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Rigorous upscaling of the infinite adsorption rate reactive flow under dominant Peclet number through a pore.

this article is dedicated to 70th birthday of professor I. Aganović

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Abstract: In this paper we present a rigorous derivation of the effective model for enhanced diffusion through a narrow and long 2D pore. The analysis uses the anisotropic singular perturbation technique. Starting point is a local pore scale model describing the transport by convection and diffusion of a reactive solute. The solute particles undergo an adsorption process at the lateral tube boundary, with high adsorption rate. The transport and reaction parameters are such that we have large, dominant Peclet number with respect to the ratio of characteristic transversal and longitudinal lengths (the small parameter ε). We give a formal derivation of the model using the anisotropic multiscale expansion with respect to ε. Error estimates for the approximation of the physical solution, by the upscaled one, are presented in the energy norm as well as in L^∞ and L^1 norms with respect to the space variable. They give the approximation error as a power of ε and guarantee the validity of the upscaled model through the rigorous mathematical justification of the effective behavior for small ε.

1 Introduction

Taylor's dispersion is one of the most well-known examples of the role of transport in dispersing a flow carrying a dissolved solute. The simplest setting for observing it, is the injection of a solute into a slit channel. The solute is transported by Poiseuille's flow. In fact this problem could be studied in three distinct regimes: a) *diffusion-dominated mixing*, b) *Taylor dispersion-mediated mixing* and c) *chaotic advection.*

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In the first problem, the velocity is small and Peclet's number of order one or smaller. Molecular diffusion plays the dominant role in solute dispersion. This case is well-understood even for reactive flows (see e.g. the papers [4],[5], [9], [3]).

If the flow rate is increased so that Peclet's number Pe>> 1, then there is a time scale at which transversal molecular diffusion smears the contact discontinuity into a plug. In the fundamental paper [18], Taylor found an effective long-time axial diffusivity, proportional to the square of the transversal Peclet number and occurring in addition to the molecular diffusivity. If Taylor's effective dispersion is used in the 1D model, obtained by section averaging, as the effective diffusion coefficient, then the numerical experiences show good agreement with the solution of the complete physical problem.

In the third regime, we observe the turbulent mixing.

Our goal is the study of reactive flows through slit channels in the regime of Taylor dispersion-mediated mixing, using anisotropic singular perturbations. Despite a huge literature on the subject, with over 2000 citations to date, mathematical results on the subject are rare. We mention articles [2], [10], [14] and [17], but they address the mechanical dispersion in the absence of chemical reactions.

In this article we continue our research from the article [11], where a slit flow under dominant Peclet and Damkohler numbers was considered in the case of an irreversible, first order, heterogeneous chemical reaction with equilibrium between the liquid and the concentrations of adsorbed solutes.

Here we concentrate our attention to the case when the adsorption rate constant is infinitely large.

Let us write the precise setting of the problem: We consider the transport of a reactive solute by diffusion and convection by Poiseuille's velocity in a semi-infinite 2D channel. The solute particles do not react among themselves. Instead they undergo an adsorption process at the lateral boundary. We consider the following model for the solute concentration c^*:

a) transport through channel $\Omega^* = \{(x^*, y^*) : 0 < x^* < +\infty, |y^*| < H\}$

$$\frac{\partial c^*}{\partial t^*} + q(y^*)\frac{\partial c^*}{\partial x^*} - D^*\frac{\partial^2 c^*}{\partial (x^*)^2} - D^*\frac{\partial^2 c^*}{\partial (y^*)^2} = 0 \quad \text{in } \Omega^*, \tag{1}$$

where $q(z) = Q^*(1 - (z/H)^2)$ and where Q^* (velocity) and D^* (molecular diffusion) are positive constants.

b) reaction at channel wall $\Gamma^* = \{(x^*, y^*) : 0 < x^* < +\infty, |y^*| = H\}$

$$-D^*\partial_{y^*} c^* = K_e\frac{\partial c^*}{\partial t^*} \quad \text{on } \Gamma^*, \tag{2}$$

where K_e is the linear adsorption equilibrium constant.

The natural way of analyzing this problem is to introduce appropriate scales. This requires characteristic or reference values for the parameters in variables involved. The obvious transversal length scale is H. For all other quantities we use reference values denoted by the subscript R. Setting

$$c = \frac{c^*}{\hat{c}},\ x = \frac{x^*}{L_R},\ y = \frac{y^*}{H},\ t = \frac{t^*}{T_R},\ Q = \frac{Q^*}{Q_R},\ D = \frac{D^*}{D_R}, \tag{3}$$

where L_R is the " observation distance ", we obtain the dimensionless equations

$$\frac{\partial c}{\partial t} + \frac{Q_R T_R}{L_R}Q(1 - y^2)\frac{\partial c}{\partial x} - \frac{D_R T_R}{L_R^2}D\frac{\partial^2 c}{\partial x^2} - \frac{D_R T_R}{H^2}D\frac{\partial^2 c}{\partial y^2} = 0 \quad \text{in } \Omega \tag{4}$$

and

$$-\frac{D D_R T_R}{H K_e}\frac{\partial c}{\partial y} = \frac{\partial c}{\partial t} \quad \text{on } \Gamma, \tag{5}$$

where

$$\Omega = (0, +\infty) \times (-1, 1) \ \text{ and } \quad \Gamma = (0, +\infty) \times \{-1, 1\}. \tag{6}$$

