Asymptotic equations for the terminal phase of glass fiber drawing and their analysis

Thierry Clopeau[1], Angiolo Farina[2], Antonio Fasano[2], Andro Mikelić[1] *

[1] *Université de Lyon, Lyon, F-69003, France;*
Université Lyon 1, Institut Camille Jordan,
UFR Mathématiques, Site de Gerland, Bât. A,
50, avenue Tony Garnier, 69367 Lyon Cedex 07, FRANCE

[2] *Dipartimento di Matematica "Ulisse Dini", Viale Morgagni 67/A, I-50134 Firenze, ITALIA*

Abstract

In this article, we study mathematical modeling of thermal drawing of glass fibers. We give a derivation of the effective model from the generalized Oberbeck-Boussinesq equations with free boundary, using singular perturbation expansion. We generalize earlier approaches by taking the isochoric compressible model, with density depending on the temperature, and we handle correctly the viscosity, which changes over several orders of magnitude. For obtained effective system of nonlinear differential equations, we prove existence of a stationary solution for the boundary value problem. We impose only physically realistic assumptions on the data (viscosity taking large values with cooling). Finally we present numerical simulations with realistic data.

Key words: glass fiber drawing, singular perturbation, non-isothermal elongational free boundary flow
PACS: 0.2.30.Jr ; 47.10.A- ; 47.10.ad ; 47.55.nb ; 47.85.M

* Corresponding author.
Email address: clopeau@univ-lyon1.fr ; farina@math.unifi.it; fasano@math.unifi.it; mikelic@univ-lyon1.fr (Thierry Clopeau[1], Angiolo Farina[2], Antonio Fasano[2], Andro Mikelić[1]).

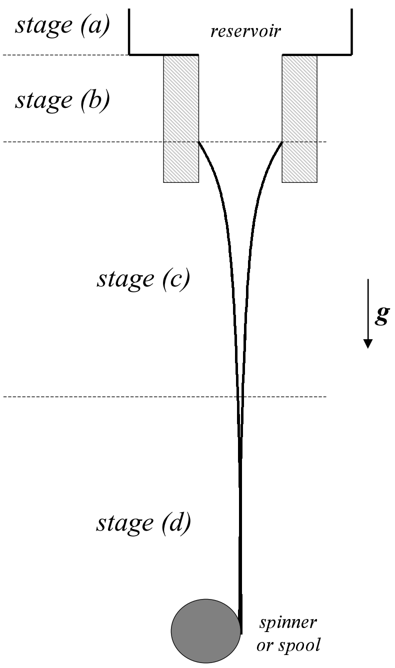


Fig. 1. *The fiber drawdown.*

1 Introduction

In industrial production of glass fibres, the molten glass flows into a bushing with many nozzles. The hot glass melt, with the temperature of approximatively 1500^0 K, is then drawn down into a fiber by a drawing force. The result is production in parallel of many fibers, which are cooled and collected on the rotating drum.

In this article, we are considering the drawdown of a single glass fiber. It is drawn from a dye and wound up on the collecting drum. A mathematical model of the manufacturing procedure contains necessarily both free boundary hydrodynamics and thermal processes. Their coupling comes from the temperature dependent viscosity, density and surface tension.

In the process of a glass fiber manufacturing, we may distinguish four stages, as shown at Figure 1:

(a) The flow of the very hot glass in the reservoir, feeding the fiber production system.
(b) The non-isothermal flow through the dye, with rigid lateral boundaries.
(c) The viscous jet flow, wetting the outer surface of the dye, rapidly changing physical parameters owing to the fast cooling.
(d) The motion of the glass fiber, pulled from below at high velocity by a device called spinner or spool.

Our study does not concern stages (a), (b) and (c), but is rather focused on the stage (d), that is on the free boundary flow of the fiber in the air, away from the dye exit. Our goal is to derive a mathematical model, using a singular perturbation argument and making some simplifying assumptions, in order to discuss the influence of some physical parameters involved.

Next, we study the obtained equations in the stationary regime and prove their solvability and some properties. Finally, we reconstruct the velocity and stress fields at the starting interface.

2 Derivation of the effective equations governing the terminal phase of a single molten glass fibre drawing

It is considered that the stage (d) starts when viscosity is sufficiently large (larger than 10^5 Pa sec) and the fiber radius is already rather small (smaller than hundred micrometers). Contrary to the stage (c), where one should treat the 3D Navier-Stokes system, with the free boundary, coupled with the nonlinear conduction of heat, here we have a long filament (several meters) of molten glass. In the stage (c) one needs an industrial code to solve the corresponding partial differential equations. Here an alternative is possible.

Fundamental equations, describing the stage (c), are the temperature dependent incompressible Navier-Stokes equations with free boundary coupled with the energy equations. Their derivation from the first principles is in the article [2], where also the Oberbeck-Boussinesq approximation we use is justified rigorously, by passing to the singular limit when expansivity parameter goes to zero.

In the engineering literature on the fiber drawing, this system is frequently approximated by a quasi 1D approximation for viscous flows, in which the radius of the free boundary $r = R(z, \alpha, t)$, axial speed w and the temperature ϑ are independent of the radial variable and depend only on the axial coordinate z and of the time t. That is possible in the situations where the fibre of liquid glass is long and thin, so that the principal flow is directed along the axis z and the velocity is basically one-dimensional. In particular, this property to be of "low thickness" leads to the "approximation of lubrication" of Reynolds.

The strategy of the "lubrication approximation" is to expand the velocity field with respect ε, being the ratio of the characteristic thickness R_E in the radial direction with the characteristic axial length of fibre L. The terms of order zero should be sufficient to describe the movement. Effective equations are then derived starting from the coefficients of expansion with respect to ε, which depend only on axial variable z (and of t). This idea is traditionally applied to flows through thin domains. Treating the flows with a free boundary is much more complicated. Initially, the radius describes the position of the free boundary and the smallness of the expansion parameter ε will depend on the solution itself. In the second place, it is not obvious which forces in the equations must be taken into account (for example, torsion can be important or negligible).

The historical references in the subject are papers [6], [7] and [9], where the "equations of Matovich-Pearson" are formally obtained. They describe the axially symmetrical movement of the viscous filaments, and read

$$\frac{\partial \mathbf{A}}{\partial t} + \frac{\partial (w\mathbf{A})}{\partial z} = 0; \qquad \frac{\partial}{\partial z}(3\mu(T)\mathbf{A}\frac{\partial w}{\partial z}) + \frac{\partial(\sigma(T)\sqrt{\mathbf{A}})}{\partial z} = 0, \tag{1}$$

where$\mathbf{A} = \mathbf{A}(z,t)$ is the area of fibre section, w = w(z, t) is effective axial velocity, 3μ is Trouton's viscosity and σ is the surface tension. μ and σ depend on the temperature, and it is necessary to add the equation for the temperature $T = T(z,t)$. For the derivation of the equations (1), one can consult the book [13]. They were obtained under the assumptions (H)

- i) The viscous forces dominate inertia
- ii) Effects of the surface tension are in balance with the normal stress at the free boundary.
- iii) The heat conduction is small compared with the heat convection in the fiber.
- iv) All the phenomena are axially symmetrical and the fiber is nearly straight.

We present in this section the derivation of the model of Matovich-Pearson for the thermal case. Calculation is based on corresponding derivation for the isothermal case, by an asymptotic development. For the details, one can see [15]. The thermal case was previously considered in the articles [4] and [5]. Nevertheless, in the two references mentioned, the viscosity which changes enormously with the change of the temperature, was regarded as a quantity of order one. We will present a derivation consistent with large viscosity variations.

We start with the generalized system of Oberbeck-Boussinesq, which we write in the dimensionless form:

2.1 Generalized Oberbeck-Boussinesq equations in non-dimensional form

We suppose the axial symmetry and the equations will be written in the cylindrical coordinates (r,z). Our unknowns are the following: velocity is $\mathbf{v} = v_z\mathbf{e}_z + v_r\mathbf{e}_r$; hydrodynamic pressure is p; temperature is T; fiber radius (being the distance from the symmetry axis) is $R = R(t, z)$; specific heat is $c_p = c_p(T)$; density is $\rho = \rho(T)$; surface tension is $\sigma = \sigma(T)$; viscosity is $\mu = \mu(T)$; thermal conductivity is $\lambda = \lambda(T)$.

The characteristic parameter values are the following:

constant axial velocity of the spooler V_f; extrusion temperature T_E
extrusion density $\rho_E = \rho(T_E)$; ambient temperature T_∞
glass transition temperature T_g; global temperature drop $\Delta T = T_E - T_\infty$;
axial extrusion velocity v_E; extrusion heat transfer coefficient h_E;
characteristic fiber length L; characteristic radius $R_E = R(\cdot, 0) = const.$

All quantities evaluated at the extrusion temperature T_E are denoted with a suffix E. For the heat transfer coefficient, we adopt the formula of Kase-Matsuo (see [4]):

$$h = \frac{\lambda_\infty}{R(z,t)} C \Big(\frac{2\rho_\infty v_z(z,t) R(z,t)}{\mu_\infty} \Big)^m, \tag{2}$$

where the subscript ∞ denotes the parameters of the surrounding air and $C > 0$ and $0 < m < 0.4$, are constants determined from the experimental data.

Concerning characteristic length, following [4], one takes $L = \frac{\rho_E R_E c_{pE} v_E}{2h_E}(1 - T_g/T_E)$. Other possibility is to take the distance between the spooler and extrusion dye.

