_2 + D_2\int_{\Omega_f} g_2\Delta c_2 - \int_{\Omega_f} g_2 v\nabla c_2\end{aligned} \tag{6.17}$$

Here, we denote by n the outer unit normal to the underlying domain. Now, by straightforward calculations we obtain

$$(6.18)\quad \frac{D_2}{K}\int_{\Omega_s}\partial_t\nabla c_2\nabla c_2 + D_2\int_{\Omega_f}\partial_t\nabla c_2\nabla c_2 + \frac{D_2^2}{K}\int_{\Omega_s}|\Delta c_2|^2$$
$$+D_2^2\int_{\Omega_f}|\Delta c_2|^2 = \frac{D_2}{K}\int_{\partial\Omega_s}\partial_t c_2\nabla c_2\cdot n + \int_{\partial\Omega_f}\partial_t c_2(D_2\nabla c_2 - vc_2)\cdot n$$
$$+\int_{\Omega_f}\partial_t\nabla c_2 vc_2 + 2D_2\int_{\Omega_f}\Delta c_2 v\nabla c_2 - D_2\int_{\Omega_s} g_3\Delta c_2$$
$$-D_2\int_{\Omega_f} g_2\Delta c_2 + \int_{\Omega_f} g_2 v\nabla c_2 - \int_{\Omega_f} v^2|\nabla c_2|^2$$

Using the equation for c_2 on Ω_f, the term on the right hand side containing $\partial_t\nabla c_2$ has to be transformed as follows

$$(6.19)\qquad \int_{\Omega_f}\partial_t\nabla c_2 vc_2 \;=\; \frac{d}{dt}\int_{\Omega_f}\nabla c_2 vc_2 - \int_{\Omega_f}\nabla c_2\partial_t vc_2$$
$$-\;\int_{\Omega_f}\nabla c_2 v(D_2\Delta c_2 + g_2 - v\nabla c_2)$$

Insearting (6.19) into (6.18), we obtain

$$(6.20)\quad \frac{1}{2}\frac{d}{dt}\int_{\Omega}\left\{\frac{D_2}{K}\chi_{\Omega_s} + D_2\chi_{\Omega_f}\right\}|\nabla c_2|^2 + \int_{\Omega}\left\{\frac{D_2^2}{K}\chi_{\Omega_s} + D_2^2\chi_{\Omega_f}\right\}|\Delta c_2|^2$$
$$= \frac{D_2}{K}\int_{\partial\Omega_s}\partial_t c_2\nabla c_2\cdot n + \int_{\partial\Omega_f}\partial_t c_2(D_2\nabla c_2 - vc_2)\cdot n$$
$$+D_2\int_{\Omega_f}\Delta c_2 v\nabla c_2 - D_2\int_{\Omega_s} g_3\Delta c_2 - D_2\int_{\Omega_f} g_2\Delta c_2$$
$$+\frac{d}{dt}\int_{\Omega_f}\nabla c_2 vc_2 - \int_{\Omega_f}\nabla c_2\partial_t vc_2$$

Let us now estimate the terms on the right hand side of (6.20). Using the flux continuity at the interface Γ, and the boundary conditions for the concentrations c_1, c_2, we get

$$(6.21)\qquad \frac{D_2}{K}\int_{\partial\Omega_s}\partial_t c_2\nabla c_2\cdot n + \int_{\partial\Omega_f}\partial_t c_2(D_2\nabla c_2 - vc_2)\cdot n =$$
$$\frac{D_2}{K}\int_{\partial\Omega_s\cap\Gamma_1}\partial_t c_{2D}\nabla c_2\cdot n + \int_{\partial\Omega_f\cap\Gamma_1}\partial_t c_{2D}(D_2\nabla c_2 - vc_2)\cdot n$$
$$-\int_{\partial\Omega_f\cap\Gamma_2}\partial_t c_2 vc_2\cdot n$$

Since $c_{2D} \in H^2(\Gamma_1 \times (0,T))$ and $v \in L^\infty(0,T;H^1(\Omega_f))$, we have the estimate

$$\begin{aligned}(6.22)\qquad & \left| \frac{D_2}{K} \int_{\partial\Omega_s \cap \Gamma_1} \partial_t c_{2D} \nabla c_2 \cdot n + \int_{\partial\Omega_f \cap \Gamma_1} \partial_t c_{2D} (D_2 \nabla c_2 - v c_2) \cdot n \right| \\ & \leq C \left(||\Delta c_2(t)||_{L^2(\Omega_s \cup \Omega_f)} + ||c_2(t)||^2_{H^1(\Omega_f)} \right)\end{aligned}$$

To estimate the last term on the right hand side of (6.21) we note that by Theorem 3 we have that $\partial_t v \in L^\infty(0,T;H^1(\Omega_f))$. Thus, using similar arguments as in (5.31), we have

$$\begin{aligned}(6.23)\qquad & -\int_{\partial\Omega_f \cap \Gamma_2} \partial_t c_2 v c_2 \cdot n = \\ & -\partial_t \int_{\partial\Omega_f \cap \Gamma_2} |c_2|^2 v \cdot n + \int_{\partial\Omega_f \cap \Gamma_2} |c_2|^2 \partial_t v \cdot n \\ & \leq -\partial_t \int_{\partial\Omega_f \cap \Gamma_2} |c_2|^2 v \cdot n + ||\partial_t v||_{L^\infty(0,T;H^1(\Omega_f))} ||c_2(t)||^2_{H^1(\Omega_f)}\end{aligned}$$

In order to estimate the next term on the right hand side in (6.20), we recall that by elliptic regularity for transmission problems, see e.g. estimate (16.12) in ([14], pages 205-223), we have

$$(6.24)\quad ||c_2(t)||_{H^2(\Omega_s \cup \Omega_f)} \leq C \left\{ ||c_2(t)||_{H^1(\Omega)} + ||\Delta c_2(t)||_{L^2(\Omega_s \cup \Omega_f)} + c_0(t) \right\}$$

where $c_0(t)$ represents the boundary conditions at $\partial\Omega$, and it is an element of $L^\infty(0,T)$. Thus, we can estimate

$$\begin{aligned}(6.25)\qquad & \left| \int_{\Omega_f} \Delta c_2 v \nabla c_2 \right| \leq ||\Delta c_2(t)||_{L^2(\Omega_f)} ||v(t)||_{L^6(\Omega_f)} ||\nabla c_2(t)||_{L^3(\Omega_f)} \\ & \leq ||\Delta c_2(t)||_{L^2(\Omega_f)} ||v(t)||_{L^6(\Omega_f)} ||\nabla c_2(t)||^{\frac{1}{2}}_{L^2(\Omega_f)} ||\nabla c_2(t)||^{\frac{1}{2}}_{L^6(\Omega_f)} \\ & \leq C ||\Delta c_2(t)||_{L^2(\Omega_f)} ||v(t)||_{L^6(\Omega_f)} ||\nabla c_2(t)||^{\frac{1}{2}}_{L^2(\Omega_f)} \times \\ & \times \left\{ ||c_2(t)||^{\frac{1}{2}}_{H^1(\Omega)} + ||\Delta c_2(t)||^{\frac{1}{2}}_{L^2(\Omega_s \cup \Omega_f)} + (c_0(t))^{\frac{1}{2}} \right\}\end{aligned}$$