The equations involve the time scales:

T_L = characteristic longitudinal time scale = $\dfrac{L_R}{Q_R}$,

T_T = characteristic transversal time scale = $\dfrac{H^2}{D_R}$,

T_C = superficial chemical reaction time scale $= \dfrac{K_e}{\varepsilon Q_R}$,

and the non-dimensional number $\mathbf{Pe} = \dfrac{L_R Q_R}{D_R}$ (Peclet number). In this paper we fix the reference time by setting $T_R = T_C = T_L$ and $K = K_e/H = T_C/T_L = O(1)$. K is the transversal Damkohler number. We are going to investigate the behavior of (4)-(5) with respect to the small parameter $\varepsilon = \dfrac{H}{L_R}$. Specifically, as in [11], we will derive expressions for the effective values of the dispersion coefficient and velocity, and an effective 1-D convection-diffusion equation for small values of ε. To carry out the analysis need to compare the dimensionless numbers with respect to ε. For this purpose we set $\mathbf{Pe} = \varepsilon^{-\alpha}$. Introducing the dimensionless numbers in equations (4)-(5) and considering constant initial/boundary conditions yields the problem :

$$\frac{\partial c^\varepsilon}{\partial t} + Q(1-y^2)\frac{\partial c^\varepsilon}{\partial x} = D\varepsilon^\alpha \frac{\partial^2 c^\varepsilon}{\partial x^2} + D\varepsilon^{\alpha-2}\frac{\partial^2 c^\varepsilon}{\partial y^2} \quad \text{in } \Omega^+ \times (0,T) \tag{7}$$

$$-D\varepsilon^{\alpha-2}\frac{\partial c^\varepsilon}{\partial y} = -D\frac{1}{\varepsilon^2 \mathbf{Pe}}\frac{\partial c^\varepsilon}{\partial y} = K\frac{\partial c^\varepsilon}{\partial t} \quad \text{on } \Gamma^+ \times (0,T) \tag{8}$$

$$c^\varepsilon(x,y,0) = 1 \quad \text{for } (x,y) \in \Omega^+, \tag{9}$$

$$c^\varepsilon(0,y,t) = 0 \quad \text{for } (y,t) \in (0,1)\times(0,T), \tag{10}$$

$$\frac{\partial c^\varepsilon}{\partial y}(x,0,t) = 0, \quad \text{for } (x,t) \in (0,+\infty)\times(0,T). \tag{11}$$

The later condition results from the $y-$symmetry of the solution. Further

$$\Omega^+ = (0,+\infty)\times(0,1), \quad \Gamma^+ = (0,+\infty)\times\{1\},$$

and T is an arbitrary chosen positive number.

We study the behavior of this problem as $\varepsilon \searrow 0$, while keeping the coefficients Q, D and K all $\mathcal{O}(1)$.

We note that more realistic boundary conditions at the inlet boundary are discussed in [7]. Nevertheless, our effective equation does not depend on the inlet boundary conditions.

In this paper we prove that the correct upscaling of the problem (7)-(11) gives the 1D parabolic problem :

$$(EFF) \qquad \begin{cases} \partial_t c + \dfrac{2Q}{3(1+K)}\partial_x c = (D\varepsilon^\alpha + \\ \dfrac{8}{945}\dfrac{Q^2}{D}\varepsilon^{2-\alpha} + \dfrac{4Q^2}{135D}\dfrac{K(7K+2)}{(1+K)^2}\varepsilon^{2-\alpha})\dfrac{\partial_{xx}c}{1+K} \text{ in } (0,+\infty)\times(0,T) \\ c|_{x=0} = 0,\ c|_{t=0} = 1,\ \partial_x c \in L^2((0,+\infty)\times(0,T)). \end{cases}$$

We note that for $K = 0$ and $\alpha = 1$ this is exactly the effective model of Taylor [18].

What is known concerning the derivation of the effective problem (EFF), with or without chemical reactions? Below we give a short overview.

⋄ In the absence of chemical reactions, R. Aris [1] presented a formal derivation using the method of moments.

⋄ For the case of reactive flows with a first order equilibrium chemical reaction with adsorbed solutes, we refer to [11], where the problem is rigorously solved. It covers also the classical Taylor's dispersion, which corresponds to $K = 0$.