We will restrict our considerations to the stationary case, even if the generalization to the non-stationary case is straightforward. Our first difficulty is that viscosity changes over several orders of magnitude. This motivates us to write the Oberbeck-Boussinesq equations from [2] in $\Omega = \{z \in (0, L); \quad 0 \le r < R(z)\}$, in the following form:

$$\nabla\Big(\frac{p}{\mu(T)} + \frac{2}{3} \text{ div } \mathbf{v}\Big) - \nabla^2 \mathbf{v} + \Big(\Big(\frac{p}{\mu(T)} + \frac{2}{3} \text{ div } \mathbf{v}\Big) I - 2D(\mathbf{v})\Big) \nabla \log \mu(T) -$$
$$\nabla \text{ div } \mathbf{v} = \frac{\rho(T)}{\mu(T)} g \mathbf{e}_z - \frac{\rho(T)}{\mu(T)} (\mathbf{v}\nabla)\mathbf{v} \quad \text{in} \quad \Omega \tag{3}$$
$$\text{div } (\rho(T)\mathbf{v}) = 0 \quad \text{in} \quad \Omega \tag{4}$$
$$\rho(T)\mathbf{v} c_p(T) \nabla T = \text{ div } (\lambda(T) \nabla T) \quad \text{in} \quad \Omega. \tag{5}$$

The stress tensor is now

$$\Sigma = -\mu(T) \Bigg(\Big(\frac{p}{\mu(T)} + \frac{2}{3} \text{ div } \mathbf{v}\Big) I - 2D(\mathbf{v}) \Bigg). \tag{6}$$

The natural small parameter is $\varepsilon = \frac{R_E}{L}$. We introduce the following dimensionless quantities:

$$r = R_E\,\tilde{r},\, t = \frac{L}{v_E}\tilde{t},\; v_z = v_E\,\tilde{v}_z,\; v_r = \varepsilon v_E\,\tilde{v}_r,\; \frac{T - T_\infty}{\Delta T} = \tilde{T},$$
$$\mu = \mu_E\tilde{\mu}, \sigma = \sigma_E\tilde{\sigma},\; h = h_E\tilde{h},\; \frac{p}{\mu(T)} + \frac{2}{3}\,\mathrm{div}\,\mathbf{v} = \frac{v_E\tilde{p}}{L},\; \lambda = \lambda_E\tilde{\lambda}.$$

Equations in non-dimensional form will contain the following dimensionless numbers:

Re= $\dfrac{\rho_E v_E L}{\mu_E}$ is Reynolds number; **Ca**= $\dfrac{\mu_E v_E}{\sigma_E}$ is capillary number;
Pe= $\dfrac{c_{pE}\rho v_E L}{\lambda_E}$ is Peclet's number; **Bi**= $\dfrac{h_E R_E}{\lambda_E}$ is Biot's number;
Fr= $\dfrac{v_E}{\sqrt{gL}}$ is Froude's number; **Bo**= $\dfrac{\mathbf{Re}}{\mathbf{Fr}^2}$ is Bond's number

and draw ratio is $E = \dfrac{V_f}{v_E}$. $\alpha = \log E$ is usually known as Hencky strain. In the text, which follows, we will omit the wiggles. Then the axially symmetric generalized Oberbeck-Boussinesq system takes the following form:

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = -v_z\partial_z \log\rho(T) - v_r\partial_r \log\rho(T), \quad \text{in} \quad \Omega; \tag{7}$$

$$\varepsilon^2\mathbf{Re}\frac{\rho(T)}{\mu(T)}(v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}) = -\frac{\partial p}{\partial r} + \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \varepsilon^2\frac{\partial^2 v_r}{\partial z^2}\right) - \frac{\partial \log\mu}{\partial r}(p - 2\frac{\partial v_r}{\partial r}) + \frac{\partial \log\mu}{\partial z}(\varepsilon^2\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}) - \frac{\partial}{\partial r}(v_z\partial_z \log\rho(T) + v_r\partial_r \log\rho(T)), \quad \text{in} \quad \Omega; \tag{8}$$

$$\varepsilon^2\mathbf{Re}\frac{\rho(T)}{\mu(T)}(v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}) = -\varepsilon^2\frac{\partial p}{\partial z} + \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \varepsilon^2\frac{\partial^2 v_z}{\partial z^2}\right) - \varepsilon^2\frac{\partial \log\mu}{\partial z}(p - 2\frac{\partial v_z}{\partial z}) + \frac{\partial \log\mu}{\partial r}(\varepsilon^2\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}) + \varepsilon^2\frac{\mathbf{Re}}{\mathbf{Fr}^2}\frac{\rho(T)}{\mu(T)} - \frac{\partial}{\partial z}(v_z\partial_z \log\rho(T) + v_r\partial_r \log\rho(T)), \quad \text{in} \quad \Omega; \tag{9}$$

$$\varepsilon^2\mathbf{Pe}\rho(T)c_p(T)(v_r\frac{\partial T}{\partial r} + v_z\frac{\partial T}{\partial z}) = \frac{1}{r}\frac{\partial}{\partial r}(r\lambda(T)\frac{\partial T}{\partial r}) + \varepsilon^2\frac{\partial}{\partial z}(\lambda(T)\frac{\partial T}{\partial z}), \text{ in } \Omega. \tag{10}$$

Next, we have

$$v_z\frac{\partial R(z)}{\partial z} = v_r \quad \text{on} \quad r = R(z) \quad \text{(the kinematic condition)} \tag{11}$$

and the dynamic conditions at the free boundary read:

$$\mu(T)\mathbf{Ca}\biggl(2\varepsilon^2\frac{\partial R(z)}{\partial z}(\frac{\partial v_r}{\partial r}-\frac{\partial v_z}{\partial z})+(1-\varepsilon^2(\frac{\partial R}{\partial z})^2)(\varepsilon^2\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r})\biggr)=$$
$$\varepsilon(\frac{\partial R(z)}{\partial z}\frac{\partial \sigma(T)}{\partial r}+\frac{\partial \sigma(T)}{\partial z})\sqrt{1+\varepsilon^2(\frac{\partial R(z)}{\partial z})^2} \quad \text{on} \;\; r=R(z); \tag{12}$$
$$\varepsilon\mathbf{Ca}\mu(T)\biggl((1+\varepsilon^2(\frac{\partial R(z)}{\partial z})^2)p-2\Bigl(\frac{\partial v_r}{\partial r}+\varepsilon^2(\frac{\partial R(z)}{\partial z})^2\frac{\partial v_z}{\partial z}-\varepsilon^2\frac{\partial R(z)}{\partial z}\frac{\partial v_r}{\partial z}-$$
$$\frac{\partial R(z)}{\partial z}\frac{\partial v_z}{\partial r})\biggr)=\frac{\sigma(T)}{R(z)}\sqrt{1+\varepsilon^2(\frac{\partial R(z)}{\partial z})^2}-\varepsilon^2\sigma(T)\frac{\partial^2 R(z)}{\partial z^2}/\sqrt{1+\varepsilon^2(\frac{\partial R(z)}{\partial z})^2}$$
$$\text{on} \;\; r=R(z). \tag{13}$$

Finally, the heat transfer to the environment is given by

$$\frac{\partial T}{\partial r}-\varepsilon^2\frac{\partial R}{\partial z}\frac{\partial T}{\partial z}=-\mathbf{Bi}\frac{h}{\lambda(T)}T\sqrt{1+\varepsilon^2(\frac{\partial R(z)}{\partial z})^2} \quad \text{on} \;\; r=R(z); \tag{14}$$

2.2 Lubrication approximation

We study long and very viscous fibers. Therefore, the parameter ε is small and earlier mentioned hypothesis (H) from the book [13] holds true. As we will see in the section with simulations, the experimental data from [4] and [12] imply that

$$\mathbf{Re}\sim\varepsilon, \quad \mathbf{Bo}\sim 1, \quad \mathbf{Ca}\sim\frac{1}{\varepsilon}, \quad \mathbf{Pe}\sim\frac{1}{\varepsilon}, \quad \mathbf{Bi}\sim\varepsilon \quad \text{as} \quad \varepsilon\to 0. \tag{15}$$

We note that the same assumption is used in [5].

In order to perform the asymptotic analysis of our equations, we expand all unknown functions with respect to ε, i.e. for an arbitrary function $f=f(r,z)$ we set $f=f^{(0)}+\varepsilon^2\mathbf{Pe}\ f^{(1)}+\varepsilon^2 f^{(2)}+\dots$. We note that contrary to the viscosity μ, the non-dimensional $\log\mu$ is of order 1.