To shorten the notation, we introduce the functions

$$\begin{aligned} h_1(t) &= ||v(t)||_{L^6(\Omega_f)} ||\nabla c_2(t)||^{\frac{1}{2}}_{L^2(\Omega_f)} ||c_2(t)||^{\frac{1}{2}}_{H^1(\Omega)} \\ h_2(t) &= ||v(t)||_{L^6(\Omega_f)} ||\nabla c_2(t)||^{\frac{1}{2}}_{L^2(\Omega_f)} \end{aligned}$$

By Theorem 1, $h_1 \in L^2(0,T)$ and $h_2 \in L^4(0,T)$. Thus, the right hand side in (6.25) can be estimated as

$$\left|\int_{\Omega_f} \Delta c_2 v \nabla c_2\right| \leq C\left(h_1(t)+h_2(t)\right)||\Delta c_2(t)||_{L^2(\Omega_f)} + h_2(t)||\Delta c_2(t)||^{3/2}_{L^2(\Omega_f\cup\Omega_s)} \leq \delta||\Delta c_2(t)||^2_{L^2(\Omega_f\cup\Omega_s)} + C(\delta)h(t) \tag{6.26}$$

with $\delta > 0$ and $h \in L^1(\Omega)$. Next

$$\left|D_2\int_{\Omega_s} g_3\Delta c_2 + D_2\int_{\Omega_f} g_2\Delta c_2\right| \leq \hat{g}(t)||\Delta c_2(t)||_{L^2(\Omega_f\cup\Omega_s)}, \tag{6.27}$$

where $\hat{g} \in L^2(0,T)$, due to the Lipschitz property of the nonlinearities g_2 and g_3. Finally, the last term on the right hand side in (6.20) can be estimated using similar arguments as in (5.30) and the fact that by Theorem 3, $\partial_t v \in L^\infty(0,T;H^1(\Omega_f))$. Thus, we have

$$\left|\int_{\Omega_f} \nabla c_2 \partial_t v c_2\right| \leq C||c_2(t)||^2_{H^1(\Omega_f)} \tag{6.28}$$

Now, for δ small enough, integrating with respect to time in (6.20) and using the estimates above, we obtain

$$\int_\Omega \left\{\frac{D_2}{K}\chi_{\Omega_s} + D_2\chi_{\Omega_f}\right\}|\nabla c_2(t)|^2 + \int_0^t\int_\Omega\left\{\frac{D_2^2}{K}\chi_{\Omega_s} + D_2^2\chi_{\Omega_f}\right\}|\Delta c_2|^2 \leq C\left(1+\int_{\Omega_f}\nabla c_2(t)v(t)c_2(t) - \int_{\partial\Omega_f\cap\Gamma_2}|c_2(t)|^2 v(t)\cdot n\right) \tag{6.29}$$

We estimate the last two terms on the right hand side in (6.29) using similar arguments like in (5.30) and (5.31) respectively, and obtain

$$\left|\int_{\Omega_f}\nabla c_2(t)v(t)c_2(t)\right| + \left|\int_{\partial\Omega_f\cap\Gamma_2}|c_2(t)|^2 v(t)\cdot n\right| \leq \delta||\nabla c_2(t)||^2_{L^2(\Omega_f)} + C(\delta)||c_2(t)||^2_{L^2(\Omega_f)}$$

The first term is absorbed into the left hand side of (6.29), and the second one can be estimated by $C(\delta)||c_2(t)||^2_{L^\infty(0,T;L^2(\Omega_f))}$. Thus, finally we can conclude that

$$\nabla c_2 \in L^\infty(0,T;L^2(\Omega)) \tag{6.30}$$

$$\Delta c_2 \in L^2(0,T;L^2(\Omega_f)) \quad \text{and} \quad \Delta c_2 \in L^2(0,T;L^2(\Omega_s)) \tag{6.31}$$

In the last step of the proof, we use the equations for c_2 to conclude that

$$\partial_t c_2 \in L^2(0,T;L^2(\Omega)). \tag{6.32}$$

Thus, Theorem 6 is proved.

Now, we switch to the study of c_1.

Theorem 7 *Let $c_{10} \in C^2(\bar{\Omega}_s)$ and $\nabla c_1 \cdot n = 0$ on $\partial\Omega_s$. Then $c_1 \in W_q^{2,1}(\Omega_s \times (0,T))$, for all $q > 1$.*

Proof: By Lemma 4 we have that $D_1(c_2)$ is Hölder continuous and takes values between two positive constants. The reaction term $g_1(c_1,c_2)$ is bounded due to the Lipschitz-continuity of g_1. Hence Theorem 9.1 in ([14], pages 341-342) yields $c_1 \in W_q^{2,1}(\Omega_s \times (0,T))$, for all $q > 1$, and the theorem is proved.

By Sobolev's embedding, see Lemma 3.3 in ([13], page 80), we obtain

Corollary 8 *c_1 and ∇c_1 are Hölder continuous.*

7 Stability and uniqueness results

In this section, we give the stability and uniqueness of the regular solutions to the problem (4.1)-(4.22). To this end, we consider solutions $(u^{(j)}, c_1^{(j)}, c_2^{(j)})$, $j = 1,2$, corresponding to the data $(\mathcal{S}_1^{(j)}, \mathcal{S}_2^{(j)}, \mathcal{S}_3^{(j)})$, $j = 1,2$ for the displacements and $g_1^{(j)}, g_2^{(j)}, g_3^{(j)}$, $c_{2D}^{(j)}, c_{10}^{(j)}$, and $c_{20}^{(j)}$, $j = 1,2$, for the concentrations.

Let us start with the calculations for the displacements. We introduce $\delta u := u^{(1)} - u^{(2)}$. δc_1 and $\delta(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ are defined analogously. Then, δu is a solution of the following problem

$$\begin{aligned} &\int_\Omega \frac{\partial^2 \delta u}{\partial t^2}(t)\varphi dx + 2\int_{\Omega_f} D(\frac{\partial \delta u}{\partial t}(t)) : D(\varphi)\, dx + \\ &\int_{\Omega_s} A(\mathcal{F}(c_1^{(1)}))D(\delta u(t)) : D(\varphi) = \int_{\Gamma_2} \delta(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)\varphi\, dS, \\ &- \int_{\Omega_s} \left\{ A(\mathcal{F}(c_1^{(1)})) - A(\mathcal{F}(c_1^{(2)})) \right\} D(u^{(2)}(t)) : D(\varphi) \end{aligned} \tag{7.1}$$

for all $\varphi \in V$, a.e. in $(0,T)$, and

$$\nabla \cdot \left(\frac{\partial \delta u}{\partial t} \right) = 0, \qquad \text{in } \Omega_f \times (0,T), \tag{7.2}$$

$$\delta u(x,0) = 0, \quad \frac{\partial \delta u}{\partial t}(x,0) = 0, \qquad \text{in } \Omega. \tag{7.3}$$