⋄ Flow with chemistry, as described by equation (2), is considered by M.A. Paine, R.G. Carbonell and S. Whitaker [13]. They noted that the equation for the difference between the physical and averaged concentrations is not closed, since it contains a dispersive source term $\dfrac{\partial}{\partial x} < \overline{q}_x \overline{c} >$. Then they multiplied the equation for $\overline{c}$ by $\overline{q}_x$ and got an equation for $< \overline{q}_x \overline{c} >$. Nevertheless, a dispersive transport term $\dfrac{\partial}{\partial x} < \overline{q}_x^2 \overline{c} >$ and clearly the procedure enters the same difficulty as the method of moments: there is an infinite system of equations. Paine et al used the "single-point" closure schemes of turbulence modeling by Launder to obtain a closed model for the averaged concentration. We note that their effective equations contain non-local terms depending of the solution and in fact the effective coefficients aren't really given.

It should be noted that the real interest is to derive *dispersion equations* for reactive flows through porous media and our results are just the first step in that direction. Our technique is strongly motivated by the paper by J. Rubinstein and R. Mauri [17], where effective dispersion and convection in porous media is studied using the homogenization technique.

Averaging the concentration in a tube with dissolution/precipitation occurring on the wall and with **Pe**= $\mathcal{O}(1)$, is considered in [5].

Plan of the paper is the following : In the Section 2 we study the homogenized problem. It turns out that it has an explicit solution having the form of moving Gaussian as the 1D boundary layers of parabolic equations, when viscosity goes to zero (see [8]). Its behavior with respect to ε and t is very singular.

Then in Section 3 we give a justification of a lower order approximation, using a simple energy argument. In fact such approximation doesn't use Taylor's dispersion formula and gives an error of the same order in $L^\infty(L^2)$ as the solution to the linear transport equation.

In the Section 4 we give a formal derivation of the upscaled problem (EFF), using the approach from [17].

Then in Section 5 we prove that the effective concentration satisfying the corresponding 1D parabolic problem, with Taylor's diffusion coefficient and the reactive correction, is an approximation in $L^\infty(L^2)$ for the physical concentration.

The validation of our result by numerical simulations is in the preprint [6].

To satisfy the curiosity of the reader not familiar with singular perturbation techniques, we give here the simplified version of the results stated in Theorem 3. For simplicity, we compare only the physical concentration c^ε with c. Throughout the paper $H(x)$ is Heaviside's function:

$$H(x) = 1,\ x > 0, \quad H(x) = 0,\ x \le 0. \tag{12}$$

Furthermore, using the elementary parabolic theory one concludes that the problem (7)-(11) has a unique bounded variational solution c^ε, with square integrable gradient in x and y. c^ε belongs to C^∞ for $x > 0$ and it stabilizes to 1 exponentially fast when $x \to \infty$.

Theorem 1 *Let c be given by (EFF). Then we have*

$$\|t^3\big(c^\varepsilon - c\big)\|_{L^2(0,T;L^2_{loc}(\Omega^+))} \le C\big(\varepsilon^{2-5\alpha/4}H(1-\alpha) + \varepsilon^{3/2-3\alpha/4}H(\alpha-1)\big) \tag{13}$$

$$\|t^3\partial_y c^\varepsilon\|_{L^2(0,T;L^2_{loc}(\Omega^+))} \le C\big(\varepsilon^{2-5\alpha/4}H(1-\alpha) + \varepsilon^{3/2-3\alpha/4}H(\alpha-1)\big) \tag{14}$$

$$\|t^3\partial_x\big(c^\varepsilon - c\big)\|_{L^2(0,T;L^2_{loc}(\Omega^+))} \le C\big(\varepsilon^{2-7\alpha/4}H(1-\alpha) + \varepsilon^{3/2-5\alpha/4}H(\alpha-1)\big) \tag{15}$$

Note that in estimate (14) c has disappeared since it is only x and t dependent. This estimate is better than estimate (15) because of the large $\mathcal{O}(\varepsilon^{\alpha-2})$ transversal diffusivity. After doing additional estimates, as in [11], we get

Corollary 1

$$\|t^3(c^\varepsilon - c)\|_{L^\infty((0,T)\times\Omega^+)} \le \begin{cases} C\varepsilon^{2-3\alpha/2}, & \textit{if}\ \ \alpha < 1, \\ C\varepsilon^{3/2-\alpha-\delta},\ \forall\delta > 0, & \textit{if}\ \ 2 > \alpha \ge 1. \end{cases} \tag{16}$$

$$\|t^3\big(c^\varepsilon - c\big)\|_{L^2(0,T;L^1_{loc}(\Omega^+))} \le C\varepsilon^{2-\alpha} \tag{17}$$

Our result could be restated in dimensional form:

Theorem 2 *Let us suppose that* $L_R > \max\{D_R/Q_R, Q_R H^2/D_R, H\}$. *Then the upscaled dimensional approximation for (1) reads*

$$(1+K)\frac{\partial c^{*,eff}}{\partial t^*} + \frac{2}{3}Q^*\frac{\partial c^{*,eff}}{\partial x^*} = D^*\Big(1 + \big(\frac{8}{945} + \frac{4}{135}\frac{K(7K+2)}{(1+K)^2}\big)\mathbf{Pe}_T^2\Big)\frac{\partial^2 c^{*,eff}}{\partial (x^*)^2}, \tag{18}$$

where $\mathbf{Pe}_T = \dfrac{Q^* H}{D^*}$ *is the transversal Peclet number and* $K = K_e/H$ *is the transversal Damkohler number.*

One could try to get even higher order approximations. Unfortunately, our procedure from Section 4 then leads to higher order differential operators and it is not clear if they are easy to handle. In the absence of the boundaries, determination of the higher order terms using the computer program REDUCE was undertaken in [10].