Let $\Omega_0=\{\ 0\le r<R^{(0)}(z),\ 0<z<1\ \}$. After inserting the expansions for unknowns into the system (7)-(14), we find out that the zero order terms in the system (7)-(10) are:

$$\frac{\partial v_r^{(0)}}{\partial r}+\frac{v_r^{(0)}}{r}+\frac{\partial v_z^{(0)}}{\partial z}=-v_z^{(0)}\partial_z\log\rho(T^{(0)})-v_r^{(0)}\partial_r\log\rho(T^{(0)})\ \text{ in }\ \Omega_0; \tag{16}$$

$$\frac{\partial}{\partial r}(v_z^{(0)}\partial_z\log\rho(T^{(0)})+v_r^{(0)}\partial_r\log\rho(T^{(0)}))=-\frac{\partial p^{(0)}}{\partial r}+\frac{\partial^2 v_r^{(0)}}{\partial r^2}+\frac{1}{r}\frac{\partial v_r^{(0)}}{\partial r}-\frac{v_r^{(0)}}{r^2}$$
$$+\frac{\partial\log\mu(T^{(0)})}{\partial r}(2\frac{\partial v_r^{(0)}}{\partial r}-p)+\frac{\partial\log\mu(T^{(0)})}{\partial z}\frac{\partial v_z^{(0)}}{\partial r}\ \text{ in }\ \Omega_0; \tag{17}$$

$$0=\frac{\partial^2 v_z^{(0)}}{\partial r^2}+\frac{1}{r}\frac{\partial v_z^{(0)}}{\partial r}+\frac{\partial\log\mu(T^{(0)})}{\partial r}\frac{\partial v_z^{(0)}}{\partial r}\ \text{ in }\ \Omega_0; \tag{18}$$

$$0=\frac{1}{r}\frac{\partial T^{(0)}}{\partial r}+\frac{\partial^2 T^{(0)}}{\partial r^2}\ \text{ in }\ \Omega_0, \tag{19}$$

with the boundary conditions

$$v_r^{(0)}=v_z^{(0)}\frac{\partial R^{(0)}}{\partial z}\quad\text{on}\ \ r=R^{(0)}(z); \tag{20}$$

$$\frac{\partial v_z^{(0)}}{\partial r}=0\quad\text{on}\ \ r=R^{(0)}(z); \tag{21}$$

$$p^{(0)}=\frac{1}{\varepsilon\mathbf{Ca}}\frac{\sigma(T^{(0)})}{\mu(T^{(0)})R^{(0)}}+2(\frac{\partial v_r^{(0)}}{\partial r}-\partial_z R^{(0)}\frac{\partial v_z^{(0)}}{\partial r})\quad\text{on}\ \ r=R^{(0)}(z); \tag{22}$$

$$\frac{\partial T^{(0)}}{\partial r}=0\quad\text{on}\ \ r=R^{(0)}(z); \tag{23}$$

Using (19), we obtain $T^{(0)}=T^{(0)}(z)$. Next, equation (18) yields $v_z^{(0)}=v_z^{(0)}(z)$.

Radial component of the velocity is calculated using the equation (16). We integrate it and get

$$\frac{\partial v_z^{(0)}}{\partial r}=0=\frac{\partial T^{(0)}}{\partial r},\ v_r^{(0)}(r,z)=-\frac{r}{2}(\frac{\partial v_z^{(0)}}{\partial z}+\partial_z\log\rho(T^{(0)})v_z^{(0)}) \tag{24}$$

Consequently, equation (17) reads $\frac{\partial p^{(0)}}{\partial r}=0$ and, after using the boundary value condition (22), we obtain

$$p^{(0)}=\frac{1}{\varepsilon\mathbf{Ca}}\frac{\sigma(T^{(0)})}{\mu(T^{(0)})R^{(0)}}+2\frac{\partial v_r^{(0)}}{\partial r}=\frac{1}{\varepsilon\mathbf{Ca}}\frac{\sigma(T^{(0)})}{\mu(T^{(0)})R^{(0)}}-\frac{\partial v_z^{(0)}}{\partial z}-\partial_z\log\rho(T^{(0)})v_z^{(0)}. \tag{25}$$

The kinematic condition (20) now transforms to

$$0=\frac{\partial}{\partial z}\Big(\rho(T^{(0)})v_z^{(0)}(R^{(0)})^2\Big). \tag{26}$$

At the order $\mathcal{O}(\varepsilon^2\ \mathbf{Pe}\ \ )$, we have the following boundary value problem for the temperature correction $T^{(1)}$:

$$\rho(T^{(0)})c_p(T^{(0)})v_z^{(0)}\frac{\partial T^{(0)}}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\lambda(T^{(0)})\frac{\partial T^{(1)}}{\partial r}) \text{ in } \Omega_0, \tag{27}$$

$$\varepsilon^2\mathbf{Pe}\lambda(T^{(0)})\frac{\partial T^{(1)}}{\partial r} + \mathbf{Bi}\frac{(v_z^{(0)}R^{(0)})^m T^{(0)}}{\varepsilon^2 R^{(0)}} = 0, \text{ on } r = R^{(0)}(z), \tag{28}$$

where m is the exponent from Kase-Matsuo's formula (2). Using Fredholm's alternative, we get the following effective energy equation:

$$\rho(T^{(0)})(R^{(0)})^2 v_z^{(0)} c_p(T^{(0)})\frac{\partial T^{(0)}}{\partial z} + \frac{2\mathbf{Bi}}{\varepsilon^2\mathbf{Pe}}(v_z^{(0)}R^{(0)})^m T^{(0)} = 0 \text{ on } (0,1). \tag{29}$$

Next, at the order $\mathcal{O}(\varepsilon^2)$, equation (9) and condition (12) give

$$\frac{\mathbf{Re}}{2}\frac{\rho^2(T^{(0)})}{\mu(T^{(0)})}\partial_z\frac{(v_z^{(0)})^2}{\rho(T^{(0)})} + \frac{\partial p^{(0)}}{\partial z} + \frac{\partial \log\mu(T^{(0)})}{\partial z}(p^{(0)} - 2\frac{\partial v_z^{(0)}}{\partial z}) = \frac{\partial^2 v_z^{(0)}}{\partial z^2} + \frac{\partial^2 v_z^{(2)}}{\partial r^2} + \frac{1}{r}\frac{\partial v_z^{(2)}}{\partial r} + \mathbf{Bo}\,\frac{\rho(T^{(0)})}{\mu(T^{(0)})} - \frac{\partial}{\partial z}(v_z^{(0)}\partial_z \log\rho(T^{(0)})) \quad \text{in } \Omega_0. \tag{30}$$

$$2\frac{\partial R^{(0)}(z)}{\partial z}(\frac{\partial v_r^{(0)}}{\partial r} - \frac{\partial v_z^{(0)}}{\partial z}) + \frac{\partial v_r^{(0)}}{\partial z} + \frac{\partial v_z^{(2)}}{\partial r} = \frac{1}{\varepsilon\mathbf{Ca}\mu(T^{(0)})}\frac{\partial\sigma(T^{(0)})}{\partial z} \text{ on } r = R^{(0)}(z). \tag{31}$$

Now (24) yields

$$\partial_z(p^0 - \partial_z v_z^0) = \partial_z(\frac{1}{\varepsilon\mathbf{Ca}}\frac{\sigma(T^{(0)})}{\mu(T^{(0)})R^{(0)}} - 2\frac{\partial v_z^{(0)}}{\partial z} - \partial_z\log\rho(T^{(0)})v_z^{(0)}) \tag{32}$$

and integration of the equation (30) yields

$$\partial_r v_z^{(2)}|_{r=R^{(0)}} = \frac{R^{(0)}}{2}\Big(\frac{\mathbf{Re}}{2}\frac{\rho^2(T^{(0)})}{\mu(T^{(0)})}\partial_z\frac{(v_z^{(0)})^2}{\rho(T^{(0)})} - \mathbf{Bo}\,\frac{\rho(T^{(0)})}{\mu(T^{(0)})} - \frac{\partial\log\mu(T^{(0)})}{\partial z}(3\frac{\partial v_z^{(0)}}{\partial z} - \frac{1}{\varepsilon\mathbf{Ca}}\frac{\sigma(T^{(0)})}{\mu(T^{(0)})R^{(0)}} + \partial_z\log\rho(T^{(0)})v_z^{(0)}) + \partial_z(\frac{1}{\varepsilon\mathbf{Ca}}\frac{\sigma(T^{(0)})}{\mu(T^{(0)})R^{(0)}} - 2\frac{\partial v_z^{(0)}}{\partial z})\Big). \tag{33}$$

On the other hand, after inserting (24) into the boundary condition (31) we get

$$\partial_r v_z^{(2)}|_{r=R^{(0)}} = \frac{1}{\varepsilon\mathbf{Ca}\mu(T^{(0)})}\frac{\partial\sigma(T^{(0)})}{\partial z} + \frac{R^{(0)}}{2}\Big(\partial_{zz}v_z^0 + \partial_z(v_z^0\partial_z\log\rho(T^{(0)}))\Big) + \partial_z R^{(0)}(3\frac{\partial v_z^{(0)}}{\partial z} + v_z^0\partial_z\log\rho(T^{(0)})). \tag{34}$$

After comparing equations (33)-(34), we obtain the effective momentum equation:

$$\frac{\partial}{\partial z}\left(3\mu(T^{(0)})(R^{(0)})^2\frac{\partial v_z^{(0)}}{\partial z}+\mu(T^{(0)})(R^{(0)})^2 v_z^{(0)}\partial_z\log\rho(T^{(0)})+\right.$$
$$\left.\frac{1}{\varepsilon\mathbf{Ca}}\sigma(T^{(0)})R^{(0)}\right)=\frac{\mathbf{Re}}{2}(\rho(T^{(0)})R^{(0)})^2\partial_z\frac{(v_z^{(0)})^2}{\rho(T^{(0)})}-\mathbf{Bo}\,\rho(T^{(0)})(R^{(0)})^2. \quad (35)$$

As conclusion, we summarize our results in dimensional form:

Proposition 1 *Let $v_{eff}=v_E v_z^{(0)}$ be the effective axial velocity, $R_{eff}=R_E R^{(0)}$ the effective fiber radius and $T_{eff}=T_E T^{(0)}$ the effective temperature. Let us suppose that the quantities Q_0 (the mass flow), R_f (the final fiber radius), V_f (the pulling velocity), F_L (the traction force) and T_E (the extrusion temperature) are given positive constants. Then all other relevant physical quantities are determined by $\{v_{eff},R_{eff},T_{eff}\}$ and given by:*

$$\textit{effective radial velocity: } v_r^{eff}(r,z)=-\frac{r}{2}\Big(\partial_z v_{eff}(z)+v_{eff}\partial_z\log\rho(T_{eff}(z))\Big); \quad (36)$$

$$\textit{effective pression: } p_{eff}(z)=\frac{\sigma(T_{eff}(z))}{R_{eff}(z)}-\mu(T_{eff}(z))\partial_z v_{eff}(z)-$$
$$\frac{\mu(T_{eff}(z))}{3}v_{eff}(z)\partial_z\log\rho(T_{eff}(z)); \quad (37)$$

$$\textit{effective axial stress: } \Sigma_{eff}(r,z)\mathbf{e}_z=\mu(T_{eff}(z))\Big(3\partial_z v_{eff}(z)+$$
$$v_{eff}(z)\partial_z\log\rho(T_{eff}(z))\Big)\mathbf{e}_z-\frac{\sigma(T_{eff}(z))}{R_{eff}(z)}\mathbf{e}_z+\mu(T_{eff}(z))\Big(3\partial_z v_{eff}(z)$$
$$+v_{eff}(z)\partial_z\log\rho(T_{eff}(z))\Big)r\frac{\partial_z R_{eff}(z)}{R_{eff}(z)}\mathbf{e}_r+r\frac{\partial_z\sigma(T_{eff}(z))}{R_{eff}(z)}\mathbf{e}_r. \quad (38)$$

$$\textit{effective traction : } \mathcal{F}_{eff}=\frac{\pi R_{eff}^2}{g}\left(\mu(T_{eff}(z))\Big(3\partial_z v_{eff}(z)\right.$$
$$\left.+v_{eff}(z)\partial_z\log\rho(T_{eff}(z))\Big)-\frac{\sigma(T_{eff}(z))}{R_{eff}(z)}\right). \quad (39)$$

Functions $\{v_{eff},R_{eff},T_{eff}\}$ are given by the Cauchy problem

$$v_{eff}(z)R_{eff}^2(z)\rho(T_{eff}(z))=\frac{Q_0}{\pi}=V_f R_f^2\rho(T_g),\ 0<z<L; \quad (40)$$

$$\frac{\partial}{\partial z}\left(3\mu(T_{eff}(z))(R_{eff}(z))^2\frac{\partial v_{eff}}{\partial z}+\sigma(T_{eff}(z))R_{eff}(z)+\right.$$
$$\left.\mu(T_{eff}(z))(R_{eff}(z))^2 v_{eff}\partial_z\log\rho(T_{eff}(z))\right)=\left(\frac{\rho(T_{eff}(z))}{2}\partial_z\frac{v_{eff}^2}{\rho(T_{eff}(z))}\right.$$
$$\left.-g\right)\rho(T_{eff}(z))(R_{eff}(z))^2,\ 0<z<L; \quad (41)$$

$$\frac{Q_0}{\pi} c_p(T_{eff}) \frac{\partial T_{eff}}{\partial z} + C\lambda_\infty \Big(\frac{2\rho_\infty v_{eff} R_{eff}}{\mu_\infty} \Big)^m (T_{eff} - T_\infty) = 0, \ 0 < z < L; \quad (42)$$

$$R_{eff}(L) = R_f, \ v_{eff}(L) = V_f, \ \mathcal{F}_{eff}(L) = F_L, \ T_{eff}(L) = T_E. \quad (43)$$

Finally, we have

$$v_z(r,z) = v_{eff}(z) + \mathcal{O}(\varepsilon^2 \mathbf{Pe}\ ); \qquad v_r(r,z) = v_r^{eff}(r,z) + \mathcal{O}(\varepsilon^2 \mathbf{Pe}\ );$$
$$R(r,z) = R_{eff} + \mathcal{O}(\varepsilon^2 \mathbf{Pe}\ ); \qquad p(r,z) = p_{eff}(z) + \mathcal{O}(\varepsilon^2 \mathbf{Pe}\ );$$
$$\Sigma(r,z) = \Sigma_{eff}(r,z) + \mathcal{O}(\varepsilon^2 \mathbf{Pe}\ ).$$

We note that for $\varepsilon^2 \mathbf{Pe}$ close to 1 it would be important to include the Taylor dispersion effects into equation (42). Taylor dispersion effects for reactive flows are studied in [10] and [11] and in references therein.

3 Solvability of the boundary value problems for the effective equations

Clearly, the values at the extrusion boundary could be replaced by the values at the interface S_E between the stages (c) and (d) of the fiber drawing process.

In the industrial simulations, it makes sense to solve the full 3D Navier-Stokes system in the stage (c), to solve the equations (40)-(42) corresponding to the stage (d) and to couple them at the interface S_E. Coupling at the interface requires construction of the boundary layer.

Conditions, which clearly hold true, are continuities of the temperature field and of the axial component of the velocity.

After [14], a typical iterative procedure for the Navier-Stokes equations with free boundary is the following: for a given free boundary, we solve the Navier-Stokes equations with normal stress given at the lateral boundary. Then we update position of the free boundary using the kinematic free boundary condition. Iterations are repeated until the stabilization. Such procedure requires solving the equations (40)-(42) with the boundary conditions (43) replaced by

$$v_{eff}(L) = V_f, \ v_{eff}(0) = v_E, \ T_{eff}(0) = T_E. \quad (44)$$

In this section, we study the boundary value problem (40)-(42), (44). We simply drop the subscript *eff* and set $Q = Q_0/\pi$.

In the absence of the gravity and inertia effects, with constant density and with the heat transfer coefficient depending only on the temperature, the problem was solved in [5]. Using the temperature as variable, it was possible to write an explicit solution for the radius and prove existence and uniqueness. In the general situation, the approach from [5] is not possible any more. Nevertheless, a suitable change of the unknown function will be useful in our existence proof. We prove an existence result under the following physical properties on the coefficients:

(H1) Functions μ, ρ and $\dfrac{\sigma}{\rho^{1/3}}(T)$ are defined on $\mathbb{R}$, bounded from above and from below by positive constants and decreasing. We suppose them infinitely derivable. $\rho_f = \min_{T_\infty \le T \le T_E} \rho(T)$ and $\rho_E = \rho(T_E)$.
(H2) $0 < v_E = v|_{z=0} < V_f = v|_{z=L}$ and $T_E > T_\infty$.
(H3) c_p is infinitely derivable strictly positive function with $c_{p,min} = \min\limits_{T_\infty \le T \le T_E} c_p(T)$.

We introduce the new unknown w by

$$w = \log \frac{V_f \rho_f^{1/3}}{v\rho^{1/3}(T)}. \tag{45}$$

Let $G = V_f \rho_g^{1/3}$ and $C_1 = \dfrac{C\lambda_\infty}{Q}(\dfrac{2\rho_\infty\sqrt{G}}{\mu_\infty})^m$. Then the boundary value problem (40)-(42), (44) transforms to

$$\frac{\partial}{\partial z}\Big(-3\frac{\mu(T)}{\rho(T)}\frac{\partial w}{\partial z} + \frac{1}{\sqrt{QG}}\frac{\sigma(T)}{\rho^{1/3}(T)}e^{w/2}\Big) =$$

$$-\frac{g}{G}\rho^{1/3}(T)e^{w} - G\rho^{-1/3}e^{-w}\partial_z w - \frac{5G}{6}\rho^{-4/3}e^{-w}\partial_z\rho, \; 0 < z < L; \tag{46}$$

$$c_p(T)\frac{\partial T}{\partial z} + C_1\rho^{-2m/3}(T)e^{-mw/2}(T - T_\infty) = 0, \; 0 < z < L; \tag{47}$$

$$w(0) = w_0 = \log\frac{V_f\rho_f^{1/3}}{v_E\rho^{1/3}} > 0, \; w(L) = 0, \; T(0) = T_E. \tag{48}$$

Remark 1 *In the rest of the section we will obtain existence of C^∞-solutions to problem (46)-(48), such that $w \le w_0$. Then the velocity v is given by $v = V_f(\dfrac{\rho_f}{\rho(T)})^{1/3}e^{-w}$. It is a C^∞-function and satisfies $v(z) \ge v_E$ on $[0, L]$. We note as well that $\rho_f = \rho(T(L))$ is not given. For simplicity, we suppose w_0 known. Otherwise, we should do one more fixed point calculation for ρ_f, which does not pose problems.*

Definition 1 *The corresponding variational formulation for problem (46)-(48) is:*

Find functions $w \in H^1(0, L)$ and $T \in H^1(0, L)$, $\partial_z T \le 0$, such that the *boundary conditions (48) are satisfied and we have*

$$\int_0^L 3\frac{\mu(T)}{\rho(T)}\frac{\partial w}{\partial z}\frac{\partial \varphi}{\partial z}\ dz - \int_0^L \frac{1}{\sqrt{QG}}\frac{\sigma(T)}{\rho^{1/3}(T)} e^{\min\{w,w_0\}/2}\frac{\partial\varphi}{\partial z}\ dz +$$
$$\int_0^L \frac{g}{G}\rho^{1/3}(T) e^{\min\{w,w_0\}}\varphi\ dz + \int_0^L G\rho^{-1/3}(T)e^{-w}\frac{\partial w}{\partial z}\varphi\ dz$$
$$+\int_0^L \frac{5G}{6}\rho^{-4/3}(T)e^{-w}\partial_z\rho\varphi\ dz = 0, \quad \forall \varphi\in H_0^1(0,L). \tag{49}$$
$$\frac{\partial T}{\partial z} = -\frac{C_1}{c_p(T)}\rho^{-2m/3}(T)e^{-m\min\{w,w_0\}/2}(T-T_\infty),\ 0<z<L. \tag{50}$$

It looks as we modified equations (46)-(47), but we will immediately prove that solutions to problem (48), (49) and (50) satisfy the starting equations.

