Modeling effective interface laws for transport phenomena between an unconfined fluid and a porous medium using homogenization

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Abstract

In this chapter of the special issue of the journal "Transport in Porous Media", on the topic "Flow and transport above permeable domains", we present modeling of flow and transport above permeable domains using the homogenization method. Our goal is to develop a heuristic approach which can be used by the engineering community for treating this type of problems and which has a solid mathematical background. The rigorous mathematical justification, of the presented results, is given in the corresponding articles of the authors. The plan is as follows: We start with the Introduction were we give an overview and comparison with interface conditions obtained using other approaches. In Section 2, we give a very short derivation of the Darcy law by homogenization, using the two-scale expansion in the typical pore size parameter ε. It gives us the definition of various auxiliary functions and typical effective properties as permeability. In Section 3, we introduce our approach to the effective interface laws on a simple 1D example. The approximation is obtained heuristically using the two steps strategy. For the 1D problem we calculate the approximation and the effective interface law explicitly and show that it is valid at order $O(\varepsilon^2)$. Next, in Section 4 we give a derivation of the Beavers-Joseph-Saffman interface condition and of the pressure jump condition, using homogenization. We construct the corresponding boundary layer and present a heuristic calculation, leading to the interface law and being based on the rigorous mathematical result. In addition, we show the invariance

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of the law with respect to the small variations in the choice of the interface position. Finally, there is a short concluding section.

1 Introduction

Homogenization is a mathematical tool that allows changing the scale in problems containing several characteristic scales. Typical examples of its utilisation are finding effective models for composite materials, in optimal shape design etc. Another important example, which we are interested in, is the fluid mechanics of the flow through porous media.

A porous medium has at least two length scales: a macroscopic scale (the reservoir scale) and a microscopic scale (the pore scale). In the practical simulations of flows through porous media, we use differential equations at the reservoir scale. The pore structure is present only implicitly through quantities like permeability, porosity, tortuosity etc.

Number of known laws from the dynamics of fluids in porous media was derived using homogenization. The most well known example is Darcy's law, being the effective equation for one phase flow through a rigid porous medium. Its formal derivation using the 2-scale expansion goes back to the classical paper [10] by Ene and Sanchez-Palencia. This derivation was made mathematically rigorous by Tartar in [25]. For detailed derivation in the case of a periodic porous medium we refer to the review papers [2] by Allaire and [21] by Mikelić. Rigorous derivation of Darcy's law for a random statistically homogeneous porous medium is due to Beliaev and Kozlov (see [5]).

Darcy's law is valid for a creeping flow through a porous medium. For its derivation, the periodicity of the porous medium was required. The periodicity condition can be relaxed to a kind of statistical homogeneity and ergodicity, but clearly, such assumptions break down close to the boundaries. Deviations from Darcy's law are expected only in thin layers near the interfaces. Nevertheless, presence of such interfaces can significantly change the structure of coefficients and the flow could obey particular effective constitutive laws.

In this article, we are interested in obtaining such laws for one phase viscous creeping laws through porous media. We will call them **interface laws** and our plan is to address the topic using the combination of homogenization and boundary layer approach, introduced by Jäger and Mikelić in the papers [12], [14], [15], [16] and [17].

The simplest possible problem is to find relationship between the seepage velocity and the pressure gradient for an incompressible viscous flow through a domain consisting of two different periodic porous media separated by an interface. The argument from derivation of Darcy's law (see § 2) is local and we get the Darcy law in every porous piece. However, due to the different geometric structures, the permeability matrices are different. In order to couple the flow we need conditions at the interface. From the incompressibility condition, we conclude immediately the **continuity of the normal components of the seepage velocities**. Another physically natural interface condition is the **continuity of the effective pressure field**. However, it is usually imposed without discussion. We note that the form of the homogenized stress tensor around the interface is not clear. Consequently, the pressure continuity is not a mathematically evident condition. For its rigorous derivation, we refer to [15], where the pressure continuity was obtained after using the corresponding boundary layers from [12]. It is interesting to note the deterioration of the error estimate. In general, the approximation of the pressure is of order $O(\varepsilon^{1/8})$, contrary to the order $O(\varepsilon)$ obtained for the Darcy law in the absence of the boundaries. The same deterioration is present when introducing the velocity corrector. It is due to the jump of the normal derivatives of the effective pressure at the interface. For discussion of these interface conditions, sometimes called the *refraction at a boundary between two porous bodies*, at physical level of rigor, we refer to [8].

The above-described cases are obvious from modeling point of view. Study of the simultaneous flow through both a pure fluid and a porous medium is much more challenging. It occurs in a wide range of industrial processes and natural phenomena. Here the problem is finding effective boundary conditions at a naturally permeable wall. Namely, if the flow region contains a porous medium, a channel with a free flow and an interface between them, then one wishes to have an effective model. As before we are considering a slow viscous and incompressible flow. Clearly, the effective flow through a porous medium will be described by the Darcy law. In the channel, the free fluid flow remains governed by the Stokes system (or by the Navier-Stokes system if the inertia effects in the free fluid are important). We note that this means that one should couple two systems of partial differential equations, one being a second order system for the velocity and a first order equation for the pressure, respectively, and the other being a scalar second order equation for the pressure and a first order system for the seepage velocity. The coupling conditions should be imposed at the interface. One coupling condition is very simple. It is a consequence of the incompressibility and says that we have the continuity of the normal mass flux. This is not enough for determination of the effective flow and one should specify more conditions. Classically, the tangential velocity of the free fluid velocity was set to zero at the interface. This condition corresponds to an impervious boundary and could not be justified, neither from mathematical nor modeling or experimental point of view.

Beavers and Joseph concluded experimentally in [4] that the difference, between the slip velocity of the free fluid and the tangential component of the seepage velocity at the interface, was proportional to the shear stress from the free fluid. This law was justified at a physical level of rigor by Saffman in [24], where it was observed that the seepage velocity contribution could be neglected and wrote the law in the form

$$\sqrt{k^\varepsilon}\frac{\partial v_\tau}{\partial \nu} = \alpha v_\tau + O(k^\varepsilon). \tag{1}$$

Here α is a dimensionless parameter depending on the geometrical structure of the porous medium, ε is the characteristic pore size and $k^\varepsilon = \varepsilon^2 k$ is the (scalar) permeability. ν denotes the unit normal vector at the interface and v_τ is the slip velocity of the free fluid in the channel. Saffman used a statistical approach to extend Darcy's law to non-homogeneous porous media. However, it should be noted that his argument is not entirely satisfactory since he made an *ad hoc* hypothesis about the representation of the averaged interfacial forces as a linear integral functional of the velocity, with an unknown kernel, being equal to Dirac's measure in a porous medium, zero in the free fluid and a given function around the interface. A similar argument is developed in [8], where Slattery's relationship

$$\frac{\partial P}{\partial x_i} = \mu\big(- \sum_{j=1}^{3} r^0_{ij} U_j + \sum_{j,l=1}^{3} r^1_{ijl}\frac{\partial U_j}{\partial x_l} + \sum_{j,l,m=1}^{3} r^2_{ijlm}\frac{\partial^2 U_j}{\partial x_l \partial x_m} + \dots\big) \tag{2}$$

between the pressure gradient and a combination of derivatives of the seepage velocity was assumed. Here r^0_{ij}, r^1_{ijl} and r^2_{ijlm} are the macroscopic resistivity tensors. The asymptotic matching at the boundary leads once more to the law (1). Neither paper [24] nor [8] contain construction of the boundary layers describing the flow behavior close to the interface. The Saffman modification of the law by Beavers and Joseph is widely accepted.

In the papers [10] and [18], Ene, Lévy and Sanchez-Palencia have undertaken the effort to find the effective interface laws by a formal asymptotic argument. They have considered two essentially different cases. The case of the flow in a cavity, lying inside of a porous matrix, was considered in [18]. By comparing the orders of the magnitude of characteristic quantities, it was found out that the effective pressure should be constant at the interface. This conclusion was rigorously justified in [12], after constructing the appropriate boundary layers.