2 Study of the upscaled diffusion-convection equation on the half-line

For $\bar{Q}, \bar{D}$ and $\varepsilon > 0$, we consider the problem

$$\begin{cases} \partial_t u + \bar{Q}\partial_x u = \gamma \bar{D} \partial_{xx} u \ \text{ in } \ (0,+\infty)\times(0,T), \\ \partial_x u \in L^2((0,+\infty)\times(0,T)), \\ u(x,0) = 1 \ \text{ in } \ (0,+\infty), \quad u(0,t) = 0 \ \text{ at } \ x = 0. \end{cases} \tag{19}$$

The unique solution is obtained using the Laplace transform and reads

$$u(x,t) = 1 - \frac{1}{\sqrt{\pi}} \left[e^{\dfrac{\bar{Q}x}{\gamma \bar{D}}} \int_{\frac{x+t\bar{Q}}{2\sqrt{\gamma \bar{D} t}}}^{+\infty} e^{-\eta^2} \, d\eta + \int_{\frac{x-t\bar{Q}}{2\sqrt{\gamma \bar{D} t}}}^{+\infty} e^{-\eta^2} \, d\eta \right] \tag{20}$$

The explicit formula allows us to find the exact behavior of u with respect to γ. We note that for $\alpha \in [0,1]$, we will set $\gamma = \varepsilon^{\alpha}$. For $\alpha \in [1,2)$, we choose $\gamma = \varepsilon^{2-\alpha}$. The derivatives of u are found using the computer program MAPLE and then their norms are estimated. Since the procedure is standard, we don't give the details. In more general situations there are no explicit solutions and these estimates could be obtained using the technique and results from [8].

First, by the maximum principle we have

$$0 \le u(x,t) \le 1 \tag{21}$$

Next we estimate the difference between $\chi_{x<\bar{Q}t}$ and u. We have

Lemma 1 *Function u satisfies the estimates*

$$\int_0^{\infty} |\chi_{\{x>\bar{Q}t\}} - u(t,x)| \, dx = 3\sqrt{\gamma \bar{D} t} + C\gamma \tag{22}$$

$$\|\chi_{\{x>\bar{Q}t\}} - u\|_{L^\infty(0,T;L^p((0,+\infty)))} \le C\gamma^{1/(2p)}, \quad \forall p \in (1,+\infty). \tag{23}$$

Proof We estimate the difference between $\chi_{x<\bar{Q}t}$ and $1-u$. We have

$$\int_0^{\infty} |\chi_{\{x<\bar{Q}t\}} - 1 + u(t,x)| \, dx = \frac{1}{\sqrt{\pi}} (I_1 + I_2 + I_3 + I_4) \tag{24}$$

$$I_1 = \int_{t\bar{Q}}^{\infty} \int_{\frac{x-t\bar{Q}}{2\sqrt{\varepsilon \bar{D} t}}}^{+\infty} e^{-\eta^2} \, d\eta dx = \sqrt{\varepsilon \bar{D} t} \int_0^{\infty} 2\eta e^{-\eta^2} \, d\eta = \sqrt{\varepsilon \bar{D} t} \tag{25}$$

$$I_2 = \int_{t\bar{Q}}^{\infty} \int_{\frac{x+t\bar{Q}}{2\sqrt{\varepsilon \bar{D} t}}}^{+\infty} e^{\bar{Q}x/(\varepsilon \bar{D}) - \eta^2} \, d\eta dx = \frac{\varepsilon \bar{D}}{\bar{Q}} \int_{\bar{Q}\sqrt{t/(\varepsilon \bar{D})}}^{\infty} \big(\exp\{\frac{\bar{Q}}{\varepsilon \bar{D}}(2\sqrt{\varepsilon \bar{D} t}\eta - t\bar{Q})\} - \exp\{\frac{\bar{Q}^2 t}{\varepsilon \bar{D}}\}\big) e^{-\eta^2} \, d\eta \le \frac{\varepsilon \bar{D}\sqrt{\pi}}{2\bar{Q}} \tag{26}$$

$$I_3=\int_0^{t\bar Q}\big(\sqrt{\pi}-\int_{\frac{x-t\bar Q}{2\sqrt{\varepsilon\bar D t}}}^{+\infty}e^{-\eta^2}\,d\eta\big)\,dx\le\int_{\frac{-\sqrt{t}\bar Q}{2\sqrt{\varepsilon\bar D}}}^{+\infty}2\sqrt{\varepsilon\bar D t}e^{-\eta^2}\,\eta d\eta\le\sqrt{\varepsilon\bar D t} \tag{27}$$