Proposition 2 *Let μ, σ and ρ satisfy* **(H1)-(H2)** *and let $\{w,T\}$ be a variational solution to (48), (49) and (50). Then we have $w\leq w_0$ and $T_E\geq T\geq T_\infty$.*

Proof: We test (49) by $\varphi=(w-w_0)_+$. Then we have

$$\int_0^L 3\frac{\mu}{\rho}(\frac{\partial\varphi}{\partial z})^2\ dz - \int_0^L \frac{1}{\sqrt{QG}}\frac{\sigma}{\rho^{1/3}}e^{\min\{w,w_0\}/2}\frac{\partial\varphi}{\partial z}\ dz + \int_0^L \frac{g}{G}\rho^{1/3}e^{\min\{w,w_0\}}\varphi\ dz$$
$$+\int_0^L G\rho^{-1/3}e^{-w}\frac{\partial\varphi}{\partial z}\varphi\ dz + \int_0^L \frac{5G}{6}\rho^{-4/3}e^{-w}\partial_z\rho\varphi\ dz = 0. \tag{51}$$

Obviously the first, third and fifth term are nonnegative and it is enough to discuss the second and fourth term. We evaluate them one by one:

$$-\int_0^L \frac{\sigma(T)}{\rho^{1/3}(T)}e^{\min\{w,w_0\}/2}\partial_z\varphi\ dz = e^{w_0/2}\int_0^L \partial_z\Big(\frac{\sigma(T)}{\rho^{1/3}(T)}\Big)\varphi\ dz \tag{52}$$
$$G\int_0^L \frac{1}{\rho^{1/3}(T)}e^{-w}\frac{\partial w}{\partial z}\varphi\ dz = -Ge^{-w_0}\int_0^L \rho^{-1/3}(T)\partial_z\{\int_0^\varphi e^{-\xi}\xi\ d\xi\}\ dz =$$
$$\frac{G}{3}\int_0^L \rho^{-4/3}(T)\partial_z\rho(T)\{\int_0^\varphi e^{-\xi}\xi\ d\xi\}\ dz. \tag{53}$$

Since $\partial_z T\leq 0$, hypothesis **(H1)** implies $\partial_z(\sigma(T)\rho^{-1/3}(T))\geq 0$ and $\partial_z\rho(T)\geq 0$. Therefore, all terms in the variational equality (51) are nonnegative. We conclude that $\partial_z(w-w_0)_+=0$ a.e. on $(0,L)$ and the Proposition is proved. $\Box$

Corollary 1 *Under hypothesis* **(H1)-(H2)**, *any variational solution $\{w,T\}$ to (48), (49) and (50) solves equations (46)-(47).*

Lemma 1 *Let $\tilde T$ be the solution for*

$$\frac{\partial\tilde T}{\partial z} = -\frac{C_1}{c_p(\tilde T)}\rho^{-2m/3}(\tilde T)e^{-mw_0/2}(\tilde T-T_\infty),\ 0<z<L;\quad \tilde T(0)=T_E. \tag{54}$$

Then $\tilde{T}(z) \geq T(z)$ *and*

$$\mathcal{A} = \int_0^L \frac{dz}{\mu(\tilde{T}(z))} \geq \int_0^L \frac{dz}{\mu(T(z))}. \tag{55}$$

Lemma 2 *Let* $\varphi \in H_0^1(0,L)$, *let* ρ *and* μ *be given functions from hypothesis* **(H1)**. *Then we have*

$$\|\varphi\|_{L^\infty(0,L)} \leq \rho_{max}^{1/2} \mathcal{A}^{1/2} \Big(\int_0^L \frac{\mu}{\rho} |\partial_z \varphi(z))|^2 \ dz \Big)^{1/2}. \tag{56}$$

Proof: First, we have the following simple inequality

$$|\varphi(z)| = |\int_0^z \partial_\xi \varphi \ d\xi| = |\int_0^z \sqrt{\frac{\mu}{\rho}} \partial_\xi \varphi \sqrt{\frac{\rho}{\mu}} \ d\xi| \leq$$
$$\Big(\int_0^z \frac{\rho}{\mu} \ d\xi \Big)^{1/2} \Big(\int_0^z \frac{\mu(T)}{\rho(T)} |\partial_\xi \varphi(\xi)|^2 \ d\xi \Big)^{1/2}, \tag{57}$$

where T is calculated using equation (50) with w replaced by φ. Now it is straightforward to get

$$\|\varphi\|_{L^\infty(0,L)} \leq \Big(\int_0^L \frac{\rho}{\mu} \ dz \Big)^{1/2} \Big(\int_0^L \frac{\mu}{\rho} |\partial_z \varphi(z)|^2 \ dz \Big)^{1/2}. \tag{58}$$

Next, we apply the inequality (55) and (56) follows immediately. □

Theorem 1 *Let us suppose hypothesis* **(H1)-(H3)**. *Let* $\kappa = \max_{T_\infty \leq T \leq T_E} |\partial_T \rho(T)|$ *and let* $\mathcal{A} = \int_0^L \frac{dz}{\mu(\tilde{T}(z))}$. *Then there is* $\delta_0 > 0$ *such that for* $\kappa \mathcal{A} < \delta_0$, *problem (46)-(48) admits a solution* $\{w, T\} \in C^\infty[0,L]^2$, *such that* $\partial_z T \leq 0$ *and* $w(z) \leq w_0$.

Remark 2 *We see that in the case of constant density, the necessary condition from Theorem 1 is always fulfilled. Furthermore, since viscosity takes large values with temperature decrease,* $\mathcal{A}$ *is a very small quantity. Consequently, Theorem 1 covers all situations of practical interest.*

Proof: We start by studying variational problem (48), (49) and (50). Following classical references, (see e.g. [8]) we will introduce Galerkin's approximation. It is enough to prove that Galerkin's approximation admits a solution and gives an a priori H^1-estimate. Then the compactness argument from the classical theory would give existence for the full variational problem. Let $A > 0$, let $\{e_j\}$ be a C^∞-basis for $H_0^1(0,L)$ and let $N \in \mathbb{N}$. We study the following finite dimensional problem (Galerkin's approximation for the problem (48),

(49) and (50):

$$\text{Find}\ \ u_N=\sum_{i=1}^N \alpha_i e_i(z)+\ell(z),\ \text{with}\ \ \ell(z)=w_0\Big(1-\frac{\displaystyle\int_0^z \frac{\rho(T_N)}{\mu(T_N)}\ d\xi}{\displaystyle\int_0^L \frac{\rho(T_N)}{\mu(T_N)}\ d\xi}\Big),$$
$$\text{and}\ \ T_N\in H^1(0,L),\ T(0)=T_E,\ \text{such that}$$
$$\int_0^L 3\frac{\mu(T_N)}{\rho(T_N)}\frac{\partial u_N}{\partial z}\frac{\partial e_j}{\partial z}\ dz-\int_0^L \frac{1}{\sqrt{QG}}\frac{\sigma(T_N)}{\rho^{1/3}(T_N)}e^{\min\{w_0,u_N\}/2}\frac{\partial e_j}{\partial z}\ dz+$$
$$\int_0^L \frac{g}{G}\rho^{1/3}(T_N)e^{\min\{w_0,u_N\}}e_j\ dz+\int_0^L G\rho^{-1/3}(T_N)e^{-u_N}\frac{\partial u_N}{\partial z}e_j\ dz$$
$$+\int_0^L \frac{5G}{6}\rho^{-4/3}(T_N)e^{-u_N}\partial_z\rho e_j\ dz=0,\quad j=1,\dots,N. \tag{59}$$
$$\frac{\partial T_N}{\partial z}=-\frac{C_1}{c_p(T_N)}\rho^{-2m/3}(T_N)e^{-m\min\{w_0,u_N\}/2}(T_N-T_\infty)=0,\ 0<z<L. \tag{60}$$

The obtained approximation is a system of non-linear algebraic equations for $\alpha_i,\ i=1,\dots,N$. Brouwer's fixed point theorem guarantees solvability of that system under the condition, that for a convenient $B>0$, we have

$$(\Phi_N(\vec{\alpha}),\vec{\alpha})\geq 0,\quad \forall\vec{\alpha}\in\mathbb{R}^N,\quad \text{such that}\ \ |\vec{\alpha}|=B, \tag{61}$$