The second case corresponds to the flow considered by Beavers and Joseph. In the paper [10] the continuity of the effective pressure was deduced, but without a rigorous argument or an asymptotic expansion. From modeling point of view, this interface law is acceptable. It can be considered as an alternative to (1), however it should be noted that the well posedness of the averaged problem is not clear.

In this review paper we will justify the law (1) by the technique developed in [14] for Laplace's operator and then in [12] for the Stokes system. Numerical calculation of the boundary layers in the conditions of the experiment by Beavers and Joseph is in [17].

The experiment by Beavers and Joseph took into consideration only flows tangential to a naturally permeable wall (a porous bed). General situation is much more complicated and many types of interfacial conditions have been used, e.g. continuity of the tangential velocity but discontinuity of the tangential shear stress introduced in the papers [23] by Ochoa-Tapia and Whitaker, or continuity of both the tangential velocity and the tangential shear stress from [22] by Neale and Nader, or discontinuity of both the tangential velocity and the tangential shear stress from [6] by Cieszko and Kubik. General question of determining of practical and relevant first order interfacial conditions between a pure fluid and a porous matrix remains an open question that could be treated using the technique we developed in [12]. The numerical implementation of effective interface couplings is in [9].

Let us also mention the derivation of the effective laws for flows through sieves and filters. We mention only the papers [7] and [13]. The paper [13] is on the effective equations for a viscous incompressible flow through a filter of a finite thickness and it uses the boundary layers developed in [12]. For the numerical implementing of the interface condition for the industrial filters, we refer to [11].

2 Homogenization approach to Darcy's law

We consider a two dimensional periodic porous medium $\Omega = (0, L)^2$ with a periodic arrangement of the pores. The formal description goes along the following lines:

(H1) First, we define the geometrical structure inside the unit cell $Y = (0, 1)^2$. Let Y_s (the solid part) be a closed strictly included subset of $\bar{Y}$, and $Y_F = Y \backslash Y_s$ (the fluid part). Now we make the periodic repetition of Y_s all over $\mathbb{R}^2$ and set $Y_s^k = Y_s + k$, $k \in \mathbb{Z}^2$. Obviously, the obtained set $E_s = \bigcup_{k\in\mathbb{Z}^2} Y_s^k$ is a closed subset of $\mathbb{R}^2$ and $E_F = \mathbb{R}^2 \backslash E_s$ in an open set in $\mathbb{R}^2$. We suppose that Y_s has a boundary of class $C^{0,1}$, which are locally located on one side of their boundary. Obviously, E_F is connected and E_s is not. For description of the geometry in 3D, we refer to [1].

Now we see that Ω is covered with a regular mesh of size ε, each cell being a cube Y_i^ε, with $1 \le i \le N(\varepsilon) = |\Omega|\varepsilon^{-2}[1 + o(1)]$. Each cube Y_i^ε is homeomorphic to Y, by linear homeomorphism Π_i^ε, being composed of translation and a homothety of ratio $1/\varepsilon$.

We define $Y_{S_i}^\varepsilon = (\Pi_i^\varepsilon)^{-1}(Y_s)$ and $Y_{F_i}^\varepsilon = (\Pi_i^\varepsilon)^{-1}(Y_F)$. For sufficiently small $\varepsilon > 0$ we consider the set $T_\varepsilon = \{k \in \mathbb{Z}^2 | Y_{S_k}^\varepsilon \subset \Omega\}$ and define

$$O_\varepsilon = \bigcup_{k\in T_\varepsilon} Y_{S_k}^\varepsilon, \quad S^\varepsilon = \partial O_\varepsilon, \quad \Omega^\varepsilon = \Omega \backslash O_\varepsilon = \Omega \cap \varepsilon E_F$$

Obviously, $\partial\Omega^\varepsilon = \partial\Omega \cup S^\varepsilon$. The domains O_ε and Ω^ε represent, respectively, the solid and fluid parts of a porous medium Ω. For simplicity, we suppose $L/\varepsilon \in \mathbb{N}$.

A very important property of the porous media is a variant of Poincaré's inequality:

Lemma 1. *Let* $W^\varepsilon = \{z \in H^1(\Omega^\varepsilon)^2,\ z = 0 \ \text{ on } \ \partial\Omega^\varepsilon \setminus \partial\Omega \ \text{ and } \ z \ \text{ is } \ L-\text{periodic}\ \}$. *Then we have*

$$\int_\Omega |w|^2 dx \ \leq \ \frac{\varepsilon^2}{\lambda_1(Y_F)} \int_\Omega |\nabla_x w|^2 dx \qquad \forall w \in W^\varepsilon, \tag{3}$$

where $\lambda_1(Y_F)$ *is the smallest eigenvalue of* $-\Delta$ *on* $W = \{z \in H^1(Y_F)^2,\ z = 0 \ \text{ on } \ \partial Y_s \ \text{ and } \ z \ \text{ is } \ 1-\text{periodic}\ \}$.

In the case of a periodic porous medium, (3) is proved by a simple rescaling argument (see e.g. [21]).

For the case of a stochastic porous medium we refer to the article [5] by Beliaev and Kozlov.

Having defined the geometrical structure of the porous medium, we precise the flow problem. Here we consider the slow viscous incompressible flow of a single fluid through a porous medium. We suppose the no-slip condition at the boundaries of the pores (i.e. a rigid porous medium). Then we describe it by the following non-dimensional steady Stokes system in Ω^ε (the fluid part of the porous medium Ω):

$$-\Delta v^\varepsilon + \nabla p^\varepsilon = f \qquad \text{in} \qquad \Omega^\varepsilon \tag{4}$$

$$\operatorname{div} v^\varepsilon = 0 \qquad \text{in} \qquad \Omega^\varepsilon \tag{5}$$

$$v^\varepsilon = 0 \quad \text{on} \quad \partial\Omega^\varepsilon \setminus \partial\Omega, \qquad \{v^\varepsilon, p^\varepsilon\} \quad \text{is } L-\text{periodic} \tag{6}$$

Here the non-dimensional f stands for the effects of external forces or an injection at the boundary or a given pressure drop, and it corresponds to the physical forcing term multiplied by the ratio between Reynolds' number and Froude's number squared. v^ε denotes the non-dimensional velocity and p^ε is the non-dimensional pressure.

The variational form of the problem (4)-(6) is:

Find $v^\varepsilon \in W^\varepsilon$, div $v^\varepsilon = 0$ in Ω^ε and $p^\varepsilon \in L^2(\Omega^\varepsilon)$ such that

$$\int_{\Omega^\varepsilon} \nabla v^\varepsilon \nabla \varphi \, dx - \int_{\Omega^\varepsilon} p^\varepsilon \ \operatorname{div} \ \varphi \, dx = \int_{\Omega^\varepsilon} f \varphi \, dx \qquad \forall \varphi \in W^\varepsilon. \tag{7}$$

Then for $f \in L^2(\Omega^\varepsilon)^2$, the elementary elliptic variational theory gives the existence of the unique velocity field $v^\varepsilon \in W^\varepsilon$, div $v^\varepsilon = 0$ in Ω^ε, which solves (7) for every $\varphi \in W^\varepsilon$, div $\varphi = 0$ in Ω^ε. The construction of the pressure field goes through De Rham's theorem (see e.g. book [26]).

We are interested in behavior of the solutions with respect to ε. First, we extend the velocity field after extension by zero to the solid part. Furthermore, it would be more comfortable to work with the pressure field p^ε defined on Ω. Following the approach from [19], we define the pressure extension $\tilde{p}^\varepsilon$ by

$$\tilde{p}^\varepsilon = \begin{cases} p^\varepsilon & \text{in} \qquad \Omega_\varepsilon \\ \frac{1}{|Y^\varepsilon_{F_i}|} \int_{Y^\varepsilon_{F_i}} p^\varepsilon & \text{in the} \quad Y^\varepsilon_{S_i} \qquad \text{for each } i \end{cases} \tag{8}$$

where $Y^\varepsilon_{F_i}$ is the fluid part of the cell Y^ε_i. Note that solid part of the porous medium is a union of all $Y^\varepsilon_{S_i}$.