$$I_4=\int_0^{t\bar Q}\int_{\frac{x+t\bar Q}{2\sqrt{\varepsilon\bar D t}}}^{+\infty}e^{\bar Q x/(\varepsilon\bar D)-\eta^2}\,d\eta dx=\frac{\varepsilon\bar D}{\bar Q}\Big(\int_{\frac{\sqrt{t}\bar Q}{\sqrt{\varepsilon\bar D}}}^{+\infty}(\exp\{\frac{\bar Q^2 t}{\varepsilon\bar D}\}-1)e^{-\eta^2}\,d\eta+$$
$$\int_{\frac{\sqrt{t}\bar Q}{2\sqrt{\varepsilon\bar D}}}^{\frac{\sqrt{t}\bar Q}{\sqrt{\varepsilon\bar D}}}(\exp\{\frac{\bar Q}{\varepsilon\bar D}(2\sqrt{\varepsilon t\bar D}\eta-t\bar Q)\}-1)e^{-\eta^2}\,d\eta\Big)\le\sqrt{\varepsilon\bar D t}+\frac{\varepsilon\bar D\sqrt{\pi}}{2\bar Q} \tag{28}$$

and the estimate (22) is proved.

We prove (23) analogously. □

For the derivatives of u the following estimates hold

Lemma 2 *Let ζ be defined by*

$$\zeta(t)=\begin{cases}(\frac{t}{\bar D\gamma})^r & \text{for } 0\le t\le\bar D\gamma,\\ 1 & \text{otherwise,}\end{cases} \tag{29}$$

with $r\ge q\ge 1$. Then we have

$$\|\zeta(t)(|\partial_t u|+|\partial_x u|)\|_{L^q((0,T)\times(0,+\infty))}\le C(\gamma\bar D)^{\min\{1/(2q)-1/2,2/q-1\}},\ q\ne 3 \tag{30}$$

$$\|\zeta(t)(|\partial_t u|+|\partial_x u|)\|_{L^3((0,T)\times(0,+\infty))}\le C\big((\gamma\bar D)^{-1}\log(\frac{1}{\gamma\bar D})\big)^{1/3} \tag{31}$$

Proof Here we should estimate the derivatives of u. We have

$$\int_0^\infty|\partial_x u|^q\,dx=\int_0^\infty(\varepsilon\bar D t\pi)^{-q/2}\exp\{-\frac{q(x-t\bar Q)^2}{4\varepsilon\bar D t}\}\cdot$$
$$|\frac{\bar Q\sqrt{t}}{\sqrt{\varepsilon\bar D}}\exp\{\frac{q(x+t\bar Q)^2}{4\varepsilon\bar D t}\}\int_{\frac{x+t\bar Q}{2\sqrt{\varepsilon\bar D t}}}^\infty e^{-\eta^2}\,d\eta-1|^q\,dx\le$$
$$\int_0^\infty(\varepsilon\bar D t\pi)^{-q/2}\exp\{-\frac{q(x-t\bar Q)^2}{4\varepsilon\bar D t}\}\,dx\le 2(\varepsilon\bar D t\pi)^{(1-q)/2}q^{-1/2} \tag{32}$$

$$\int_0^\infty|\partial_t u|^q\,dx=(4\varepsilon\bar D\pi)^{-q/2}t^{-3q/2}\int_0^\infty x^q\exp\{-\frac{q(x-t\bar Q)^2}{4\varepsilon\bar D t}\}\,dx$$
$$\le C_{0,q}(\varepsilon\bar D)^{(1-q)/2}t^{1/2-q}\max\{t,\bar D\varepsilon\}^{q/2} \tag{33}$$

and (30)-(31) follows. □

Now we estimate the second derivatives :

Lemma 3 *Let ζ be defined by (29). Then the second derivatives of u satisfy the estimates*

$$\|\zeta(t)u_{tt}\|_{L^q((0,T)\times(0,+\infty))}+\|\zeta(t)u_{tx}\|_{L^q((0,T)\times(0,+\infty))}+\|\zeta(t)u_{xx}\|_{L^q((0,T)\times(0,+\infty))}$$
$$\le C_q(\gamma\bar D)^{\min\{1/(2q)-1,2/q-2\}},\ q\ne 3/2 \tag{34}$$

$$\|\zeta(t)u_{tt}\|_{L^{3/2}((0,T)\times(0,+\infty))}+\|\zeta(t)u_{tx}\|_{L^{3/2}((0,T)\times(0,+\infty))}+\|\zeta(t)u_{xx}\|_{L^{3/2}((0,T)\times(0,+\infty))}$$
$$\le C\big((\gamma\bar D)^{-1}\log(\frac{1}{\gamma\bar D})\big)^{2/3} \tag{35}$$