Theorem 9 *The following estimates hold for all $t \in [0,T]$:*

$$\begin{aligned} &||\partial_t \delta u||_{L^\infty(0,t;L^2(\Omega))^3} + ||D(\partial_t \delta u)||_{L^2(0,t;L^2(\Omega_f))^9} + \\ &||D(\delta u)||_{L^\infty(0,t;L^2(\Omega_s))^9} \\ &\leq C \left\{ ||\delta(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)||_{H^1(0,T;L^2(\Gamma_2))^3} + ||\delta c_1||_{L^1(0,t;L^\infty(\Omega_s))} \right\} \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} &||\partial_{tt} \delta u||_{L^\infty(0,t;L^2(\Omega))^3} + ||D(\partial_{tt} \delta u)||_{L^2(0,t;L^2(\Omega_f))^9} + \\ &||D(\partial_t \delta u)||_{L^\infty(0,t;L^2(\Omega_s))^9} \\ &\leq C \left\{ ||\delta(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)||_{H^2(0,T;L^2(\Gamma_2))^3} + ||\delta c_1||_{L^1(0,t;L^\infty(\Omega_s))} \right\} . \end{aligned} \tag{7.5}$$

Proof: We test equation (7.1) by $\varphi = \frac{\partial}{\partial t} \delta u$ and as in the second step of the proof of Theorem 1, we get

$$\begin{aligned} &\int_\Omega |\partial_t \delta u(t)|^2 dx + \int_0^t \int_{\Omega_f} |D(\partial_t \delta u)|^2 \, dx d\tau + \\ &\int_{\Omega_s} A(\mathcal{F}(c_1^{(1)})) D(\delta u)(t) : D(\delta u)(t) dx \\ &\leq ||\delta(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)||^2_{H^1(0,T;L^2(\Gamma_2))^3} + \\ &\left| \int_0^t \int_{\Omega_s} \left\{ A(\mathcal{F}(c_1^{(1)})) - A(\mathcal{F}(c_1^{(2)})) \right\} D(u^{(2)}) : D(\partial_t \delta u) dx d\tau \right| \end{aligned} \tag{7.6}$$

To estimate the last term on the right hand side, we first transform it as follows:

$$\begin{aligned} &\int_0^t \int_{\Omega_s} \left\{ A(\mathcal{F}(c_1^{(1)})) - A(\mathcal{F}(c_1^{(2)})) \right\} D(u^{(2)}) : D(\partial_t \delta u) dx d\tau \\ &= \int_{\Omega_s} \left\{ A(\mathcal{F}(c_1^{(1)})) - A(\mathcal{F}(c_1^{(2)})) \right\} D(u^{(2)}(t)) : D(\delta u(t)) dx \\ &- \int_0^t \int_{\Omega_s} \left\{ A(\mathcal{F}(c_1^{(1)})) - A(\mathcal{F}(c_1^{(2)})) \right\} D(\partial_t u^{(2)}) : D(\delta u) dx d\tau \\ &- \int_0^t \int_{\Omega_s} \left\{ \frac{dA}{d\mathcal{F}}|_{c_1^{(1)}} \frac{d\mathcal{F}}{dt}(c_1^{(1)}) - \frac{dA}{d\mathcal{F}}|_{c_1^{(2)}} \frac{d\mathcal{F}}{dt}(c_1^{(2)}) \right\} D(u^{(2)}) : D(\delta u) \end{aligned} \tag{7.7}$$

Now, we estimate the three terms on the right hand side of (7.7) separately.

$$\begin{aligned}(7.8)\qquad & \left|\int_{\Omega_s}\left\{A(\mathcal{F}(c_1^{(1)}))-A(\mathcal{F}(c_1^{(2)}))\right\}D(u^{(2)}(t)):D(\delta u(t))dx\right| \\ & \leq ||D(\delta u(t))||_{L^2(\Omega_s)^9}||D(u^{(2)})||_{L^\infty(0,T;L^2(\Omega_s))^9}\times \\ & \times C||\mathcal{K}\star_t\delta c_1||_{L^\infty(0,T;L^\infty(\Omega_s))} \\ & \leq C||D(\delta u(t))||_{L^2(\Omega_s)^9}||\delta c_1||_{L^1(0,t;L^\infty(\Omega_s))}\end{aligned}$$

$$\begin{aligned}(7.9)\qquad & \left|\int_0^t\int_{\Omega_s}\left\{A(\mathcal{F}(c_1^{(1)}))-A(\mathcal{F}(c_1^{(2)}))\right\}D(\partial_t u^{(2)}):D(\delta u)dxd\tau\right| \\ & \leq ||D(\delta u)||_{L^2(0,t;L^2(\Omega_s))^9}||D(\partial_t u^{(2)})||_{L^2(0,T;L^2(\Omega_s))^9}\times \\ & \times C||\mathcal{K}\star_t\delta c_1||_{L^\infty(0,T;L^\infty(\Omega_s))} \\ & \leq C||D(\delta u)||_{L^2(0,t;L^2(\Omega_s))^9}||\delta c_1||_{L^1(0,t;L^\infty(\Omega_s))}\end{aligned}$$

$$\begin{aligned}(7.10)\, & \left|\int_0^t\int_{\Omega_s}\left\{\frac{dA}{d\mathcal{F}}|_{c_1^{(1)}}\frac{d\mathcal{F}}{dt}(c_1^{(1)})-\frac{dA}{d\mathcal{F}}|_{c_1^{(2)}}\frac{d\mathcal{F}}{dt}(c_1^{(2)})\right\}D(u^{(2)}):D(\delta u)\right| \\ & \leq\int_0^t\int_{\Omega_s}\left|\frac{dA}{d\mathcal{F}}|_{c_1^{(1)}}\right|\left|\mathcal{K}'\star_\tau(F(c_1^{(1)})-F(c_1^{(2)}))\right|\times \\ & \times|D(u^{(2)})(\tau)||D(\delta u)(\tau)|dxd\tau+ \\ & \int_0^t\int_{\Omega_s}\left|\frac{dA}{d\mathcal{F}}|_{c_1^{(1)}}-\frac{dA}{d\mathcal{F}}|_{c_1^{(2)}}\right|\left|\frac{d\mathcal{F}}{dt}(c_1^{(2)})\right||D(u^{(2)})(\tau)||D(\delta u)(\tau)|dxd\tau+ \\ & \leq C||D(\delta u)||_{L^2(0,t;L^2(\Omega_s))^9}||D(u^{(2)})||_{L^2(0,T;L^2(\Omega_s))^9}||\delta c_1||_{L^1(0,t;L^\infty(\Omega_s))}\end{aligned}$$

After plugging (7.8) - (7.10) into (7.7), we obtain

$$\begin{aligned}(7.11)\qquad & \int_\Omega|\partial_t\delta u(t)|^2dx+\int_0^t\int_{\Omega_f}|D(\partial_t\delta u)|^2\,dxd\tau+ \\ & \int_{\Omega_s}A(\mathcal{F}(c_1^{(1)}))D(\delta u)(t):D(\delta u)(t)dx \\ & \leq C\left\{||\delta(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)||^2_{H^1(0,T;L^2(\Gamma_2))^3}+||\delta c_1||_{L^1(0,t;L^\infty(\Omega_s))}\right\}\end{aligned}$$

which proves (7.4). Next, we differentiate (7.1) with respect to time. It yields

$$(7.12)\ \int_\Omega \partial_t^3 \delta u(t)\varphi dx + 2\int_{\Omega_f} D(\partial_t^2 \delta u(t)) : D(\varphi)\, dx +$$
$$\int_{\Omega_s} A(\mathcal{F}(c_1^{(1)}))D(\partial_t \delta u(t)) : D(\varphi)dx \ = \int_{\Gamma_2} \partial_t \delta(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)\varphi\, dS,$$
$$-\int_{\Omega_s} \frac{dA}{d\mathcal{F}}|_{(c_1^{(1)})} \frac{d\mathcal{F}}{dt}(c_1^{(1)}) D(\delta u(t)) : D(\varphi)dx$$
$$-\int_{\Omega_s} \left\{ A(\mathcal{F}(c_1^{(1)})) - A(\mathcal{F}(c_1^{(2)})) \right\} D(\partial_t u^{(2)}(t)) : D(\varphi)dx$$
$$-\int_{\Omega_s} \left\{ \frac{dA}{d\mathcal{F}}|_{(c_1^{(1)})} \frac{d\mathcal{F}}{dt}(c_1^{(1)}) - \frac{dA}{d\mathcal{F}}|_{(c_1^{(2)})} \frac{d\mathcal{F}}{dt}(c_1^{(2)}) \right\} D(u^{(2)}(t)) : D(\varphi)dx$$