Then after [25], [19], [2] and [21], we have

Theorem 2. *(a priori estimates for the velocity and the pressure fields in* Ω*). Let the velocity field* v^ε *be extended by zero to the solid part. Then we have*

$$\|v^\varepsilon\|_{L^2(\Omega^\varepsilon)^2} + \varepsilon \|\nabla v^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} \leq C\varepsilon^2. \tag{9}$$

Let $\tilde{p}^\varepsilon$ be defined by (8). Then it satisfies the estimates

$$\|\tilde{p}^\varepsilon - \frac{1}{|\Omega|}\int_\Omega \tilde{p}^\varepsilon dx\|_{L^2(\Omega)} \leq \frac{1}{|Y_F|}\|p^\varepsilon - \frac{1}{|\Omega \cap \varepsilon E_F|}\int_{\Omega\cap\varepsilon E_F} p^\varepsilon dx\|_{L^2(\Omega\cap\varepsilon E_F)} \leq C. \tag{10}$$

Theorem 2 gives a priori estimates for the velocity and the pressure. As a consequence of (9) and (10), we postulate the following asymptotic expansion

$$v^\varepsilon(x) = \varepsilon^2 v^0(x,y) + \varepsilon^3 v^1(x,y) + \dots, \quad y = \frac{x}{\varepsilon} \tag{11}$$

$$p^\varepsilon(x) = p^0(x,y) + \varepsilon p^1(x,y) + \dots, \quad y = \frac{x}{\varepsilon}. \tag{12}$$

This expansion takes care of the disparity of the two length scales in the problem. Furthermore, since the geometry is periodic, it is natural to suppose a **periodic** dependence on the fast scale y.

Having two scales we should transform the derivatives. We have

$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y; \quad \text{div} = \text{div}_x + \frac{1}{\varepsilon}\,\text{div}_y \quad \text{and} \quad \Delta = \Delta_x + \frac{2}{\varepsilon}\,\text{div}_x\nabla_y + \frac{1}{\varepsilon^2}\Delta_y,$$

where the subscript indicates the variable involved in the differentiation.

Substituting the expansions (11)-(12) into (4)-(6), the following equations are obtained:

- $O(\varepsilon^{-1})$:

$$\nabla_y p^0(x,y) = 0 \quad \text{in} \quad \Omega \times Y_F \tag{13}$$

- $O(1)$:

$$-\Delta_y v^0(x,y) + \nabla_y p^1(x,y) + \nabla_x p^0(x,y) = f(x) \quad \text{in} \quad \Omega \times Y_F \tag{14}$$

- $O(\varepsilon)$:

$$\text{div}_y v^0(x,y) = 0 \quad \text{in} \quad \Omega \times Y_F \tag{15}$$

$$-\Delta_y v^1(x,y) + \nabla_y p^2(x,y) + \nabla_x p^1(x,y) = 0 \quad \text{in} \quad \Omega \times Y_F \tag{16}$$

- $O(\varepsilon^2)$:

$$\text{div}_x v^0(x,y) + \text{div}_y v^1(x,y) = 0 \quad \text{in} \quad \Omega \times Y_F \tag{17}$$

$$v^0(x,y) = 0 \text{ on } \Omega \times (\partial Y_F \setminus \partial Y); \tag{18}$$

$$\{v^0(x,y), p^1(x,y)\} \text{ is } 1-\text{periodic in } y, \tag{19}$$

$$-\Delta_y v^2(x,y) + \nabla_y p^3(x,y) + \nabla_x p^2(x,y) = 0 \quad \text{in} \quad \Omega \times Y_F \tag{20}$$

- $O(\varepsilon^3)$: ...

  We are interested only in the lowest order approximation, corresponding to the homogenized problem. First, we note that (13) is equivalent to $p^0 = p^0(x)$, i.e. the zeroth order approximation of the pressure does not depend on y .

Next, there is v^1 satisfying (17) if and only if

$$\operatorname{div}_x \int_{Y_F} v^0(x,y)\,dy = 0 \quad \text{in} \quad \Omega. \tag{21}$$

We can now summarize the conditions that $\{v^0, p^0, p^1\}$ should satisfy in $\Omega \times Y_F$. We take (13), (14), (15), (18) and (21) and obtain the following problem

$$\begin{cases} -\Delta_y v^0(x,y) + \nabla_y p^1(x,y) + \nabla_x p^0(x) = f(x) & \text{in} \quad \Omega \times Y_F \\ \operatorname{div}_y v^0(x,y) = 0 & \text{in} \quad \Omega \times Y_F \\ v^0(x,y) = 0 & \text{on } \ \Omega \times (\partial Y_F \setminus \partial Y) \\ \{v^0(x,y), p^1(x,y)\} & \text{is } 1-\text{periodic in } y \\ \operatorname{div}_x \int_{Y_F} v^0(x,y)\,dy = 0 & \text{in} \quad \Omega \end{cases} \tag{22}$$

The quantity $q^0(x) = \int_{Y_F} v^0(x,y)\ dy$ is the **seepage velocity** (or the specific discharge) for the filtration through the porous medium Ω. $|Y_F|$ is the **porosity** of Ω and the average velocity is the seepage velocity divided by the porosity.

We still miss another boundary condition in x-variable. It is reasonable to impose

$$\{p^0, \int_{Y_F} v^0\ dy\} \quad \text{is} \quad L-\text{periodic}\ . \tag{23}$$

In general fixing the boundary conditions at the outer boundary $\partial\Omega$ is a difficult problem. The detailed study of the boundary layers for the homogenization of the Stokes flow in a porous medium, with a general boundary, is in [20]. In this paper we do not discuss those questions and this is the reason why we have chosen the periodic conditions at the outer boundary, for the ε-problem.

System (22)-(23) is called the **Stokes system with two pressures**. We are going to show that it has a unique solution $\{v^0, p^0, p^1\}$ in an appropriate functional space. Then it is natural to consider it as the homogenized problem corresponding to (4)-(6) and we shall justify the approximation in the next subsection. Furthermore, one should find a relationship between (22)-(23) and Darcy's law in theories of groundwater flows, stating that the seepage velocity q^0 is proportional to the pressure gradient $\nabla_x p^0$. We start with the study of the problem (22)-(23). We start with **elimination of two pressures**.

We consider the following auxiliary problems in Y_F:

For $1 \le i \le 2$, find $\{w^i, \pi^i\} \in H^1_{per}(Y_F)^2 \times L^2(Y_F)$, $\int_{Y_F} \pi^i(y)\,dy = 0$, such that

$$\begin{cases} -\Delta_y w^i(y) + \nabla_y \pi^i(y) = e^i & \text{in} \ \ Y_F \\ \operatorname{div}_y w^i(y) = 0 & \text{in} \ \ Y_F \\ w^i(y) = 0 & \text{on} \ \ (\partial Y_F \setminus \partial Y) \end{cases} \tag{24}$$

Obviously, these problems always admit a unique solution. Let us introduce the **permeability matrix** K by

$$K_{ij} = \int_{Y_F} \nabla_y w^i \nabla_y w^j\ dy = \int_{Y_F} w^i_j\ dy,\ 1 \le i,j \le 2. \tag{25}$$

Permeability tensor K is symmetric and positive definite. Consequently, the **drag tensor** K^{-1} is also positive definite.