Proof Now we estimate the second derivatives :

$$\int_0^\infty |\partial_{tt}u|^q\ dx = (8\sqrt{\pi})^{-q}(\varepsilon\bar{D})^{-3q/2}t^{-7q/2}\int_0^{+\infty} x^q|x^2 - t^2\bar{Q}^2 - 6\varepsilon\bar{D}t|^q\cdot$$
$$\exp\{-\frac{q(x-t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\ dx = \left\{\begin{array}{c} x - t\bar{Q} = 2z\sqrt{\varepsilon\bar{D}t/q} \\ dx = 2\sqrt{\varepsilon\bar{D}t/q}\ dz \end{array}\right\} = (8\sqrt{\pi})^{-q}(\varepsilon\bar{D})^{-3q/2}t^{-7q/2}\cdot$$
$$\int_{-\frac{\bar{Q}}{2}\sqrt{tq/(\varepsilon\bar{D})}}^{+\infty}(t\bar{Q}+2z\sqrt{\varepsilon\bar{D}t/q})^q|2z\sqrt{\varepsilon\bar{D}t/q}(2\bar{Q}t+2z\sqrt{\varepsilon\bar{D}t/q}) - 6\varepsilon\bar{D}t|^q\cdot$$
$$2e^{-z^2}\sqrt{\varepsilon\bar{D}t/q}\ dz \le C_q(\varepsilon\bar{D})^{1/2-q}t^{1/2-2q}\max\{t,\bar{D}\varepsilon\}^q \tag{36}$$

$$\int_0^\infty |\partial_{tx}u|^q\ dx = (4\sqrt{\pi})^{-q}(\varepsilon\bar{D})^{-3q/2}t^{-5q/2}\int_0^{+\infty} |x(x-\bar{Q}t) - 2\varepsilon\bar{D}t|^q \exp\{-\frac{q(x-t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\ dx$$
$$= \left\{\begin{array}{c} x - t\bar{Q} = 2z\sqrt{\varepsilon\bar{D}t/q} \\ dx = 2\sqrt{\varepsilon\bar{D}t/q}\ dz \end{array}\right\} = (4\sqrt{\pi})^{-q}(\varepsilon\bar{D})^{-3q/2}t^{-5q/2}\int_{-\frac{\bar{Q}}{2}\sqrt{tq/(\varepsilon\bar{D})}}^{+\infty} e^{-z^2}\cdot$$
$$|(2\bar{Q}t+2z\sqrt{\varepsilon\bar{D}t/q})2z\sqrt{\varepsilon\bar{D}t/q} - 2\varepsilon\bar{D}t|^q 2\sqrt{\varepsilon\bar{D}t/q}\ dz \le$$
$$C_q(\varepsilon\bar{D})^{1/2-q}t^{1/2-3q/2}\max\{t,\bar{D}\varepsilon\}^{q/2} \tag{37}$$

Estimating $\partial_{xx}u$ is slightly more complicated. We analyse the expression for $\partial_{xx}u$ and find out that for $x \le \bar{Q}t$

$$\frac{x(x-\bar{Q}t)}{2t\sqrt{t}(x+\bar{Q}t)} - \frac{2\bar{Q}^2D\varepsilon t\sqrt{t}}{(x+Qt)^3} \le \frac{Q^2}{\sqrt{\varepsilon\bar{D}}}\exp\{\frac{(x+t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\int_{\frac{x+t\bar{Q}}{2\sqrt{\varepsilon\bar{D}t}}}^{+\infty} e^{-\eta^2}\ d\eta +$$
$$\frac{x}{2t\sqrt{t}} - \frac{\bar{Q}}{\sqrt{t}} \le \frac{x(x-\bar{Q}t)}{2t\sqrt{t}(x+\bar{Q}t)} \le 0$$

and for $x > \bar{Q}t$

$$0 \le \frac{\bar{Q}}{2\sqrt{t}} - \frac{Q^2}{\sqrt{\varepsilon\bar{D}}}\exp\{\frac{(x+t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\int_{\frac{x+t\bar{Q}}{2\sqrt{\varepsilon\bar{D}t}}}^{+\infty} e^{-\eta^2}\ d\eta \le \frac{\bar{Q}(x-\bar{Q}t)}{2\sqrt{t}(x+\bar{Q}t)} + \frac{2\bar{Q}^2D\varepsilon t\sqrt{t}}{(x+Qt)^3}.$$