Now, we test (7.12) by $\partial_t^2 \delta u$ and, as in the second step of the proof of Theorem 1, we find out

$$(7.13)\ \int_\Omega |\partial_t^2 \delta u(t)|^2 dx + \int_0^t \int_{\Omega_f} |D(\partial_t^2 \delta u)|^2\, dx d\tau +$$
$$\int_{\Omega_s} A(\mathcal{F}(c_1^{(1)}))D(\partial_t \delta u)(t) : D(\partial_t \delta u)(t) dx$$
$$\leq ||\delta(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)||^2_{H^2(0,T;L^2(\Gamma_2))^3} +$$
$$\left| \int_0^t \int_{\Omega_s} \left\{ A(\mathcal{F}(c_1^{(1)})) - A(\mathcal{F}(c_1^{(2)})) \right\} D(\partial_t u^{(2)}) : D(\partial_t^2 \delta u) dx d\tau \right| +$$
$$\left| \int_{\Omega_s} \left\{ \frac{dA}{d\mathcal{F}}|_{(c_1^{(1)})} \frac{d\mathcal{F}}{dt}(c_1^{(1)}) - \frac{dA}{d\mathcal{F}}|_{(c_1^{(2)})} \frac{d\mathcal{F}}{dt}(c_1^{(2)}) \right\} D(u^{(2)}(t)) : D(\partial_t^2 \delta u) \right|$$

Let us estimate the last two terms on the right hand side. First, proceeding as in (7.8) - (7.10) and using the estimates from section 5 for the displacements, we obtain

$$(7.14)\ \left| \int_0^t \int_{\Omega_s} \left\{ A(\mathcal{F}(c_1^{(1)})) - A(\mathcal{F}(c_1^{(2)})) \right\} D(\partial_t u^{(2)}) : D(\partial_t^2 \delta u) dx d\tau \right|$$
$$\leq \left| \int_{\Omega_s} \left\{ A(\mathcal{F}(c_1^{(1)})) - A(\mathcal{F}(c_1^{(2)})) \right\} D(\partial_t u^{(2)}(t)) : D(\partial_t \delta u(t)) dx \right|$$
$$+ \left| \int_0^t \int_{\Omega_s} \left\{ A(\mathcal{F}(c_1^{(1)})) - A(\mathcal{F}(c_1^{(2)})) \right\} D(\partial_t^2 u^{(2)}) : D(\partial_t \delta u) dx d\tau \right|$$
$$+ \left| \int_0^t \int_{\Omega_s} \left\{ \frac{dA}{d\mathcal{F}}|_{(c_1^{(1)})} \frac{d\mathcal{F}}{dt}(c_1^{(1)}) - \frac{dA}{d\mathcal{F}}|_{(c_1^{(2)})} \frac{d\mathcal{F}}{dt}(c_1^{(2)}) \right\} \times \right.$$
$$\left. \times D(u^{(2)}(t)) : D(\partial_t \delta u) \right| \leq C||D(\partial_t \delta u(t))||_{L^2(\Omega_s)} ||\delta c_1||_{L^1(0,t;L^\infty(\Omega_s))}$$

Next

$$\begin{aligned}(7.15)\quad &\left|\int_0^t\int_{\Omega_s}\left\{\frac{dA}{d\mathcal{F}}|_{(c_1^{(1)})}\frac{d\mathcal{F}}{dt}(c_1^{(1)})-\frac{dA}{d\mathcal{F}}|_{(c_1^{(2)})}\frac{d\mathcal{F}}{dt}(c_1^{(2)})\right\}\times\right.\\
&\left.\times D(u^{(2)}(t)):D(\partial_t^2\delta u)\right|\le\left|\int_{\Omega_s}\left\{\frac{dA}{d\mathcal{F}}|_{(c_1^{(1)})}\frac{d\mathcal{F}}{dt}(c_1^{(1)})\right.\right.\\
&\left.\left.-\frac{dA}{d\mathcal{F}}|_{(c_1^{(2)})}\frac{d\mathcal{F}}{dt}(c_1^{(2)})\right\}D(u^{(2)}(t)):D(\partial_t\delta u(t))\right|\\
&+\left|\int_0^t\int_{\Omega_s}\left\{\frac{dA}{d\mathcal{F}}|_{(c_1^{(1)})}\frac{d\mathcal{F}}{dt}(c_1^{(1)})-\frac{dA}{d\mathcal{F}}|_{(c_1^{(2)})}\frac{d\mathcal{F}}{dt}(c_1^{(2)})\right\}\times\right.\\
&\left.\times D(\partial_t u^{(2)}):D(\partial_t\delta u)\right|\\
&+\left|\int_0^t\int_{\Omega_s}\left\{\frac{d^2A}{d\mathcal{F}^2}|_{(c_1^{(1)})}\left(\frac{d\mathcal{F}}{dt}(c_1^{(1)})\right)^2-\frac{d^2A}{d\mathcal{F}^2}|_{(c_1^{(2)})}\left(\frac{d\mathcal{F}}{dt}(c_1^{(2)})\right)^2\right.\right.\\
&\left.\frac{dA}{d\mathcal{F}}|_{(c_1^{(1)})}\frac{d^2\mathcal{F}}{dt^2}(c_1^{(1)})-\frac{dA}{d\mathcal{F}}|_{(c_1^{(2)})}\frac{d^2\mathcal{F}}{dt^2}(c_1^{(2)})\right\}\times\\
&\left.\times D(u^{(2)}):D(\partial_t\delta u)\right|\\
&\le C\left\{||D(\partial_t\delta u(t))||_{L^2(\Omega_s)}+||D(\partial_t\delta u)||_{L^2(0,t;L^2(\Omega_s))}\right\}\times\\
&\times||\delta c_1||_{L^1(0,t;L^\infty(\Omega_s))}\end{aligned}$$

After plugging (7.14)-(7.15) into (7.13), we get

$$\begin{aligned}(7.16)\quad &\int_\Omega|\partial_t^2\delta u(t)|^2dx+\int_0^t\int_{\Omega_f}|D(\partial_t^2\delta u)|^2\,dxd\tau+\\
&\int_{\Omega_s}A(\mathcal{F}(c_1^{(1)}))D(\partial_t\delta u)(t):D(\partial_t\delta u)(t)dx\\
&\le C\left\{||\delta(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)||^2_{H^2(0,T;L^2(\Gamma_2))^3}+||\delta c_1||_{L^1(0,t;L^\infty(\Omega_s))}\right\},\end{aligned}$$

which proves (7.5).