Theorem 3. *(Darcy's law). We have*

$$v^0(x,y) = \sum_{j=1}^{2} w^j(y)(f_j(x) - \frac{\partial p^0(x)}{\partial x_j}) \qquad x \in \Omega,\, y \in Y_F \tag{26}$$

$$q^0(x) = \int_{Y_F} v^0(x,y)\; dy = K(f(x) - \nabla_x p^0(x))\,(\textbf{Darcy's law}), \tag{27}$$

where $\{w^j, \pi^j\}$ *are given by (24) and the permeability matrix* K *by (25).*

Approximation order obtained by the 2-scale extension indicates that the L^2-norm of $r^\varepsilon = v^\varepsilon/\varepsilon^2 - v^0(x, x/\varepsilon)$ should be bounded as $C\varepsilon$. The idea is to estimate it using the Stokes equations with appropriate forces. However, the presence of *bad* incompressibility effects in the velocity $v^0(x, x/\varepsilon)$ forces us to add a corrector for div v^0 as in [12] and in [20]. Now we suppose $f \in C^\infty_{per}(\Omega)^2$ and set

$$v^{0,\varepsilon}(x) = v^0(x, x/\varepsilon);\; x \in \Omega^\varepsilon; \qquad v^{0,\varepsilon}(x) = 0 \quad x \in \Omega \setminus \Omega^\varepsilon \tag{28}$$

Let $C^\infty_{per}(\Omega)^2$ be the space of infinitely derivable L-periodic functions on Ω and $L^2_0(\Omega) = \{z \in L^2(\Omega) \mid \int_\Omega z\; dx = 0\}$. Now we can state the result :

Theorem 4. *(see [21]) Let* $f \in C^\infty_{per}(\Omega)^2$ *and* $\operatorname{div} f = 0$*. Then we have*

$$\|\frac{v^\varepsilon}{\varepsilon^2} - v^{0,\varepsilon}\|_{L^2(\Omega)} \leq C\varepsilon, \tag{29}$$

where v^ε *is a solution for (4)-(6) extended by zero to* Ω*. Furthermore, there exists an extension* $\widetilde{\Pi}^\varepsilon$ *of* $\Pi^\varepsilon = p^\varepsilon - p^0$ *such that*

$$||\widetilde{\Pi}^\varepsilon||_{L^2_0(\Omega)} = \inf_{C \in \mathbb{R}} ||\widetilde{\Pi}^\varepsilon + C||_{L^2(\Omega)} \leq C\varepsilon. \tag{30}$$

3 Interface conditions for a 1D example

In this section we try to explain our approach to interface laws on a simple 1D example.

Let $\Omega_1 = (-\infty, 0)$ and $\Omega_2 = (0, \infty)$. Interface between Ω_1 and Ω_2 is the point $\Sigma = \{0\}$. Let $Y = (0,1)$ and $Z^* = (0,a)$, $0 < a < 1$. Then the "fluid" part of Ω_1 is given by $\Omega^\varepsilon_{1F} = \cup_{k=1}^{\infty} \varepsilon(a-k, 1-k)$. The 1D "pore space" is now $\Omega^\varepsilon = \Omega^\varepsilon_{1F} \cup \Sigma \cup \Omega_2$.

Let $f \in C^\infty_0(\mathbb{R})$ be a given function. We consider the problem

$$\begin{cases} -\dfrac{d^2u^\varepsilon}{dx^2} = f(x), & \text{in } \Omega^\varepsilon \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon, & \displaystyle\lim_{|x| \to +\infty} \frac{du^\varepsilon}{dx} = 0. \end{cases} \tag{31}$$

Following the calculations from Section §2, we expect the following behavior of u^ε:

$$\begin{cases} u^\varepsilon = -\dfrac{\varepsilon^2 f(x)}{2}(\dfrac{x}{\varepsilon} + k)(\dfrac{x}{\varepsilon} + k + 1 - a) + O(\varepsilon^3), & \text{for } -k + a - 1 \leq \dfrac{x}{\varepsilon} \leq -k,\; k = 0, 1, \dots \\ u^\varepsilon = \displaystyle\int_0^x t f(t)\; dt + x \int_x^\infty f(t)\; dt + C_\varepsilon, & \text{in } \Omega_2, \end{cases} \tag{32}$$

where C_ε is an unknown constant. The corresponding "permeability" is $k = \varepsilon^2(1-a)^3/12$. Two domains are linked through the interface $\Sigma = \{0\}$. Without an interface condition, the approximation in Ω_2 is not determined. We search for effective interface conditions at Σ, leading to a good approximation of u^ε by some u^{eff}.

Classical way of finding interface conditions is by using matched asymptotic expansions (MMAE). A recent reference in asymptotic methods and boundary layers in fluid mechanics is the book [27] by Zeytounian and for the detailed explications, we invite reader to consult it and references therein.

In the language of the MMAE, expansions in Ω_1 and Ω_2 give us the *outer expansions*. We should supplement it by an (local) *inner expansion* in which the independent variable is stretched out in order to capture the behavior in the neighborhood of the interface.

The MMAE approach uses the *limit matching rule*, by which asymptotic behavior of the outer expansion in the neighborhood of the interface has to be equal to asymptotic behavior of the inner expansion outside interface.

The stretched variable is $\xi = \dfrac{x}{\varepsilon^\alpha}$, $\alpha > 0$. The geometry of Ω_{1F}^ε obliges us to take $\alpha \geq 1$. Then the zero order term in the expansion is linear in ξ and the limit matching rule implies that, at the leading order,

$$u^0 = 0 \qquad \text{at the interface} \quad \Sigma = \{0\}. \tag{33}$$

Equations in the outer regions read

$$\begin{cases} u^0 = 0, & \text{in} \quad \Omega_1; \\ -\dfrac{d^2u^0}{dx^2} = f, & \text{in} \quad \Omega_2; \\ \dfrac{du^0}{dx} \to 0, & \text{as} \quad x \to +\infty. \end{cases} \tag{34}$$

The system (33)-(34) determines u^0.

It is easy to find out that

$$u^0(x) = \begin{cases} \displaystyle\int_0^x tf(t)\ dt + x\int_x^\infty f(t)\ dt, & x \geq 0; \\ u^0 = 0, & \text{in} \quad \Omega_1; \end{cases} \tag{35}$$

and

$$u^\varepsilon(x) = \int_{\varepsilon(a-1)}^x (t+\varepsilon(1-a))f(t)\ dt + (x+\varepsilon(1-a))\int_x^\infty f(t)\ dt, \quad \text{for } x \geq -\varepsilon(1-a)); \tag{36}$$

$$u^\varepsilon(x) = \int_{\varepsilon(a-1)-k\varepsilon}^x (t+\varepsilon(1-a+k))f(t)\ dt + (x+\varepsilon(k+1-a))\Big(\int_x^{-\varepsilon k} f(t)\ dt -$$

$$\frac{1}{\varepsilon(1-a)}\int_{\varepsilon(a-1)}^{-\varepsilon k}(t+\varepsilon(1-a))f(t)\ dt\Big), \quad \text{for } -k\varepsilon \geq x \geq \varepsilon(a-1)-k\varepsilon,\ k=1,2,\ldots \tag{37}$$

Now we see that

$$u^\varepsilon(x) = u^0(x) + O(\varepsilon) \quad \text{in} \quad \Omega_1. \tag{38}$$

Nevertheless in the neighborhood of the interface $\Sigma = \{0\}$ approximation for $\frac{du^\varepsilon}{dx}$ is not good and it differs at order $O(1)$.

Why the approximation deteriorates around the interface? It is due to the fact that the MMAE method, as it is used in classical textbooks, does not suit interface problems. It matches only the function values at the interface, but not the values of the normal derivative. This difficulty is not easy to circumvent because imposing matching of the values of the function and its normal derivative leads to an ill posed problem for our 2nd order equation.

In order to circumvent the difficulty, we propose the following strategy, introduced in the papers [12], [13], [14], [15], [16] and [17] by Jäger and Mikelić:

1. STEP: We match the function values, as when using the MMAE method. In our particular example this means that the first approximation $u^{0,eff}$ is given by the problem (33)-(34).