(see [8], page 53). Let $\varphi=\sum_{i=1}^N \alpha_i e_i(z)$. Then we have

$$(\Phi_N(\vec{\alpha}),\vec{\alpha})=\int_0^L 3\frac{\mu(T_N)}{\rho(T_N)}\frac{\partial u_N}{\partial z}\frac{\partial\varphi}{\partial z}\ dz-\int_0^L \frac{1}{\sqrt{QG}}\frac{\sigma(T_N)}{\rho^{1/3}(T_N)}e^{\min\{w_0,u_N\}/2}\frac{\partial\varphi}{\partial z}\ dz+$$
$$\int_0^L \frac{g}{G}\rho^{1/3}(T_N)e^{\min\{w_0,u_N\}}\varphi\ dz+\int_0^L G\rho^{-1/3}(T_N)e^{-u_N}\frac{\partial u_N}{\partial z}\varphi\ dz$$
$$+\int_0^L \frac{5G}{6}\rho^{-4/3}(T_N)e^{-u_N}\partial_z\rho\varphi\ dz=\int_0^L 3\frac{\mu(T_N)}{\rho(T_N)}(\frac{\partial\varphi}{\partial z})^2\ dz+$$
$$\int_0^L 3\frac{\mu(T_N)}{\rho(T_N)}\frac{\partial\ell}{\partial z}\frac{\partial\varphi}{\partial z}\ dz-\int_0^L \frac{1}{\sqrt{QG}}\frac{\sigma(T_N)}{\rho^{1/3}(T_N)}e^{\min\{w_0,u_N\}/2}\frac{\partial u_N}{\partial z}\ dz+$$
$$\int_0^L \frac{1}{\sqrt{QG}}\frac{\sigma(T_N)}{\rho^{1/3}(T_N)}e^{\min\{w_0,u_N\}/2}\frac{\partial\ell}{\partial z}+\int_0^L \frac{g}{G}\rho^{1/3}(T_N)e^{\min\{w_0,u_N\}}\varphi\ dz+$$
$$\int_0^L \Big(G\rho^{-1/3}(T_N)e^{-u_N}\frac{\partial u_N}{\partial z}+\frac{5G}{6}\rho^{-4/3}(T_N)e^{-u_N}\partial_z\rho\Big)(u_N-\ell(z))\ dz. \tag{62}$$

Let us inspect the terms in (62). We have

$$|\int_0^L 3\frac{\mu(T_N)}{\rho(T_N)}\frac{\partial\ell}{\partial z}\frac{\partial\varphi}{\partial z}\ dz|=|\int_0^L 3\frac{\mu(T_N)}{\rho(T_N)}\frac{\dfrac{w_0\rho(T_N)}{\mu(T_N)}}{\displaystyle\int_0^L \frac{\rho(T_N)}{\mu(T_N)}\ d\xi}\frac{\partial\varphi}{\partial z}\ dz|\leq$$

$$\int_0^L \frac{\mu(T_N)}{\rho(T_N)}(\frac{\partial\varphi}{\partial z})^2\ dz + \frac{9w_0^2}{4}\frac{1}{\displaystyle\int_0^L \frac{\rho(T_N)}{\mu(T_N)}\ d\xi} \tag{63}$$

and

$$-\int_0^L \frac{1}{\sqrt{QG}}\frac{\sigma(T_N)}{\rho^{1/3}(T_N)}e^{\min\{w_0,u_N\}/2}\frac{\partial u_N}{\partial z}\ dz = -\frac{1}{\sqrt{QG}}\frac{2\sigma(T_N)}{\rho^{1/3}(T_N)}|_{z=L}+$$
$$\frac{1}{\sqrt{QG}}\frac{2\sigma_E}{\rho_E^{1/3}}e^{w_0/2} + \int_0^L \Phi(u_N)\partial_z\frac{\sigma(T_N)}{\rho^{1/3}(T_N)}, \tag{64}$$

where $\Phi(y) = (y - w_0 + 2)\exp\{w_0/2\}$ for $y \geq w_0$ and $\Phi(y) = 2\exp\{y/2\}$ for $y < w_0$. Using hypothesis **(H1)** we see that the expression in (64) is a sum of a uniformly bounded and a nonnegative term. The second term linked to the surface tension is estimated as follows:

$$|\int_0^L \frac{1}{\sqrt{QG}}\frac{\sigma(T_N)}{\rho^{1/3}(T_N)}e^{\min\{w_0,u_N\}/2}\frac{\dfrac{w_0\rho(T_N)}{\mu(T_N)}}{\displaystyle\int_0^L \frac{\rho(T_N)}{\mu(T_N)}\ d\xi}\ dz| \leq$$
$$e^{w_0/2}\frac{|w_0|}{\sqrt{QG}}\max_{T_\infty\leq\xi\leq T_E}\frac{\sigma(\xi)}{\rho^{1/3}(\xi)}. \tag{65}$$

Next

$$|\int_0^L \frac{g}{G}\rho^{1/3}(T_N)e^{\min\{w_0,u_N\}}\varphi\ dz| \leq \frac{g}{G}\rho_{max}^{1/3}Le^{w_0}\|\varphi\|_{L^\infty(0,L)} \leq$$
$$C(\delta\int_0^L 3\frac{\mu(T_N)}{\rho(T_N)}(\frac{\partial\varphi}{\partial z})^2\ dz + \frac{\mathcal{A}}{\delta}), \quad \forall\delta > 0. \tag{66}$$

Finally, the inertia terms are transformed as follows:

$$|\int_0^L (\rho^{-1/3}(T_N)e^{-u_N}\frac{\partial u_N}{\partial z}u_N + \frac{5}{6}\rho^{-4/3}(T_N)e^{-u_N}\partial_z\rho u_N)\ dz| \leq |-\rho_{max}^{-1/3}+$$
$$\rho_E^{-1/3}(w_0+1)e^{-w_0}| + |\int_0^L \rho^{-4/3}\partial_z\rho e^{-u_N}(\frac{u_N}{2} - \frac{1}{3})\ dz| \leq$$
$$C_1 + C_2\kappa\|\varphi\|_{L^\infty(0,L)}\exp\{(\frac{m}{2}+1)\|\varphi\|_{L^\infty(0,L)}\} \tag{67}$$
$$|\int_0^L (\rho^{-1/3}(T_N)e^{-u_N}\frac{\partial u_N}{\partial z} + \frac{5}{6}\rho^{-4/3}(T_N)e^{-u_N}\partial_z\rho)\ell\ dz| \leq C_1+$$
$$C_2\kappa\exp\{(\frac{m}{2}+1)\|\varphi\|_{L^\infty(0,L)}\}. \tag{68}$$

After inserting (63)-(68) into (62), recalling that $T_N(z) \in (T_\infty, T_E)$ and using (58) from Lemma 2, we get

$$(\Phi_N(\vec{\alpha}),\vec{\alpha}) \geq \int_0^L \frac{\mu(T_N)}{\rho(T_N)}(\partial_z\varphi)^2\ dz - C_w\kappa\|\varphi\|_{L^\infty(0,L)}\exp\{(\frac{m}{2}+1)\|\varphi\|_{L^\infty(0,L)}\} - \tag{69}$$

$$\frac{9w_0^2}{4}\frac{1}{\displaystyle\int_0^L \frac{\rho(T_N)}{\mu(T_N)}\,d\xi} - \mathcal{C} \geq \frac{\|\varphi\|_{L^\infty(0,L)}}{\displaystyle\int_0^L \frac{\rho(T_N)}{\mu(T_N)}\,dz}(\|\varphi\|_{L^\infty(0,L)} -$$

$$C_w\kappa \exp\{(\frac{m}{2}+1)\|\varphi\|_{L^\infty(0,L)}\}\int_0^L \frac{\rho(T_N)}{\mu(T_N)}\,dz) - \frac{9w_0^2}{4}\frac{1}{\displaystyle\int_0^L \frac{\rho(T_N)}{\mu(T_N)}\,d\xi} - \mathcal{C}, \quad (70)$$

Last expression in (70) is positive for some value of $\|\varphi\|_{L^\infty(0,L)}$ if we can obtain

$$\|\varphi\|_{L^\infty(0,L)} - C_w\kappa \exp\{(\frac{m}{2}+1)\|\varphi\|_{L^\infty(0,L)}\} - C_2 > 0. \quad (71)$$

Hence (61) is fulfilled if the polynomial

$$f(x) = x - C_w\kappa\mathcal{A}e^{(1+m/2)x} - C_2$$

has nonnegative value at the positive root of its derivative. The root is given by

$$(1+\frac{m}{2})x_0 = -\log(C_w(\frac{m}{2}+1)) - \log(\kappa\mathcal{A}). \quad (72)$$

By direct computation we find out that $f(x_0) \geq 0$ if $\kappa\mathcal{A}$ is smaller than some constant. Now we see that (61) holds true and we have existence for Galerkin's approximation. Existence result comes along with uniform H^1 a priori estimate for u_N and a uniform C^1 estimate for T_N. As already mentioned above, now the classic compactness argument gives existence for the variational problem (48), (49) and (50).

By Corollary 1 this solution satisfies (46)-(48) in the weak sense. After plugging the solution into equation (46), we find out that $\frac{\partial^2 w}{\partial z^2} \in L^2(0,L)$. Therefore, $w \in C^1[0,L]$ and $T \in C^2[0,L]$. By repeating the procedure, we get $\{w,T\} \in C^\infty[0,L]^2$. $\square$

4 Numerical simulations of the effective boundary value problem

We start by presenting the data:

For the surrounding air, we take as the air viscosity $\mu_\infty = 53.8E-6$ Pa s; for the air density, we take $\rho_\infty = 0.232$ kg/m^3 and the thermal conductivity of the air is set to be $\lambda_\infty = 0.084$ W/mK. We take $\rho(T) = \rho_E$ and $c_p(T) = c_{pE}$.