Now, we continue with estimates for δc_1. Using the initial-boundary-value problem (4.12)-(4.14) for c_1, we obtain the following problem for δc_1.

$$\begin{aligned}(7.17)\quad &\partial_t\delta c_1-\nabla\cdot\big(D_1(c_2^{(1)})\nabla(\delta c_1)\big)=g_1(c_1^{(1)},c_2^{(1)})\\
&-g_1(c_1^{(2)},c_2^{(2)})+\big(D_1(c_2^{(1)})-D_1(c_2^{(2)})\big)\Delta c_1^{(2)}\\
&+\nabla c_1^{(2)}\left(D_1'(c_2^{(1)})\nabla c_2^{(1)}-D_1'(c_2^{(2)})\nabla c_2^{(2)}\right)=:R\qquad\text{in }\ \Omega_s\times(0,T)\\
(7.18)\quad &\nabla(\delta c_1)\cdot\nu=0\qquad\text{on }\ \partial\Omega_s\times(0,T)\\
(7.19)\quad &\delta c_1(0)=\delta c_{10}\qquad\text{in }\ \Omega_s\end{aligned}$$

The following theorem holds.

Theorem 10 *Let* $c_{10}^{(j)} \in C^2(\bar{\Omega}_s)$ *and* $\nabla c_{10}^{(j)} \cdot \nu = 0$ *on* $\Omega_s \times (0,T)$*, for* $j = 1, 2$*. Then, there exists* $\alpha \geq \alpha_0 > 0$*, such that*

$$(7.20) \qquad ||\delta c_1||_{W_2^{2,1}(Q_t^s)} \leq C \left\{ ||\delta c_2||_{W_2^{2,1}(Q_t^s)} t^\alpha + ||\delta c_{10}||_{H^1(\Omega_s)} \right\}$$

for all $t \in (0,T)$*.*

Proof: We argue as in the proof of Theorem 7. We note that $c_2^{(1)}$ is Hölder continuous. Then, Theorem 9.1 in ([14], pages 341-342) yields

$$(7.21) \qquad ||\delta c_1||_{W_2^{2,1}(Q_t^s)} \leq C \left\{ ||R||_{L^2(Q_t^s)} + ||\delta c_{10}||_{H^1(\Omega_s)} \right\}$$

Now, it remains to estimate F. Let $R = R_1 + R_2 + R_3 + R_4$, where

$$\begin{aligned} R_1 &= g_1(c_1^{(1)}, c_2^{(1)}) - g_1(c_1^{(2)}, c_2^{(2)}) \\ R_2 &= \left(D_1(c_2^{(1)}) - D_1(c_2^{(2)})\right)\Delta c_1^{(2)} \\ R_3 &= \nabla c_1^{(2)} D_1'(c_2^{(1)}) \nabla \delta c_2 \\ R_4 &= \nabla c_1^{(2)} \nabla c_2^{(2)} \left(D_1'(c_2^{(1)}) - D_1'(c_2^{(2)}) \right) \end{aligned}$$

Now, using the regularity properties from section 6, we estimate

$$(7.22) \quad ||R_1||_{L^2(Q_t^s))} \leq C \left(||\delta c_1||_{L^2(Q_t^s))} + ||\delta c_2||_{L^2(Q_t^s))} \right)$$

$$(7.23) \quad ||R_2||_{L^2(Q_t^s))} \leq C ||\Delta c_1^{(2)}||_{L^l(Q_T^s))} ||\delta c_2||_{L^{10}(Q_t^s))} t^\alpha$$

$$(7.24) \quad ||R_3||_{L^2(Q_t^s))} \leq C ||\nabla c_1^{(2)}||_{L^l(Q_t^s))} ||\nabla \delta c_2||_{L^{10/3}(Q_t^s))} t^\alpha$$

$$(7.25) \quad ||R_4||_{L^2(Q_t^s))} \leq C ||\nabla c_1^{(2)}||_{L^l(Q_T^s))} ||\delta c_2||_{L^{10}(Q_t^s))} ||\nabla c_2^{(2)}||_{L^{10/3}(Q_t^s))} t^\alpha$$

for $\alpha > 0$ and l sufficiently big. Hence, we obtain

$$\begin{aligned} (7.26) \qquad ||\delta c_1||_{W_2^{2,1}(Q_t^s)} \quad \leq \quad & C \Big\{ ||\delta c_1||_{L^2(Q_t^s)} + ||\nabla \delta c_2||_{L^{10/3}(Q_t^s)} t^\alpha \\ & + ||\delta c_2||_{L^{10}(Q_t^s)} || t^\alpha + ||\delta c_{10}||_{H^1(\Omega_s)} \Big\} \end{aligned}$$

Since the straightforward energy estimate for c_1 gives

$$\begin{aligned} (7.27) \qquad & ||\delta c_1||_{L^\infty(0,t;L^2(\Omega_s))} + ||\nabla(\delta c_1)||_{L^2(0,t;L^2(\Omega_s))} \\ & \leq C \left\{ ||\delta c_2||_{L^{10}(0,t;L^{10}(\Omega_s))} t^\alpha + ||\delta c_{10}||_{L^2(\Omega_s)} \right\}, \end{aligned}$$

after inserting (7.27) into (7.26), we obtain

$$
\begin{aligned}
(7.28)\qquad ||\delta c_1||_{W_2^{2,1}(Q_t^s)} \leq C\,\Big\{&||\delta c_{10}||_{H^1(\Omega_s)} \\
&+ ||\nabla \delta c_2||_{L^{10/3}(Q_t^s)} t^{\alpha} + ||\delta c_2||_{L^{10}(Q_t^s)}||t^{\alpha}\Big\}
\end{aligned}
$$

Now the embedding theorem, see Lemma 3.3 in ([13], page 80), implies that

$$
(7.29)\qquad ||\nabla \delta c_2||_{L^{10/3}(Q_t^s)} + ||\delta c_2||_{L^{10}(Q_t^s)} \leq C||\delta c_2||_{W_2^{2,1}(Q_t^s)}
$$

and we get (7.20).

Finally, we have to derive an estimate for δc_2. We note that δc_2 is the solution to the following problem:

$$
\begin{aligned}
&\hat{a}\partial_t \delta c_2 + \nabla\cdot(v^{(1)}\delta c_2 + \delta v c_2^{(2)})\chi_{\Omega_f} = \nabla\cdot(\hat{a}D_2\nabla(\delta c_2)) + \delta f && \text{in}\quad \Omega\times(0,T)\\
&\Big[\chi_{\Omega_f}(v^{(1)}\delta c_2 + \delta v c_2^{(2)}) - \hat{a}D_2\nabla c_2\Big]\cdot\nu = 0 && \text{on}\ \ \Gamma\times(0,T)\\
&\delta c_2\chi_{\Omega_f} = \delta c_2\chi_{\Omega_s} && \text{on}\ \ \Gamma\times(0,T)\\
&\delta c_2 = \delta c_{2D} := c_{2n}^{(1)} - c_{2D}^{(2)} && \text{on}\ \ \Gamma_1\times(0,T)\\
&\nabla(\delta c_2)\cdot e_1 = 0 && \text{on}\ \ \Gamma_2\times(0,T)\\
&\Big(\chi_{\Omega_f}(v^{(1)}\delta c_2 + \delta v c_2^{(2)}) - \hat{a}D_2\nabla c_2\Big)\cdot\nu = 0 && \text{on}\ \ \Gamma_3\times(0,T)\\
&\delta c_2(0) = \delta c_{20} := c_{20}^{(1)} - c_{20}^{(2)} && \text{in}\ \ \Omega
\end{aligned}
$$

where $\hat{a}$ is defined as in (5.63) and

$$
\delta f = \begin{cases} \delta g_2 := g_2(c_2^{(1)}) - g_2(c_2^{(2)}) & \text{in } \Omega_f \\ \delta g_3 := g_3(c_1^{(1)}, c_2^{(1)}) - g_3(c_1^{(2)}, c_2^{(2)}) & \text{in } \Omega_s \end{cases}
$$

¿From the estimate (7.20), it is clear that we need higher order estimates for δc_2. The following theorem holds.