2. STEP: At $\Sigma = \{0\}$ we have the derivative jump equal to $\frac{du^0}{dx} = \int_0^{+\infty} f(t)\ dt$. Natural stretching variable is given by the geometry and reads $y = \frac{x}{\varepsilon}$. Therefore, the correction w is given by

$$\begin{cases} -\frac{d^2w}{dy^2} = 0, & \text{in} \quad (0,+\infty); \\ [w]_\Sigma = w(+0) - w(-0) = 0; \ [\frac{dw}{dy}]_\Sigma = \frac{dw}{dy}(+0) - \frac{dw}{dy}(-0) = -\frac{du^0}{dx}(+0), & \text{on} \quad \Sigma \\ -\frac{d^2w}{dy^2} = 0, & \text{in} \quad (a-1,0); \\ w(a-1) = 0; \quad \frac{dw}{dy} \to 0, \ \text{ when } \ y \to +\infty. \end{cases} \tag{39}$$

For this simple problem we find easily that the solution is given by

$$w(y) = \begin{cases} \frac{du^0}{dx}(+0)(1-a), & \text{for} \quad y > 0; \\ \\ \frac{du^0}{dx}(+0)(1-a+y), & \text{for} \quad a-1 < y \le 0. \\ 0, & \text{for} \quad y \le -1. \end{cases} \tag{40}$$

We add this correction to u^0 and obtain $u^{1,eff}(x) = u^0(x) + \varepsilon w(\frac{x}{\varepsilon})$. It is easy to see that

$$\begin{cases} u^\varepsilon(x) = u^0(x) + \varepsilon w(\frac{x}{\varepsilon}) + O(\varepsilon^2); \\ \\ \frac{du^\varepsilon}{dx}(x) = \frac{du^0}{dx}(x) + \frac{dw}{dy}(\frac{x}{\varepsilon}) + O(\varepsilon). \end{cases} \tag{41}$$

Next we find out that $u^0(+0) + \varepsilon w(+0) = \varepsilon(1-a)\frac{du^0}{dx}(+0)$ and $\frac{du^0}{dx}(+0) + \frac{dw}{dy}(+0) = \frac{du^0}{dx}(+0)$. Consequently, we impose the following **effective interface condition** :

$$u^{eff}(+0) = \varepsilon(1-a)\frac{du^{eff}}{dx}(+0) = \sqrt{\frac{12k}{1-a}}\frac{du^{eff}}{dx}(+0). \tag{42}$$

In $(0,+\infty)$, u^{eff} satisfies the original PDE:

$$-\frac{d^2u^{eff}}{dx^2} = f, \quad \text{in} \quad \Omega_2; \qquad \frac{du^{eff}}{dx} \to 0, \text{ as } \quad x \to +\infty. \tag{43}$$

By easy direct calculation, we calculate the solution u^{eff} for (42)-(43) and find out that

$$||u^\varepsilon - u^{eff}||_{L^\infty(0,+\infty)} = \sup_{x\geq 0} |u^\varepsilon(x) - u^{eff}(x)| \leq C\varepsilon^2, \tag{44}$$

proving the pointwise $O(\varepsilon^2)$ approximation of u^ε by the solution of the problem with effective interface condition (42)-(43).

Clearly, in the case of a porous medium things are more complicated and w should be calculated using the corresponding boundary layer problem.

4 Law by Beavers and Joseph

In this section we are going to justify the law (1) by the technique developed in [14] for Laplace's operator and then in [12] for the Stokes system and presented in Section §3. We suppose the conditions of the experiment from [4], i.e. we consider the 2D laminar stationary incompressible viscous flow over a porous bed. The flow is governed by the pressure drop $p_b - p_0$ over the bed of length b. The mathematical justification of the law (1) for the Navier-Stokes system and the boundary conditions for the pressure on the inlet and outlet boundaries is in [16]. Since the inertia effects and the outer boundary layers effects, due to the choice of the pressure boundary conditions, are not of the fundamental importance for the study of the interface boundary conditions, we will make some non-essential simplifications. First, we neglect the inertial term. We note that anyhow we are not able to find the boundary behavior for the turbulent free flow. The nonlinear stability results for the laminar Navier-Stokes system are in [16]. Second, we suppose that the boundary is sufficiently long and one can suppose the periodic boundary conditions at inlet/outlet boundary. The flow is then governed by a force coming from the pressure drop and equal to $\frac{p_b - p_0}{b} e_1$.

This section is organized as follows: In §4.1 we construct the necessary boundary layer and in §4.2 the law (1) is justified.

4.1 Navier's boundary layer

As observed in hydrology, the phenomena relevant to the boundary occur in a thin layer surrounding the interface between a porous medium and a free flow. In this subsection we are going to present a sketch of the construction of the main boundary layer, used for determining the coefficient α in (1) and for a rigorous justification of the law by Beavers and Joseph. Since the law by Beavers and Joseph is an example of the Navier slip condition, we call it **Navier's boundary layer**.

Let $Y = (0,1)^2$ and $Y_s = Z^*$ a Lipschitz domain strictly contained in Y. We introduce the pore space Y_F by $Y_F = Y \setminus \bar{Z}^*$, $S = (0,1) \times \{0\}$, $Z^+ = (0,1) \times (0,+\infty)$ and the semi-infinite porous slab $Z^- = \cup_{k=1}^{\infty}(Y_F - \{0,k\})$. The flow region is then $Z_{BL} = Z^+ \cup S \cup Z^-$.

We consider the following problem:

Find $\{\beta^{bl}, \omega^{bl}\}$ with square-integrable gradients satisfying

$$-\triangle_y \beta^{bl} + \nabla_y \omega^{bl} = 0 \qquad \text{in } Z^+ \cup Z^- \tag{45}$$

$$\text{div}_y \beta^{bl} = 0 \qquad \text{in } Z^+ \cup Z^- \tag{46}$$

$$\big[\beta^{bl}\big]_S(\cdot, 0) = 0 \quad \text{and} \quad \big[\{\nabla_y \beta^{bl} - \omega^{bl} I\} e_2\big]_S(\cdot, 0) = e_1 \ \text{ on } S \tag{47}$$

$$\beta^{bl} = 0 \quad \text{on } \cup_{k=1}^{\infty} (\partial Z^* - \{0, k\}), \qquad \{\beta^{bl}, \omega^{bl}\} \text{ is } 1-\text{periodic in } y_1 \tag{48}$$

By Lax-Milgram's lemma, there is a unique $\beta^{bl} \in L^2_{loc}(Z_{BL})^2$, $\nabla_y z \in L^2(Z_{BL})^4$ satisfying (45)-(48) and $\omega^{bl} \in L^2_{loc}(Z^+ \cup Z^-)$, unique up to a constant and satisfying (45).

The goal of this subsection is to establish that the system (45)-(48) describes a boundary layer, i.e. that β^{bl} and ω^{bl} stabilize exponentially towards constants, when $|y_2| \to \infty$.

Since we are studying an incompressible flow, it is useful to prove properties of the conserved averages.

Lemma 5. *([12]). Any solution $\{\beta^{bl}, \omega^{bl}\}$ satisfies*

$$\int_0^1 \beta_2^{bl}(y_1, b)\, dy_1 = 0, \quad \forall b \in I\!R \ \text{ and } \int_0^1 \omega^{bl}(y_1, b_1)\, dy_1 = \int_0^1 \omega^{bl}(y_1, b_2)\, dy_1, \ \forall b_1 > b_2 \geq 0 \tag{49}$$

$$\int_0^1 \beta_1^{bl}(y_1, b_1)\, dy_1 = \int_0^1 \beta_1^{bl}(y_1, b_2)\, dy_1 = -\int_{Z_{BL}} |\nabla \beta^{bl}(y)|^2\, dy, \qquad \forall b_1 > b_2 \geq 0. \tag{50}$$

Proposition 6. *([12]). Let*

$$C_1^{bl} = \int_0^1 \beta_1^{bl}(y_1, 0) dy_1. \tag{51}$$

Then for every $y_2 \geq 0$ *and* $y_1 \in (0,1)$, $|\beta^{bl}(y_1, y_2) - (C_1^{bl}, 0)| \leq C e^{-\delta y_2}$, $\forall \delta < 2\pi$. (52)

Corollary 7. *([12]). Let*

$$C_\omega^{bl} = \int_0^1 \omega^{bl}(y_1, 0)\, dy_1. \tag{53}$$

Then for every $y_2 \geq 0$ *and* $y_1 \in (0,1)$, *we have* $|\, \omega^{bl}(y_1, y_2) - C_\omega^{bl} \,| \leq e^{-2\pi y_2}$. (54)

In the last step we study the decay of β^{bl} and ω^{bl} in the semi-infinite porous slab Z^-.