$$\text{Hence}\ \int_0^\infty |\partial_{xx}u|^q\ dx = \int_0^{\bar{Q}t}|\partial_{xx}u|^q\ dx + \int_{\bar{Q}t}^{+\infty}|\partial_{xx}u|^q\ dx\ \text{and}$$
$$\int_0^{\bar{Q}t}|\partial_{xx}u|^q\ dx \le 2^{q-1}(\sqrt{\pi})^{-q}(\varepsilon\bar{D})^{-3q/2}t^{-3q/2}\int_0^\infty \exp\{-\frac{q(x-t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\cdot$$
$$\Big(\frac{x^q(\bar{Q}t-x)^q}{2^q(x+\bar{Q}t)^q} + \frac{\bar{Q}^{2q}2^q(\bar{D}\varepsilon)^q}{(x+\bar{Q}t)^{3q}}\Big)\ dx = \left\{\begin{array}{c} x - t\bar{Q} = 2z\sqrt{\varepsilon\bar{D}t/q} \\ dx = 2\sqrt{\varepsilon\bar{D}t/q}\ dz \end{array}\right\} \le C_q(\varepsilon\bar{D}t)^{1/2-3q/2}$$
$$\int_{-\frac{\sqrt{q t}\bar{Q}}{2\sqrt{\varepsilon\bar{D}}}}^0 e^{-\eta^2}\Big((\varepsilon\bar{D}t)^{q/2} + (\bar{D}\varepsilon)^q\Big)\ d\eta =$$
$$C_q(\varepsilon\bar{D})^{1/2-q}t^{1/2-3q/2}\Big(\max\{\bar{D}\varepsilon,t\}\Big)^{q/2} \tag{38}$$

Furthermore

$$\int_{\bar{Q}t}^{+\infty} |\partial_{xx}u|^q \ dx \leq C_q(\varepsilon\bar{D})^{-3q/2}\int_{\bar{Q}t}^{+\infty} \exp\{-\frac{q(x-t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\Big(|\frac{x-\bar{Q}t}{t\sqrt{t}}|^q +$$

$$|\frac{\bar{Q}}{2\sqrt{t}} - \frac{Q^2}{\sqrt{\varepsilon\bar{D}}}\exp\{\frac{(x+t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\int_{\frac{x+t\bar{Q}}{2\sqrt{\varepsilon\bar{D}t}}}^{+\infty} e^{-\eta^2}\ d\eta|^q\Big) \ dx \leq$$

$$C(\varepsilon\bar{D})^{1/2-q}t^{1/2-3q/2}\Big(\max\{\bar{D}\varepsilon, t\}\Big)^{q/2} \tag{39}$$

and (34)-(35) now follows. □

For the 3rd order derivatives we have :

Lemma 4 *Let ζ be defined by (29) . Then*

$$\|\partial_{xxx}(\zeta(t)u)\|_{L^q((0,T)\times(0,+\infty))} + \|\zeta(t)\partial_{xxt}u\|_{L^q((0,T)\times(0,+\infty))}$$
$$+\|\zeta(t)\partial_{xtt}u\|_{L^q((0,T)\times(0,+\infty))} \leq C_q(\gamma\bar{D})^{2/q-3}, \ q>1 \tag{40}$$
$$\|\partial_{xxx}(\zeta(t)u)\|_{L^1((0,T)\times(0,+\infty))} + \|\zeta(t)\partial_{xxt}u\|_{L^1((0,T)\times(0,+\infty))}$$
$$+\|\zeta(t)\partial_{xtt}u\|_{L^1((0,T)\times(0,+\infty))} \leq C_1(\gamma\bar{D})^{-1}\log\frac{1}{\gamma\bar{D}} \tag{41}$$

Proof For the third derivatives we have:

$$\int_0^\infty |\partial_{xxx}u|^q \ dx = (\sqrt{\pi})^{-q}(\varepsilon\bar{D}t)^{-5q/2}\int_0^\infty \exp\{-\frac{q(x-t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\cdot$$

$$|\frac{\bar{Q}^3t^{5/2}}{\sqrt{\varepsilon\bar{D}}}\exp\{\frac{q(x+t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\int_{\frac{x+t\bar{Q}}{2\sqrt{\varepsilon\bar{D}t}}}^\infty e^{-\eta^2}\ d\eta - \bar{Q}^2t^2 + 3x\bar{Q}t/4 - x^2/4 + \varepsilon\bar{D}t/2|^q \ dx \leq$$

$$C(\varepsilon\bar{D}t)^{-5q/2}\int_0^\infty \exp\{-\frac{q(x-t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\Big(|x-\bar{Q}t|^{2q} + (\bar{D}\varepsilon t)^q\Big) \ dx =$$

$$\left\{\begin{array}{c} x - t\bar{Q} = 2z\sqrt{\varepsilon\bar{D}t/q} \\ dx = 2\sqrt{\varepsilon\bar{D}t/q}\ dz \end{array}\right\} \leq C_q(\varepsilon\bar{D}t)^{1/2-3q/2} \quad \text{and} \tag{42}$$

$$\int_0^\infty |\partial_{xxt}u|^q \ dx \leq C_q(\varepsilon\bar{D})^{-3q/2}t^{-5q/2}\int_0^\infty \exp\{-\frac{q(x-t\bar{Q})^2}{4\varepsilon\bar{D}t}\}\Big\{|\bar{Q}t-2x|^q +$$

$$|x(\bar{Q}t-x)+2\varepsilon\bar{D}t|^q|x-\bar{Q}t|^q(\varepsilon\bar{D})^{-q}t^{-q}\Big\} \ dx \leq C_q(\varepsilon\bar{D})^{1/2-3q/2}t^{1/2-2q}\max\{\varepsilon\bar{D}, t\}^{q/2} \tag{43}$$