Theorem 11 *Let $c_{20}^{(j)} \in H^1(\Omega)$ and $c_{2D}^{(j)} \in H^2(\Gamma_1\times(0,T))$ for $j = 1, 2$. Then, we have the estimate*

$$
\begin{aligned}
(7.30)\qquad ||\delta c_2||_{W_2^{2,1}(Q_t^s)} + ||\delta c_2||_{W_2^{2,1}(Q_t^f)} \leq C\,\Big\{&||\delta v||_{L^\infty(0,t;H^1(\Omega_f))} \\
&+||\partial_t \delta v||_{L^2(0,t;H^1(\Omega_f))} + ||\delta c_{2D}||_{H^2(\Gamma_1\times(0,T))} + ||\delta c_{20}||_{H^1(\Omega)} \\
&+ ||\delta c_2||_{W_2^{2,1}(Q_t^s)} t^{\alpha} + ||\delta c_{10}||_{H^1(\Omega_s)}\Big\}
\end{aligned}
$$

Proof: The proof follows the lines of the regularity Theorem 6. We first multiply the above equation for δc_2 in Ω_s by $D_2\Delta(\delta c_2)$ and in Ω_f by $\nabla\cdot(D_2\nabla(\delta c_2) - v^{(1)}\delta c_2 - \delta v c_2^{(2)})$, and integrate with respect to x. We obtain

$$\frac{D_2}{K}\int_{\Omega_s}\partial_t\delta c_2\Delta(\delta c_2) - \frac{D_2^2}{K}\int_{\Omega_s}|\Delta(\delta c_2)|^2 = D_2\int_{\Omega_s}\delta g_3\Delta(\delta c_2) \tag{7.31}$$

$$\begin{aligned}&\int_{\Omega_f}\partial_t\delta c_2\nabla\cdot(D_2\nabla(\delta c_2) - v^{(1)}\delta c_2 - \delta v c_2^{(2)})\\ &-\int_{\Omega_f}|\nabla\cdot(D_2\nabla(\delta c_2) - v^{(1)}\delta c_2 - \delta v c_2^{(2)})|^2\\ &=\int_{\Omega_f}\delta g_2\nabla\cdot(D_2\nabla(\delta c_2) - v^{(1)}\delta c_2 - \delta v c_2^{(2)})\end{aligned} \tag{7.32}$$

In (7.31) and (7.32) we integrate by parts in the terms containing the time derivative and add the two equations to get

$$\begin{aligned}&\frac{D_2}{K}\int_{\Omega_s}\partial_t\nabla(\delta c_2)\nabla(\delta c_2) + D_2\int_{\Omega_f}\partial_t\nabla(\delta c_2)\nabla(\delta c_2)\\ &+\frac{D_2^2}{K}\int_{\Omega_s}|\Delta(\delta(c_2)|^2 + D_2^2\int_{\Omega_f}|\Delta(\delta(c_2)|^2 = \frac{D_2}{K}\int_{\partial\Omega_s}\partial_t\delta c_2\nabla(\delta c_2)\cdot n\\ &+\int_{\partial\Omega_f}\partial_t\delta c_2(D_2\nabla(\delta c_2) - v^{(1)}\delta c_2 - \delta v c_2^{(2)})\cdot n\\ &+\int_{\Omega_f}\partial_t\nabla(\delta c_2)(v^{(1)}\delta c_2 + \delta v c_2^{(2)})\\ &+2D_2\int_{\Omega_f}\Delta(\delta c_2)(v^{(1)}\nabla(\delta c_2) + \delta v\nabla c_2^{(2)})\\ &-D_2\int_{\Omega_s}\delta g_3\Delta(\delta c_2) - D_2\int_{\Omega_f}\delta g_2\Delta(\delta c_2)\\ &+\int_{\Omega_f}\delta g_2(v^{(1)}\nabla(\delta c_2) + \delta v\nabla c_2^{(2)}) - \int_{\Omega_f}|v^{(1)}\nabla(\delta c_2) + \delta v\nabla c_2^{(2)}|^2\end{aligned} \tag{7.33}$$

The term on the right hand side containing $\partial_t\nabla(\delta c_2)$ has to be transformed as follows

$$\begin{aligned}&\int_{\Omega_f}\partial_t\nabla(\delta c_2)(v^{(1)}\delta c_2 + \delta v c_2^{(2)}) = \frac{d}{dt}\int_{\Omega_f}\nabla(\delta c_2)(v^{(1)}\delta c_2 + \delta v c_2^{(2)})\\ &-\int_{\Omega_f}\nabla(\delta c_2)(\partial_t v^{(1)}\delta c_2 + \partial_t\delta v c_2^{(2)}) - \int_{\Omega_f}\nabla(\delta c_2)(v^{(1)}\partial_t\delta c_2 + \delta v\partial_t c_2^{(2)})\end{aligned} \tag{7.34}$$

Now, we use the equation for δc_2 to replace $\partial_t \delta c_2$ in (7.34) and then insert the result in (7.33). We obtain

$$
\begin{aligned}
(7.35)\ & \frac{D_2}{K}\int_{\Omega_s} \partial_t \nabla(\delta c_2)\nabla(\delta c_2) + D_2 \int_{\Omega_f} \partial_t \nabla(\delta c_2)\nabla(\delta c_2) \\
& + \frac{D_2^2}{K}\int_{\Omega_s} |\Delta(\delta(c_2)|^2 + D_2^2 \int_{\Omega_f} |\Delta(\delta(c_2)|^2 = \frac{D_2}{K}\int_{\partial\Omega_s} \partial_t \delta c_2 \nabla(\delta c_2)\cdot n \\
& + \int_{\partial\Omega_f} \partial_t \delta c_2 (D_2 \nabla(\delta c_2) - v^{(1)}\delta c_2 - \delta v c_2^{(2)})\cdot n \\
& + \frac{d}{dt}\int_{\Omega_f} \nabla(\delta c_2)(v^{(1)}\delta c_2 + \delta v c_2^{(2)}) - \int_{\Omega_f} \nabla(\delta c_2)(\partial_t v^{(1)}\delta c_2 + \partial_t \delta v c_2^{(2)}) \\
& + D_2 \int_{\Omega_f} \Delta(\delta c_2) v^{(1)}\nabla(\delta c_2) + 2D_2 \int_{\Omega_f} \Delta(\delta c_2)\delta v \nabla c_2^{(2)} \\
& - D_2 \int_{\Omega_s} \delta g_3 \Delta(\delta c_2) - D_2 \int_{\Omega_f} \delta g_2 \Delta(\delta c_2) \\
& + \int_{\Omega_f} \delta g_2 \delta v \nabla c_2^{(2)} - \int_{\Omega_f} v^{(1)}\nabla(\delta c_2)\delta v \nabla c_2^{(2)} - \int_{\Omega} |\delta v \nabla c_2^{(2)}|^2
\end{aligned}
$$