Proposition 8. *([12]). Let β^{bl} and ω^{bl} be defined by (45)-(48). Then there exist positive constants C and γ_0, such that*

$$|\nabla \beta^{bl}(y_1, y_2)| + |\nabla \beta^{bl}(y_1, y_2)| \leq C e^{-\gamma_0 |y_2|}, \qquad \text{for every} \quad (y_1, y_2) \in Z^-. \tag{55}$$

Furthermore, the limit $\kappa_\infty = \lim_{k \to -\infty} \dfrac{1}{|\, Y_F \,|} \displaystyle\int_{Z_k} \omega^{bl}(y)\, dy$ *exists and we have*

$$|\omega^{bl}(y_1, y_2) - \kappa_\infty| \leq C e^{-\gamma_0 |y_2|}, \qquad \text{for every} \quad (y_1, y_2) \in Z^-. \tag{56}$$

Remark 9. *Without loosing generality we take $\kappa_\infty = 0$. If the geometry of Z^- is axially symmetric with respect to reflections around the axis $y_1 = 1/2$, then $C_\omega^{bl} = 0$. For the proof, we refer to [17]. In [17] a detailed numerical analysis of the problem (45)-(48) is given. Through numerical experiments it is shown that for a general geometry of Z^-, $C_\omega^{bl} \neq 0$.*

4.2 Justification of the law by Beavers and Joseph

We consider the laminar viscous two-dimensional incompressible flow through a domain Ω consisting of the porous medium $\Omega_2 = (0,b) \times (-L,0)$, the channel $\Omega_1 = (0,b) \times (0,h)$ and the permeable interface $\Sigma = (0,b) \times \{0\}$ between them. We assume that the structure of the porous medium is periodic and generated by translations of a cell $Y^\varepsilon = \varepsilon Y$, defined in subsection §4.1. Let χ be the characteristic function of Y_F, extended by periodicity to $I\!R^2$. We set $\chi^\varepsilon(x) = \chi(\frac{x}{\varepsilon})$, $x \in I\!R^2$, and define Ω_2^ε by $\Omega_2^\varepsilon = \{x \mid x \in \Omega_2,\ \chi^\varepsilon(x) = 1\}$. Furthermore, $\Omega^\varepsilon = \Omega_1 \cup \Sigma \cup \Omega_2^\varepsilon$ is the fluid part of Ω. It is supposed that $(b/\varepsilon, L/\varepsilon) \in I\!N^2$.

Therefore, our porous medium is supposed to consist of a large number of periodically distributed channels of characteristic length ε, being small compared with a characteristic length of the macroscopic domain. The flow is supposed to be slow and governed by the following equations

$$-\mu \triangle v^\varepsilon + \nabla p^\varepsilon = -\frac{p_b - p_0}{b} e_1 \qquad \text{in } \Omega^\varepsilon, \tag{57}$$

$$\text{div } v^\varepsilon = 0 \qquad \text{in } \Omega^\varepsilon, \tag{58}$$

$$v^\varepsilon = 0 \qquad \text{on } \partial\Omega^\varepsilon \setminus \partial\Omega \quad \text{and on} \quad (0,b) \times (\{-L\} \cup \{h\}), \tag{59}$$

$$\{v^\varepsilon, p^\varepsilon\} \qquad \text{is } b-\text{periodic in } x_1 \tag{60}$$

where $\mu > 0$ is the viscosity and p_0 and p_b are given constants. $\varepsilon > 0$ is the characteristic pore size, v^ε is the velocity and p^ε is the pressure field. Problem (57)-(60) has a unique solution $\{v^\varepsilon, p^\varepsilon\} \in H^1(\Omega^\varepsilon)^2 \times L_0^2(\Omega^\varepsilon)$ (see e.g. book [26]).

Now one would like to study of the effective behavior of the velocities v^ε and pressures p^ε as $\varepsilon \to 0$.

We follow the decomposition approach from [16] as presented in Section §3.

Following the two-scale expansions from Section §2, we expect the following behavior of u^ε:

In Ω_2^ε:

$$u^\varepsilon(x) = \frac{\varepsilon^2}{\mu} \sum_{j=1}^{2} w^j(\frac{x}{\varepsilon})(-\frac{p_b - p_0}{b}\delta_{1j} - \frac{\partial p^0(x)}{\partial x_j}) + O(\varepsilon^3), \qquad x \in \Omega; \tag{61}$$

$$p^\varepsilon(x) = p^0(x) + O(\varepsilon), \qquad x \in \Omega, \tag{62}$$

where w^j, $j = 1, 2$ are given by (24). Furthermore, let the permeability tensor K be given by (25). Then we have

$$\text{div } \big(K \nabla p^0\big) = 0 \ \text{ in } \Omega_2, \tag{63}$$

$$K\nabla p^0 \cdot e_2 = 0 \ \text{ on } (0,b) \times \{-L\} \quad \text{and} \quad p^0 \quad \text{is b-periodic in } x_1. \tag{64}$$

In Ω_1: We keep the equations (57)-(60) for $\{u^0, p^0\}$.

Obviously, without setting interface conditions at Σ, functions $\{u^0, p^0\}$ are not determined.

Let us apply the strategy presented in Section §3:

1. STEP: We match the function values, as when using the MMAE method. It comes out immediately that for the lowest order approximation $\{v^0, \pi^0\}$ we have on Σ the no-slip condition

$$v^0 = 0 \qquad \text{on} \quad \Sigma. \tag{65}$$

Now $\{v^0, \pi^0\}$ satisfies equations (57)-(60) in Ω_1, supplemented by the condition (65).

We observe that the unique solution for this problem in $H^1(\Omega_1)^2 \times L_0^2(\Omega_1)$ is the classic Poiseuille flow in Ω_1, satisfying the no-slip conditions at Σ. It is given by

$$v^0 = \left(\frac{p_b - p_0}{2b\mu} x_2(x_2 - h), 0\right) \quad \text{for} \quad 0 \le x_2 \le h; \qquad \pi^0 = 0 \quad \text{for} \quad 0 \le x_1 \le b. \tag{66}$$

We extend this solution to Ω_2 by setting $v^0 = 0$ for $-L \le x_2 \le 0$ and keeping the same form of π^0.

Does this solution approximate in some sense the solution $\{u^\varepsilon, p^\varepsilon\}$ for the original problem (57)-(60)? The idea is to construct the solution to (57)-(60) as a small perturbation to the Poiseuille flow (66). By uniqueness, this would give us the approximation result.

In order to establish it, we first need the following simple auxiliary result:

Lemma 10. *Let $\varphi \in H^1(\Omega_2^\varepsilon)$ be such that $\varphi = 0$ on $\partial\Omega_2^\varepsilon \setminus \partial\Omega_2$. Then we have*

$$\|\varphi\|_{L^2(\Omega_2^\varepsilon)} + \varepsilon^{1/2}\|\varphi\|_{L^2(\Sigma)} \le C\varepsilon \|\nabla\varphi\|_{L^2(\Omega_2^\varepsilon)^2} \tag{67}$$

Direct consequence of Lemma 10 is the following result, proved in [16], which establishes the nonlinear stability of the Poiseuille flow with respect to the perturbation by changing the impermeable boundary to a porous bed:

Proposition 11. *Let $\{v^\varepsilon, p^\varepsilon\}$ be the solution for (57)-(60) and $\{v^0, \pi^0\}$ defined by (66). Then we have*

$$\sqrt{\varepsilon}\|\nabla(v^\varepsilon - v^0)\|_{L^2(\Omega^\varepsilon)^4} + \sqrt{\varepsilon}\|p^\varepsilon - \pi^0\|_{L^2(\Omega_1)} + \|v^\varepsilon\|_{L^2(\Sigma)} + \|v^\varepsilon - v^0\|_{L^2(\Omega_1)^2} \le C\varepsilon \tag{68}$$

Therefore, we have obtained the uniform a priori estimates for $\{v^\varepsilon, p^\varepsilon\}$. Moreover, we have found that Poiseuille's flow in Ω_1 is an $O(\varepsilon)$ L^2-approximation for v^ε. Beavers and Joseph's law should correspond to the next order velocity correction. Since the Darcy velocity is of order $O(\varepsilon^2)$, we will in fact justify Saffman's version of the law.