For viscosity, we use the following expressions:

$$\mu(T) = 83e^{-0.01769(T-1227)} \text{ (value 1) and}$$

$$\mu(T) = 10^{-2.8188 + (\log_{10}\mu_E + 2.8188)(T_E - 532.28)/(T-532.28)} \text{ (value 2).}$$

	Value 1	Value 2		Value 1	Value 2
h_E	$10\ W/m^2/K$	$2.36\ W/m^2/K$	σ_E	0.1 N/m	0.37 N/m
T_E	$1227^o\ C$	$1145.15^o\ C$	L	$0.06m$	$0.06m$
ρ_E	$2.4 \cdot 10^3\ kg/m^3$	$2.735 \cdot 10^3\ kg/m^3$	T_∞	$27^o\ C$	$612^o\ C$
c_{pE}	$1046.6\ J/kg/K$	$1591.23\ J/kg/K$	λ_E	$1,0\ W/m/K$	$3\ W/m/K$
C	$1,117$	0.42	m	0.137	0.334
μ_E	$83\ Nsec/m^2$	$179.17\ Nsec/m^2$	R_E	$0.838 \cdot 10^{-3}m$	$0.838 \cdot 10^{-3}m$
v_E	$3.124 \cdot 10^{-3}m/sec$	$0.702 \cdot 10^{-3}m/sec$	T_g	$627^o\ C$	$635^o\ C$

Table 1
Data from the reference [4] (Value 1), and from the reference [12] (Value 2)

	Value 1	Value 2
V_f	14, 46 m/sec	35 m/sec
ε **Ca** $= \mu_E v_E R_E/(L\sigma_E)$	0.036	0.00475
Bo $= \rho_E g L^2/(\mu_E v_E)$	326	767.94
Re $= \rho_E v_E L/\mu_E$	$5.42 \cdot 10^{-3}$	$6.43 \cdot 10^{-4}$
Pe $= c_{pE}\rho_E v_E L/\lambda_E$	470.82	61.102
Bi $= h_E R_E/\lambda_E$	$8.38 \cdot 10^{-3}$	$1.98 \cdot 10^{-3}$
2 **Bi** / (**Pe** ε^2)	0.1845	0.332

Table 2
The non-dimensional numbers calculated using the table 1

It is clear that Reynolds number is small and that the inertia effects could be neglected in numerical simulations. Capillary number would increase significantly if taken at the computational interface and not at the extrusion orifice. High Peclet number justifies neglecting the thermal diffusion.

With idea to develop a fast solution method, we follow [5] and, instead of w given by (45), we use as unknown $u = \dfrac{1}{R\rho^{1/3}(T)}$. Momentum equation for u reads

$$\frac{\partial}{\partial z}\left(6Q\frac{\mu(T)}{\rho(T)u}\frac{\partial u}{\partial z} + \frac{\sigma(T)}{\rho^{1/3}(T)}\frac{1}{u} - \frac{Q^2}{\rho^{1/3}(T)}u^2\right) = -\frac{g}{u^2}\rho^{1/3}(T) - \frac{Q^2u^2}{2}\rho^{-4/3}\partial_z\rho,$$

$$0 < z < L; \quad u(0) = w_E = \sqrt{\frac{v_E\rho_E^{1/3}}{Q}},\ u(L) = w_L = \sqrt{\frac{V_f\rho_f^{1/3}}{Q}}. \qquad (73)$$

The principal part of our differential equation is under the z-derivative and the gravity and inertia term are kind of perturbation. Without the gravity and inertia terms, equation (73) reduces to

$$6Q\frac{\mu(T)}{\rho(T)}\frac{du}{dz} - K_0 u(z) = -\frac{\sigma(T)}{\rho^{1/3}(T)} \quad \text{on} \quad (0, L);\ u(0) = w_E;\ u(L) = w_L, \quad (74)$$

where K_0 is a constant to determine from the boundary conditions.

For a given temperature T, the solution for (74) is given by

$$u(z) = -\frac{1}{6Q}\int_0^z \frac{\sigma\rho^{2/3}}{\mu}(T(\eta)) \exp\left(\frac{K_0}{6Q}\int_\eta^z \frac{\rho}{\mu} d\zeta\right) d\eta + w_E \exp\left(\frac{K_0}{6Q}\int_0^z \frac{\rho}{\mu} d\zeta,\right) \quad (75)$$

where $Q = Q_0/\pi$, and the constant K_0 is such that $u(L) = w_L$. We note that dependence of u on K_0 is monotone and it is uniquely determined from the equation $u(L) = w_L$ (see [5]).

For a given $w = u$, the temperature field T is determined from $T(0) = T_E$ and

$$c_p(T)\frac{\partial T}{\partial z}(z) = C_1 \rho(T)^{-2m/3} w^m (T_\infty - T) \quad (76)$$

We use the Runge-Kutta method for integrating this Cauchy problem and we get the monotone decreasing solution.

Solution of the system (40)-(42), (44) is obtained through the following iterative procedure:

First step : Let $T^{(-1)}(z)$ and K_0 be given, we solve the simplified ordinary differential equation of first order for $w^0(z)$

$$\begin{cases} \dfrac{6Q\mu(T^{(-1)})}{\rho(T^{(-1)})}\dfrac{dw^0}{dz} - K_0 w^0 = -\dfrac{\sigma(T^{(-1)})}{\rho^{1/3}(T^{(-1)})} \quad \text{on } (0, L) \\ w^0(0) = w_E \end{cases} \quad (77)$$

Second step: Since $w^0(L)$ is monotone with respect to K_0, we find the value of K_0 by using the boundary condition at $z = L$:

$$w^0(L) = w_L = \sqrt{\frac{V_f \rho^{1/3}(T^{(-1)}(L))}{Q}} \quad (78)$$

Third step: Next we solve the equation for the temperature with the initial condition $T^0(0) = T_E$

$$c_p(T^0)\frac{\partial T^0}{\partial z}(z) = \frac{C\lambda_\infty}{Q}\left(\frac{2\rho_\infty Q w_0}{\mu_\infty \rho^{1/3}(T^0)}\right)^m \left(T_\infty - T^0\right). \tag{79}$$

Now we have the data for $i = 0$ and we iterate by incrementing the value of i

Start of the iterations: Let $T^i(z)$ be given and $w^{i+1,0}(z) = w^i(z)$ be given.
Full problem: Now we iterate with respect to j
Iterations: Let $w^{i+1,j}(z)$ be given. We compute $w^{i+1,j+1}(z)$ and the constant K_{j+1}, solutions of the problem

$$\begin{cases} \dfrac{6Q\mu(T^{(i)})}{\rho(T^{(i)})}\dfrac{dw^{i+1,j+1}}{dz} - \left(\displaystyle\int_0^z \frac{g\rho^{1/3}(T^{(i)})\,dz}{(w^{i+1,j})^2} + K_{j+1}\right) w^{i+1,j+1} = \\ -\dfrac{\sigma(T^{(i)})}{\rho^{1/3}(T^{(i)})} \quad \text{on } (0,L) \\ w^{i+1,j+1}(0) = w_E \\ w^{i+1,j+1}(L) = w_L \end{cases} \tag{80}$$

Convergence criterion: If $\dfrac{\|w^{i+1,j+1} - w^{i+1,j}\|}{\|w^{i+1,j}\|} \leq \varepsilon = 1.E-7$, we end iterations in j and set $j = j_i$.
New value: We set $w^{i+1}(z) = w^{i+1,j_i+1}$.
New temperature: Then we calculate $T^{(i+1)}(z)$, $T^{(i+1)}(0) = T_E$, as solution for

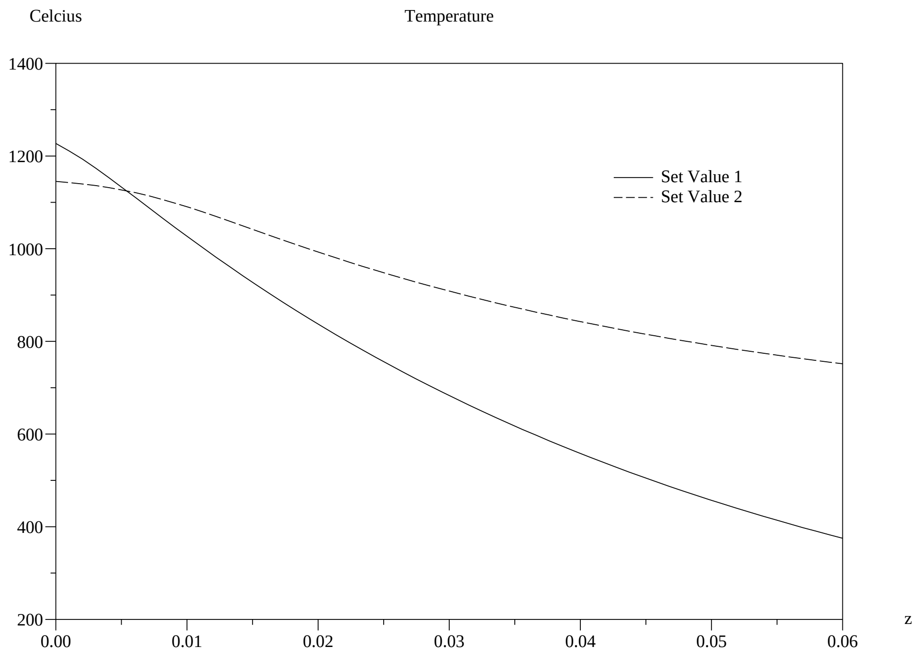
$$c_p(T^{(i+1)})\frac{dT^{(i+1)}}{dz}(z) = \frac{C\lambda_\infty}{Q}\left(\frac{2\rho_\infty Q w^{i+1}}{\mu_\infty \rho^{1/3}(T^{(i+1)})}\right)^m \left(T_\infty - T^{(i+1)}(z)\right) \tag{81}$$