We note that in comparison to (6.18) on the right hand side of (7.35) the following new terms appear:

$$
\begin{aligned}
(7.36)\quad & \int_{\partial\Omega_f\cap\Gamma_1} \partial_t \delta c_{2D} \delta v c_2^{(2)}\cdot n + \int_{\partial\Omega_f\cap\Gamma_2} \partial_t \delta c_2 \delta v c_2^{(2)}\cdot n \\
& + \frac{d}{dt}\int_{\Omega_f} \nabla(\delta c_2)\delta v c_2^{(2)} - \int_{\Omega_f} \nabla(\delta c_2)\partial_t \delta v c_2^{(2)} \\
& - \int_{\Omega_f} \nabla(\delta c_2)\delta v \partial_t c_2^{(2)} + 2D_2 \int_{\Omega_f} \Delta(\delta c_2)\delta v \nabla c_2^{(2)} + \int_{\Omega_f} \delta g_2 \delta v \nabla c_2^{(2)} \\
& - \int_{\Omega_f} v^{(1)}\nabla(\delta c_2)\delta v \nabla c_2^{(2)} - \int_{\Omega} |\delta v \nabla c_2^{(2)}|^2
\end{aligned}
$$

We estimate three of the terms in (7.36) which are more difficult. All others can be estimated using similar techniques as before. Thus, we first transform the second term as follows

$$
\begin{aligned}
(7.37)\qquad & \int_{\partial\Omega_f\cap\Gamma_2} \partial_t \delta c_2 \delta v c_2^{(2)}\cdot n = \partial_t \int_{\partial\Omega_f\cap\Gamma_2} \delta c_2 \delta v c_2^{(2)}\cdot n \\
& - \int_{\partial\Omega_f\cap\Gamma_2} \delta c_2 \partial_t \delta v c_2^{(2)}\cdot n - \int_{\partial\Omega_f\cap\Gamma_2} \delta c_2 \delta v \partial_t c_2^{(2)}\cdot n
\end{aligned}
$$

Now, using the embedding theorem for traces, see [14], and the regularity of $c_2^{(2)}$, we can estimate

$$\left|\int_{\partial\Omega_f\cap\Gamma_2} \delta c_2(t)\delta v(t)c_2^{(2)}(t)dx\right| \le C||\delta c_2(t)||_{H^1(\Omega_f)}||\delta v(t)||_{H^1(\Omega_f)}||c_2^{(2)}(t)||_{L^\infty(\bar{\Omega}_f)}$$

$$\begin{aligned}
&\left|\int_0^t\int_{\partial\Omega_f\cap\Gamma_2} \delta c_2\partial_t\delta v c_2^{(2)}\right| \\
&\qquad\le C||\delta c_2||_{L^2(0,t;H^1(\Omega_f))}||\delta v||_{L^2(0,t;H^1(\Omega_f))}||c_2^{(2)}||_{L^\infty(\Omega_t^f)} \\
&\qquad\le C||\delta c_2||_{L^2(0,t;H^1(\Omega_f))}||\delta v||_{L^2(0,t;H^1(\Omega_t^f))}||c_2^{(2)}||_{W_2^{2,1}(\Omega_t^f)}t^\alpha
\end{aligned}$$

$$\begin{aligned}
&\left|\int_0^t\int_{\partial\Omega_f\cap\Gamma_2} \delta c_2\delta v\partial_t c_2^{(2)}\right| \\
&\qquad\le C||\delta c_2||_{L^2(0,t;H^1(\Omega_f))}||\delta v||_{L^2(0,t;H^1(\Omega_f))}||\partial_t c_2^{(2)}||_{L^\infty(\bar{\Omega}_t^f)} \\
&\qquad\le C||\delta c_2||_{L^2(0,t;H^1(\Omega_f))}||\delta v||_{L^2(0,t;H^1(\Omega_f))}
\end{aligned}$$

To estimate the fourth term we proceed as follows

$$\begin{aligned}
&\left|\int_0^t\int_{\Omega_f} \nabla(\delta c_2)\partial_t\delta v c_2^{(2)}\right| \\
&\le C||\nabla\delta c_2||_{L^4(0,t;L^2(\Omega_f))}||\partial_t\delta v||_{L^{5/4}(0,t;L^2(\Omega_f))}||c_2^{(2)}||_{L^\infty(\bar{\Omega}_t^f)} \\
&\le C(\eta)||\partial_t\delta v||^2_{L^{5/4}(0,t;L^2(\Omega_f))} + \eta||\nabla\delta c_2||^2_{L^4(0,t;L^2(\Omega_f))}
\end{aligned}$$

Finally, for the sixth term in (7.36) can be estimated as follows

$$\begin{aligned}
&\left|\int_0^t\int_{\Omega_f} \Delta(\delta c_2)\delta v\nabla c_2^{(2)}\right| \\
&\le \int_0^t ||\Delta(\delta c_2)||_{L^2(\Omega_f)}||\delta v||_{L^6(\Omega_f)}||\nabla c_2^{(2)}||_{L^3(\Omega_f)} \\
&\le C||\Delta(\delta c_2)||_{L^2(0,t;L^2(\Omega_f))}||\delta v||_{L^\infty(0,t;L^6(\Omega_f))}
\end{aligned}$$

Now, using the above estimates, the estimate which was already established in the proof of Theorem 6, as well as the estimate (7.20), we obtain estimate (7.39).

Now, we can state our stability result.

Theorem 12 *The initial-boundary-value problem (4.1)-(4.22) is stable with respect to the perturbations of the data, i.e.*

$$
\begin{aligned}
&||\partial_{tt}\delta u||_{L^\infty(0,t;L^2(\Omega))^3} + ||D(\partial_{tt}\delta u)||_{L^2(0,t;L^2(\Omega_f))^9} \\
&+||D(\delta u)||_{L^\infty(0,t;L^2(\Omega))^9} + ||D(\partial_t\delta u)||_{L^\infty(0,t;L^2(\Omega_s))^9} \\
&+||\delta c_1||_{W_2^{1,2}(\Omega_t^s))} + ||\delta c_2||_{W_2^{1,2}(\Omega_t^s))} + ||\delta c_2||_{W_2^{1,2}(\Omega_t^f))} \\
&\leq C\left\{||\delta c_{10}||_{H^1(\Omega_s)} + ||\delta c_{20}||_{H^1(\Omega)} + ||\delta c_{2D}||_{H^2(\Gamma_1\times(0,T))}\right. \\
&\left. + ||\delta(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)||_{H^2(0,T;L^2(\Gamma_2))^3}\right\}
\end{aligned}
\tag{7.38}
$$