2. STEP: At the interface Σ we have the shear stress jump equal to $-\mu \frac{\partial v_1^0}{\partial x_2}|_\Sigma$. Again, natural stretching variable is given by the geometry and reads $y = \frac{x}{\varepsilon}$. The correction $\{w, p_w\}$ is given by

$$-\mu \triangle_y w + \nabla_y p_w = 0 \qquad \text{in} \qquad \Omega_1/\varepsilon \cup \Omega_2^\varepsilon/\varepsilon, \tag{69}$$

$$\operatorname{div}_y w = 0 \qquad \text{in} \qquad \Omega_1/\varepsilon \cup \Omega_2^\varepsilon/\varepsilon, \tag{70}$$

$$[w](\cdot, 0) = 0 \quad \text{and} \quad [\mu \frac{\partial w_1}{\partial y_2}](\cdot, 0) = \mu \frac{\partial v_1^0}{\partial x_2}|_\Sigma \quad \text{on} \quad \Sigma/\varepsilon, \tag{71}$$

$$w = 0 \quad \text{on } \partial(\Omega_2^\varepsilon \setminus \Omega_2)/\varepsilon, \qquad \{w, p_w\} \text{ is } b/\varepsilon - \text{periodic in } y_1. \tag{72}$$

Using periodicity of the geometry and independence of $\frac{\partial v_1^0}{\partial x_2}|_\Sigma$ on y, we have

$$w(y) = \frac{\partial v_1^0}{\partial x_2}|_\Sigma \beta^{bl}(y) \quad \text{and} \quad p_w(y) = \mu \frac{\partial v_1^0}{\partial x_2}|_\Sigma \omega^{bl}(y), \tag{73}$$

where $\{\beta^{bl}, \omega^{bl}\}$ is given by (45)-(48).

Now we set

$$\beta^{bl,\varepsilon}(x) = \varepsilon\beta^{bl}(\frac{x}{\varepsilon}) \qquad \text{and} \qquad \omega^{bl,\varepsilon}(x) = \omega^{bl}(\frac{x}{\varepsilon}), \quad x \in \Omega^\varepsilon, \tag{74}$$

$\beta^{bl,\varepsilon}$ is extended by zero to $\Omega \setminus \Omega^\varepsilon$. Let H be Heaviside's function. Then for every $q \geq 1$ we have

$$\frac{1}{\varepsilon}\|\beta^{bl,\varepsilon} - \varepsilon(C_1^{bl},0)H(x_2)\|_{L^q(\Omega)^2} + \|\omega^{bl,\varepsilon} - C_\omega^{bl}H(x_2)\|_{L^q(\Omega^\varepsilon)} + \|\nabla\beta^{bl,\varepsilon}\|_{L^q(\Omega_1\cup\Sigma\cup\Omega_2)^4} = C\varepsilon^{1/q}. \quad (75)$$

Hence, our correction is not concentrated around the interface and there are some stabilization constants. We will see that these constants are closely linked with our effective interface law.

As in [12] stabilization of $\beta^{bl,\varepsilon}$ towards a nonzero constant velocity $\varepsilon\big(C_1^{bl},0\big)$, at the upper boundary, generates a counterflow. It is given by the homogeneous Stokes system (57)-(58) in Ω_1, the periodicity condition (60), on Σ the counter velocity d is zero and on $(0,b)\times\{h\}$ it is equal to $\varepsilon\big(C_1^{bl},0\big)$. Obviously the unique solution is the two dimensional Couette flow $d = \varepsilon C_1^{bl}\dfrac{x_2}{h}e_1$.

Now, we expected that

$$v^\varepsilon = v^0 - \beta^{bl,\varepsilon}\frac{\partial v_1^0}{\partial x_2}|_\Sigma + \varepsilon C_1^{bl}\frac{\partial v_1^0}{\partial x_2}|_\Sigma H(x_2)\frac{x_2}{h}e_1 + O(\varepsilon^2), \quad (76)$$

Contrary to the scalar case considered in Section §3, here we have additional complications due to the stabilization of the boundary layer pressure to C_ω^{bl}, when $y_2 \to +\infty$. Consequently, the correction in Ω_1 is $\omega^{bl,\varepsilon} - H(x_2)C_\omega^{bl}\big)\mu\dfrac{\partial v_1^0}{\partial x_2}|_\Sigma$. At the flat interface Σ, the normal component of the normal stress reduces to the pressure field. Subtraction of the stabilization pressure constant at infinity leads to the pressure jump on Σ:

$$[p^0]_\Sigma = \pi^0(x_1,+0) - p^0(x_1,-0) = -p^0(x_1,-0) = -C_\omega^{bl}\mu\frac{\partial v_1^0}{\partial x_2}|_\Sigma \quad \text{for} \quad x_1 \in (0,b) \quad (77)$$

Now we are able to calculate the leading order porous media pressure. It satisfies the equations (63)-(64) and the condition (77). We see immediately that in our particular situation $p^0 = H(-x_2)C_\omega^{bl}\mu\dfrac{\partial v_1^0}{\partial x_2}|_\Sigma$. Therefore the pressure approximation is

$$p^\varepsilon(x) = p^0H(-x_2) - \big(\omega^{bl,\varepsilon}(x) - H(x_2)C_\omega^{bl}\big)\mu\frac{\partial v_1^0}{\partial x_2}|_\Sigma + O(\varepsilon) = \big(C_\omega^{bl} - \omega^{bl,\varepsilon}(x)\big)\mu\frac{\partial v_1^0}{\partial x_2}|_\Sigma + O(\varepsilon). \quad (78)$$

Follow the ideas from [12], these heuristic calculations could be made rigorous. Let us introduce the errors in velocity and in the pressure:

$$\mathcal{U}^\varepsilon(x) = v^\varepsilon - v^0 + \beta^{bl,\varepsilon}\frac{\partial v_1^0}{\partial x_2}|_\Sigma - \varepsilon C_1^{bl}\frac{\partial v_1^0}{\partial x_2}|_\Sigma H(x_2)\frac{x_2}{h}e_1 \quad (79)$$

$$\mathcal{P}^\varepsilon(x) = p^\varepsilon - p^0H(-x_2) + \big(\omega^{bl,\varepsilon} - H(x_2)C_\omega^{bl}\big)\mu\frac{\partial v_1^0}{\partial x_2}|_\Sigma \quad (80)$$

Then, after [16], we have the required higher order error estimate:

Theorem 12. *Let $\mathcal{U}^\varepsilon$ be defined by (79) and $\mathcal{P}^\varepsilon$ by (80). Then we have the following estimates*

$$\varepsilon\|\mathcal{P}^\varepsilon\|_{L^2(\Omega_1)} + \varepsilon\|\nabla\mathcal{U}^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|\mathcal{U}^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \varepsilon^{1/2}(\|\mathcal{U}^\varepsilon\|_{L^2(\Sigma)^2} + \|\mathcal{U}^\varepsilon\|_{L^2(\Omega_1)^2}) \leq C\varepsilon^2 \quad (81)$$