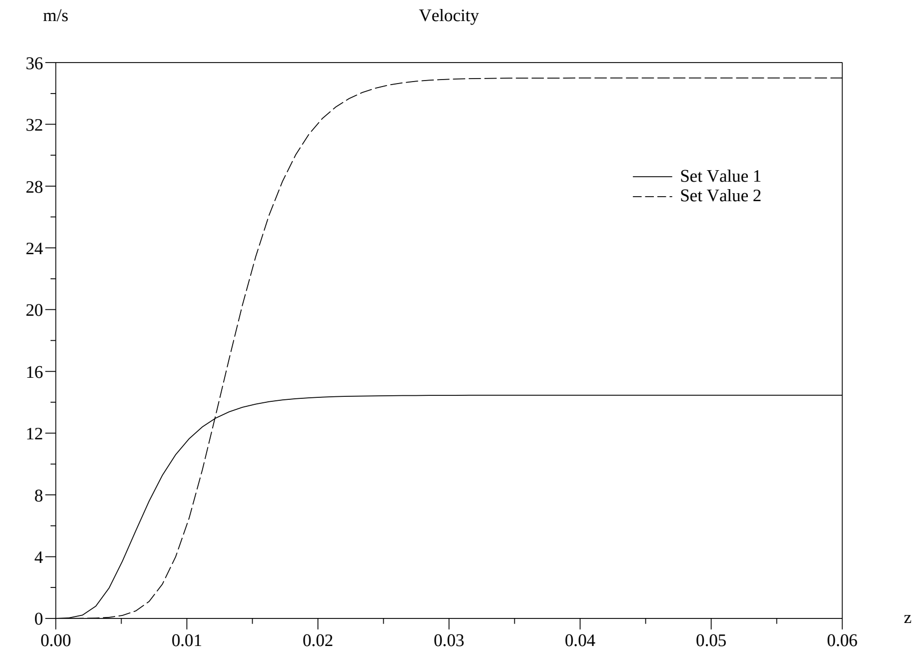
End of iteration If $\dfrac{\|T^{(i+1)} - T^{(i)}\|}{\|T^{(i)}\|} \leq \varepsilon = 1.E-7$, we end iterations in i. $u = w^{i_{final}+1}$ and $T = T^{i_{final}+1}$ are computed.

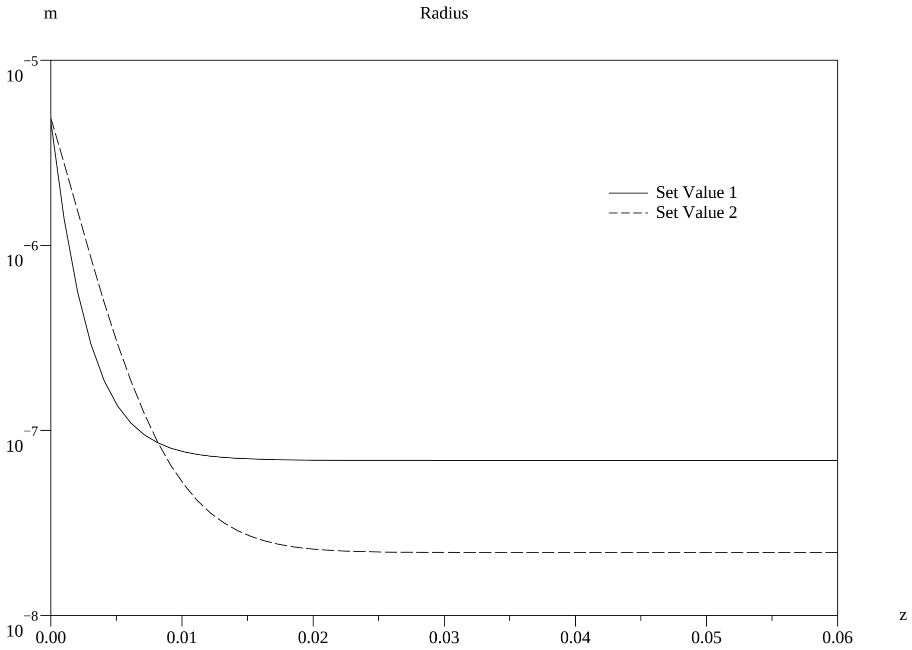
Finally, we compute the velocity as $v = \dfrac{Qu^2}{\rho^{1/3}(T)}$.

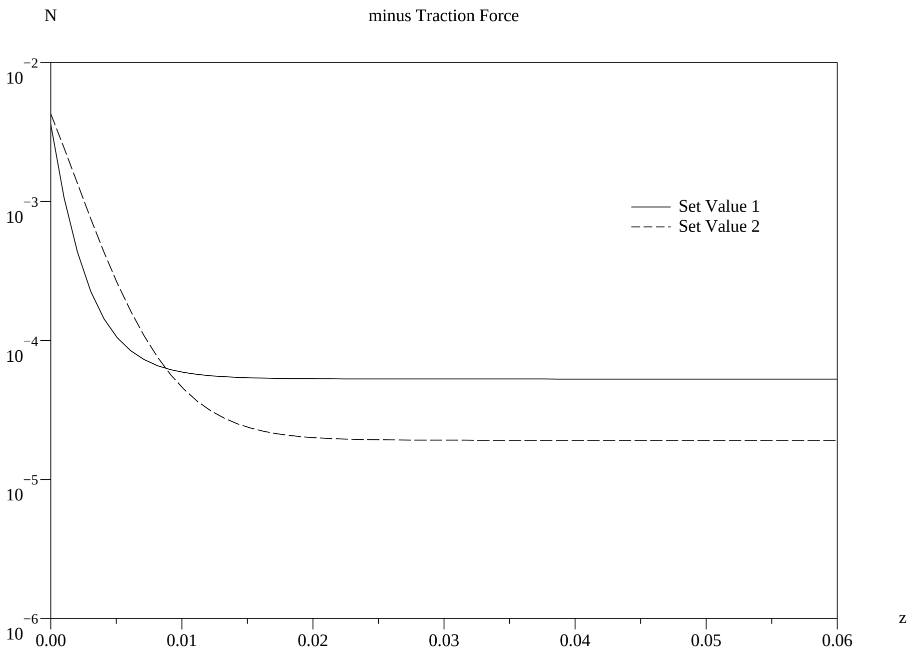
We note that the effects of inertia were neglected in the numerical procedure. A posteriori verification also gave these effects very small. The gravity effects were kept and they are important only in the neighborhood of the inlet boundary $z = 0$ and as soon as the viscosity becomes very large, the axial velocity is constant. The calculations confirm the estimates from the existence proof.

The results are shown on the figures, which follow. They correspond to two sets of values from the Tables. First figure shows the temperature fields, second the velocity fields, third the radii and the last figure displays the traction forces.


Celcius
Temperature
1400
1200
1000
800
600
400
200
0.00
0.01
0.02
0.03
0.04
0.05
0.06
z
Set Value 1
Set Value 2


m/s
Velocity
36
32
28
24
20
16
12
8
4
0
0.00
0.01
0.02
0.03
0.04
0.05
0.06
z
Set Value 1
Set Value 2





Acknowledgement: The authors are grateful to Javier Olaiz (LYON 1 INGENIERIE) for his assistance and to the (anonymous) referees for their careful reading of the manuscript and helpful remarks.

References

[1] Dewynne, J. N., Ockendon, J. R. Wilmott, P. On a mathematical model for fibre tapering. SIAM J. Appl. Math. **49** (1989), 983-990.

[2] A. Farina, A. Fasano, A. Mikelić : On the equations governing the flow of mechanically incompressible, but thermally expansible, viscous fluids, M^3AS : Math. Models Methods Appl. Sci., Vol. 18 (2008), no. 6, p. 813-858.

[3] M. Gregory Forest, H. Zhou, Unsteady analyses of thermal glass fibre drawing processes. Euro. Jnl of Applied Mathematics, Vol. 12 (2001), p. 479-496.

[4] G. Gupta, W.W. Schultz: Non-isothermal flows of Newtonian slender glass fibers. Int. J. Non-Linear Mech., Vol. 33 (1998), p. 151-163.

[5] T. Hagen, On the Effects of Spinline Cooling and Surface Tension in Fiber Spinnning, ZAMM - Z. Angew. Math. Mech., Vol. 82 (2002), no. 8, p. 545-558.

[6] S. Kase and T. Matsuo, Studies of melt spinning I. Fundamental Equations on the Dynamics of Melt Spinning, J. Polym. Sci. A, 3 (1965), pp. 2541–2554.

[7] S. Kase and T. Matsuo, Studies of melt spinning. II, Steady state and transient solutions of fundamental equations compared with experimental results, J. Polym. Sci 11, 251-287 (1967).

[8] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod , Paris, 1969.

[9] M. A. Matovich and J. R. A. Pearson, Spinning a molten threadline-steady state isothermal viscous flows, Ind. Engrg. Chem. Fundam., 8 (1969), pp. 512–520.

[10] R. Mauri, Dispersion, convection and reaction in porous media, Phys. Fluids A, Vol. 3 (1991), p. 743-755.

[11] A. Mikelić, V. Devigne, C.J. van Duijn : Rigorous upscaling of the reactive flow through a pore, under dominant Peclet and Damkohler numbers, SIAM J. Math. Anal., Vol. 38, Issue 4 (2006), p. 1262-1287.

[12] R. von der Ohe, Simulation of Glass Fiber Forming Processes, Thesis PhD (2005) - AUC Imprint: ISBN: 8791200253. - Aalborg : Department of Production, Aalborg University, 2005. pp. 192.

[13] Pearson, J. R. A. (1985) Mechanics of Polymer Processing. Elsevier.

[14] M. Renardy, An existence theorem for a free surface flow problem with open boundaries, Commun. in Partial Differential Equations, Vol. 17 (1992), pp. 1387-1405.

[15] W.W. Schultz, S.H.Davis: One-dimensional liquid fibers, J. Rheol., Vol. 26 (1982), p. 331-345.