Proof: First we note that for $0 < t \leq T_0 = T_0(\alpha)$, the estimate (7.20) from the statement of Theorem 11 gives

$$
\begin{aligned}
&||\delta c_2||_{W_2^{2,1}(Q_t^s)} + ||\delta c_2||_{W_2^{2,1}(Q_t^f)} \leq C\left\{||\partial_t\delta u||_{L^\infty(0,t;H^1(\Omega_f))}\right. \\
&+||\partial_{tt}\delta u||_{L^2(0,t;H^1(\Omega_f))} + ||\delta c_{2D}||_{H^2(\Gamma_1\times(0,T))} + ||\delta c_{20}||_{H^1(\Omega)} \\
&\left.+||\delta c_{10}||_{H^1(\Omega_s)}\right\}
\end{aligned}
\tag{7.39}
$$

Next, we start with the estimates (7.4)-(7.5) and plug inside the estimate (7.20) for $||\delta c_1||_{L^1(0,t;L^\infty(\Omega_s))}$. This gives us an estimate of the quantities at the left hand side of (7.4) and (7.5) in terms of $||\delta c_2||_{W_2^{2,1}(Q_t^s)} t^\alpha$. Finally, we replace $||\delta c_2||_{W_2^{2,1}(Q_t^s)}$ by the estimate (7.39) and obtain

$$
\begin{aligned}
&||\partial_{tt}\delta u||_{L^\infty(0,t;L^2(\Omega))^3} + ||D(\partial_{tt}\delta u)||_{L^2(0,t;L^2(\Omega_f))^9} \\
&+||D(\delta u)||_{L^\infty(0,t;L^2(\Omega))^9} + ||D(\partial_t\delta u)||_{L^\infty(0,t;L^2(\Omega_s))^9} \\
&\leq C\left\{||\delta c_{10}||_{H^1(\Omega_s)} + ||\delta c_{20}||_{H^1(\Omega)} + ||\delta c_{2D}||_{H^2(\Gamma_1\times(0,T))}\right. \\
&||\delta(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)||_{H^2(0,T;L^2(\Gamma_2))^3} \\
&\left.+ t^\alpha\left(||\partial_t\delta u||_{L^\infty(0,t;H^1(\Omega_f))^9} + ||\partial_{tt}\delta u||_{L^2(0,t;H^1(\Omega_f))^9}\right)\right\}
\end{aligned}
\tag{7.40}
$$

Now, for $0 < t \leq T_1 = T_1(\alpha) \leq T_0(\alpha)$, we conclude the estimate (7.38). Since T_0 and T_1 do not depend on the data, we can repeat the procedure and arrive at $t = T$ after a finite number of steps.

Corollary 13 *The initial-boundary-value problem (4.1)-(4.22) has a unique solution (u, c_1, c_2), with*

$$u \in W^{3,\infty}(0,T;L^2(\Omega)) \cap W^{2,\infty}(0,T;H^1(\Omega)) \cap H^3(0,T;H^1(\Omega_f)),$$

$$c_1 \in W_2^{2,1}(Q_T^s), \quad \textit{and} \quad c_2 \in W_2^{1,1}(Q) \cap W_2^{2,1}(Q_T^s \cup \Omega_T^f).$$

References

[1] Calabro, F., Zunino, P., *Analysis of Parabolic Problems on Partitioned Domains with Nonlinear Conditions at the Interface: Application to Mass Transfer through Semi-Permeable Membranes*, Mathematical Models and Methods in Applied Sciences, **16** (2006), pp. 479-501.

[2] Chambolle, A., Desjardins, B., Esteban, M. J., Grandmont, C., *Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate*, J. Math. Fluid Mech., **7** (2005), pp. 368–404.

[3] Cheng, C. H. A., Coutand, D., Shkoller, S., *Navier-Stokes Equations Interacting with a Nonlinear Elastic Biofluid Shell*, SIAM J. Math. Anal., **39** (2007), pp. 742–800.

[4] Coutand, D., Shkoller, S., *Motion of an elastic solid inside an incompressible viscous fluid*, Arch. Ration. Mech. Anal., **176** (2005), pp. 25–102.

[5] Coutand, D., Shkoller, S., *The interaction between quasilinear elastodynamics and the Navier-Stokes equations*, Arch. Ration. Mech. Anal., **179** (2006), pp. 303–352.

[6] Clopeau, T., Ferrin, J.L., Gilbert, R.P., Mikelić, A., *Homogenizing the acoustic properties of the seebed: Part II*, Mathematical and Computer Modelling, **33 (8-9)** (2001), pp. 821-841.

[7] Čanić, S., Lamponi, D., Mikelić, A., Tambača, J., *Self-Consistent Effective Equations Modeling Blood Flow in Medium-to-Large Compliant Arteries*, Multiscale Model. Simul., **3** (2005), pp. 559–596.

[8] Ferrin, J. L., Mikelić, A., *Homogenizing the acoustic properties of a porous matrix containing an incompressible inviscid fluid*, Math. Methods Appl. Sci. **26**, (2003), pp. 831-859 .

[9] Gilbert, R. P., Mikelić, A., *Homogenizing the acoustic properties of the seebed: Part I*, Nonlinear Analysis, **40** (2000), pp. 185-212.

[10] Hornung, U., Jäger, W., Mikelić, A., *Reactive Transport through an Array of Cells with Semi-Permeable Membranes*, Mathematical Modelling and Numerical Analysis (M2AN), **28** (1994), pp. 59-94.

[11] Jäger, W., Neuss-Radu, M., *Multiscale analysis of processes in complex media*, In Reactive flow, diffusion and transport. SFB-Berichtband. Editors W. Jäger, R. Rannacher, J. Warnatz. Springer (2006), pp.

[12] Jäger, W., Mikelić, A., Neuss-Radu, M., *Derivation of an effective model for reactive flow through a deformable system of cells*, Manuscript in preparation.

[13] Ladyzenskaja, O. A., Ural'ceva, N. N., *Linear and quasi-linear elliptic equations*, Academic Press, New York and London, 1968.

[14] Ladyzenskaja, O. A., Solonnikov, V. A., Ural'ceva, N. N., *Linear and quasi-linear equations of parabolic type*, American Mathematical Society, Providence, Rhode Island, (1968).

[15] Neuss-Radu, M. Jäger, W., *Homogenization of thin porous layers and applications to ion transport through channels of biological membranes*, Oberwolfach Reports, **49** (2005), pp. 2809–2812.

[16] Neuss-Radu, M. Jäger, W., *Effective transmission conditions for reaction-diffusion processes in domains separated by an interface*, SIAM J. Math. Analysis, Accepted (2007).

[17] Oleinik, O. A., Shamaev A. S., Yosifian, G. A., *Mathematical problems in elasticity and homogenization*, North-Holland, Amsterdam, (1992).

[18] Quarteroni, A., Veneziani, A., Zunino, P., *Mathematical and Numerical Modelling of Solute Dynamics in Blood Flow and Arterial Wall*, SIAM J. Numer. Analysis, **39** (2001), pp. 1488-1511.

[19] Seebach, J., Mädler, H.-J., Wojciak-Stothard, B., Schnittler H.-J., *Tyrosine phosphorylation and the small GTPase rac cross-talk in regulation of endothelial barrier function*, Thromb Haemost: Cell Signalling and Vessel Remodelling, **94** (2005), pp. 620-629.

[20] Tolstoy, I., ed. *Acoustics, elasticity, and thermodynamics of porous media, Twenty-one papers by M. A. Biot*, Acoustical Society of America, New York, (1992).

[21] Weller, F., *Modelling, Analysis and Simulation of Thrombosis and Haemostasis*, PhD thesis University of Heidelberg, (2007).