The estimate (81) allows justifying Saffman's modification of the Beavers and Joseph law (1):

$$v^\varepsilon = v^0 - \left(\beta^{bl,\varepsilon} - \varepsilon\big(C_1^{bl},0\big)H(x_2)\right)\frac{\partial v_1^0}{\partial x_2}|_\Sigma - \varepsilon C_1^{bl}\frac{\partial v_1^0}{\partial x_2}|_\Sigma H(x_2)(1-\frac{x_2}{h})e_1 + O(\varepsilon^2)$$

where v^0 is the Poiseuille velocity in Ω_1 and the third term corresponds to the counterflow generated by the boundary layer stabilization constant tangential velocity at infinity. Then on the interface Σ

$$\frac{\partial v_1^\varepsilon}{\partial x_2}|_\Sigma = \frac{\partial v_1^0}{\partial x_2}|_\Sigma - \frac{\partial \beta_1^{bl}}{\partial y_2}|_{\Sigma, y=x/\varepsilon} + O(\varepsilon) \quad \text{and} \quad \frac{v_1^\varepsilon}{\varepsilon} = -\beta_1^{bl}(x_1/\varepsilon, 0)\frac{\partial v_1^0}{\partial x_2}|_\Sigma + O(\varepsilon).$$

After averaging over Σ with respect to y_1, we obtain the familiar form of the Saffman version of the law by Beavers and Joseph

$$u_1^{eff} = -\varepsilon C_1^{bl}\frac{\partial u_1^{eff}}{\partial x_2} \quad \text{on} \quad \Sigma, \tag{82}$$

where u^{eff} is the average over the characteristic pore opening at the naturally permeable wall. The higher order terms are neglected.

Now we introduce the effective flow equations in Ω_1 through the following boundary value problem:

Find a velocity field u^0 and a pressure field p^{eff} such that

$$-\mu \triangle u^{eff} + \nabla p^{eff} = -\frac{p_b - p_0}{b} e_1 \qquad \text{in } \Omega_1, \tag{83}$$

$$\text{div } u^{eff} = 0 \qquad \text{in } \Omega_1, \tag{84}$$

$$u^{eff} = 0 \qquad \text{on } (0,b)\times\{h\}; \quad u^{eff} \text{ and } p^{eff} \quad \text{are } b-\text{periodic} \quad , \tag{85}$$

$$u_2^{eff} = 0 \qquad \text{and} \qquad u_1^{eff} + \varepsilon C_1^{bl}\frac{\partial u_1^{eff}}{\partial x_2} = 0 \quad \text{on} \quad \Sigma. \tag{86}$$

Problem (83)-(86) has a unique solution

$$u^{eff} = \left(\frac{p_b - p_0}{2b\mu}(x_2 - \frac{\varepsilon C_1^{bl} h}{h - \varepsilon C_1^{bl}})(x_2 - h), 0\right) \text{ for } 0 \le x_2 \le h; \quad p^{eff} = 0 \text{ for } 0 \le x_1 \le b. \tag{87}$$

The **effective mass flow rate** through the channel is then

$$M^{eff} = b\int_0^h u_1^{eff}(x_2)\, dx_2 = -\frac{p_b - p_0}{12\mu} h^3 \frac{h - 4\varepsilon C_1^{bl}}{h - \varepsilon C_1^{bl}}, \tag{88}$$

where $C_1^{bl} < 0$.

Let us estimate the error made when replacing $\{v^\varepsilon, p^\varepsilon, M^\varepsilon\}$ by $\{u^{eff}, p^{eff}, M^{eff}\}$.

Proposition 13. *We have*

$$\sqrt{\varepsilon}\|\nabla(v^\varepsilon - u^{eff})\|_{L^1(\Omega_1)^4} + \|v^\varepsilon - u^{eff}\|_{L^2(\Omega_1)^2} + |M^\varepsilon - M^{eff}| \le C\varepsilon^{3/2}. \tag{89}$$

Our interface is a mathematical one and it does not exist as a physical boundary. It is clear that we can take any straight line at the distance $O(\varepsilon)$ from the rigid parts as an interface. Hence, it remains to prove that the law by Beavers and Joseph does not depend on the position of the interface. We have the following auxiliary result

Lemma 14. *Let $a < 0$ and let $\beta^{a,bl}$ be the solution for (45)-(48) with S replaced by $S_a = (0,1)\times\{a\}$, Z^+ by $Z_a^+ = (0,1)\times(a,+\infty)$ and $Z_a^- = Z_{BL}\setminus(S_a \cup Z_a^+)$. Then we have*

$$C_1^{a,bl} = C_1^{bl} - a. \tag{90}$$

This simple result will imply the invariance of the obtained law on the position of the interface.

Remark 15. *Let $\Omega_{a\varepsilon} = (0,b) \times (a\varepsilon, h)$ for $a < 0$ and let $\{u^{a,eff}, p^{a,eff}\}$ be a solution for (83)-(86) in $\Omega_{a\varepsilon}$, with (86) replaced by*

$$u_2^{a,eff} = 0 \qquad \text{and} \qquad u_1^{a,eff} + \varepsilon C_1^{a,bl} \frac{\partial u_1^{a,eff}}{\partial x_2} = 0 \qquad \text{on} \qquad \Sigma_a = (0,b) \times a\varepsilon. \tag{91}$$

The unique solution $\{u^{a,eff}, p^{a,eff}\}$ for (83)-(85), (91) is given by

$$u^{a,eff} = \Big(\frac{p_b - p_0}{2b\mu} \Big((x_2 - a\varepsilon)^2 - (x_2 - a\varepsilon - \varepsilon C_1^{a,bl}) \frac{(h - a\varepsilon)^2}{h - a\varepsilon - \varepsilon C_1^{a,bl}} \Big), 0 \Big)$$

for $a\varepsilon \le x_2 \le h$ and $p^{a,eff} = p^0 = \frac{p_b - p_0}{b} x_1 + p_0$ for $0 \le x_1 \le b$. By Lemma 14, $C_1^{a,bl} = C_1^{bl} - a$ and $u^{a,eff}(x) = u^{eff}(x) + O(\varepsilon^2)$. Therefore, a perturbation of the interface position for an $O(\varepsilon)$ implies a perturbation in the solution of $O(\varepsilon^2)$. Consequently, there is a freedom in fixing position of Σ. It influences the result only at the next order of the asymptotic expansion.

5 Conclusion

We presented a derivation of the interface law by Beavers and Joseph, describing the viscous flow over a porous bed. Using a combination of the homogenization and boundary layer approach we obtain the effective equations first heuristically and then justify them rigorously. Main results are the following:

1. We derive Saffman's form of the law by Beavers and Joseph. We note that the physical permeability is $k^\varepsilon = \varepsilon^2 K$ and the proportionality constant in (82) is proportional to $\sqrt{k^\varepsilon}$. The error is of order k^ε, as remarked by Saffman in [24]. It is important to point out that the parameter α from the expression (1) is determined through the auxiliary problems (45)-(48) and (51) by $\alpha = -\frac{1}{\varepsilon C_1^{bl}} > 0$.

2. Interface between the unconfined flow and the porous bed is an artificial mathematical boundary and there is a liberty to choose it in a layer having the pore size thickness. We have shown that a perturbation of the interface position of order $O(\varepsilon)$ implies a perturbation in the solution of order $O(\varepsilon^2)$.

3. In what concerns the pressure approximation, after extending it as in Section §2, we get the uniform bound on it. Hence the effective pressure in Ω_2 is $\mu C_\omega^{bl} \frac{\partial v_1^0}{\partial x_2}|_\Sigma$ and in Ω_1 it is zero. In our effective approximation (78) the two values are linked by a pressure boundary layer and for the full approximation effective pressure is continuous. In practice, one would like to neglect the pressure boundary layer. This leads to the discontinuity of the effective pressure field at Σ and does not confirm the effective pressure interface continuity law proposed in [10]. The price to pay for neglecting the boundary layer pressure is a bad approximation for the pressure in the neighborhood of the interface Σ. Proving the error estimate for the pressure approximation in the porous bed Ω_2^ε remains an open problem.

4. We do not address here derivation of the interface conditions for flows through porous media, containing two subdomains with different microstructures. The problem could be treated by the same technique used for deriving the law by Beavers and Joseph. For details we refer to [15]. In contrast with the law by Beavers and Joseph, in this case we do not have contribution of effective parameters from the corresponding boundary layers to the leading order approximation.

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