Similarly,

$$\int_0^\infty |\partial_{xtt}u|^q \ dx \leq C_q(\varepsilon\bar{D})^{1/2-3q/2}t^{1/2-5q/2}\max\{\varepsilon\bar{D}, t\}^q \tag{44}$$

Hence we have proved (40)-(41). □

3 A simple L^2 error estimate

The simplest way to average the problem (7)-(11) is to take the mean value with respect to y. Supposing that the mean of the product is the product of the means, which is in general wrong, we get the following problem for the " averaged " concentration $c_0^{eff}(x,t)$:

$$\begin{cases} (1+K)\dfrac{\partial c_0^{eff}}{\partial t} + \dfrac{2Q}{3}\dfrac{\partial c_0^{eff}}{\partial x} = \varepsilon^\alpha D\dfrac{\partial^2 c_0^{eff}}{\partial x^2} \ \text{ in } \ (0,+\infty)\times(0,T), \\ \partial_x c_0^{eff} \in L^2((0,+\infty)\times(0,T)), \quad c_0^{eff}|_{t=0} = 1, \quad c_0^{eff}|_{x=0} = 0. \end{cases} \tag{45}$$

This is the problem (19) with $\tilde{Q} = \frac{2}{3}\frac{Q}{1+K}$ and $\bar{D} = \frac{D}{1+K}$. We will call this problem the "**simple closure approximation**". The small parameter γ is equal to ε^α. Let the operator $\mathcal{L}^\varepsilon$ be given by

$$\mathcal{L}^\varepsilon \zeta = \frac{\partial \zeta}{\partial t} + Q(1-y^2)\frac{\partial \zeta}{\partial x} - D\varepsilon^\alpha \left(\frac{\partial^2 \zeta}{\partial x^2} + \varepsilon^{-2}\frac{\partial^2 \zeta}{\partial y^2} \right) \tag{46}$$

The non-dimensional physical concentration c^ε satisfies (7)-(11), i.e.

$$\mathcal{L}^\varepsilon c^\varepsilon = 0 \quad \text{in} \ (0,+\infty)\times(0,1)\times(0,T) \tag{47}$$

$$c^\varepsilon(0,y,t) = 0 \quad \text{on} \ (0,1)\times(0,T) \tag{48}$$

$$\partial_y c^\varepsilon(x,0,t) = 0 \quad \text{on} \ (0,+\infty)\times(0,T) \tag{49}$$

$$-D\varepsilon^{\alpha-2}\partial_y c^\varepsilon(x,1,t) = K\partial_t c^\varepsilon(x,1,t) \quad \text{on} \ (0,+\infty)\times(0,T) \tag{50}$$

$$c^\varepsilon(x,y,0) = 1 \quad \text{on} \ (0,+\infty)\times(0,1) \tag{51}$$

We want to approximate c^ε by c_0^{eff}. Then

$$\mathcal{L}^\varepsilon(c_0^{eff}) = -K\partial_t c_0^{eff} + Q\partial_x c_0^{eff}(1/3 - y^2) = R^\varepsilon$$

$$\mathcal{L}^\varepsilon(c^\varepsilon - c_0^{eff}) = -R^\varepsilon \ \text{ in } \ (0,+\infty)\times(0,1)\times(0,T) \quad \text{and} \tag{52}$$

$$-D\varepsilon^{\alpha-2}\partial_y(c^\varepsilon(x,1,t) - c_0^{eff}) = K\partial_t c^\varepsilon(x,1,t) \quad \text{on} \ (0,+\infty)\times(0,T) \tag{53}$$

Let $\Psi(x) = 1/(x+1)$. Then $(\partial_x \Psi)^2/\Psi^2 \le 4\Psi^2$. We have the following proposition, which will be useful in getting the estimates :

Proposition 1 *Let g^ε, ξ_0^ε and R^ε be such that $\Psi g^\varepsilon \in H^1(\Omega^+\times(0,T))$, $\Psi\xi_0^\varepsilon \in L^2(\Omega^+)$ and $\Psi R^\varepsilon \in L^2(\Omega^+\times(0,T))$. Let ξ be a bounded function, such that $\Psi\xi \in C([0,T];L^2(\Omega^+))$, $\Psi\nabla_{x,y}\xi \in L^2(\Omega^+\times(0,T))$, and satisfying the system*

$$\mathcal{L}^\varepsilon(\xi) = -R^\varepsilon \quad in \ \ \Omega^+\times(0,T) \tag{54}$$

$$-D\varepsilon^{\alpha-2}\partial_y \xi|_{y=1} = K\partial_t \xi|_{y=1} + g^\varepsilon|_{y=1} \ \ and \ \ \partial_y\xi|_{y=0} = 0 \ \ on \ \ (0,+\infty)\times(0,T) \tag{55}$$

$$\xi|_{t=0} = \xi_0^\varepsilon \quad on \ \ \Omega^+ \quad and\, \xi|_{x=0} = 0 \ on \ \ (0,1)\times(0,T). \tag{56}$$

Then we have the following energy estimate

$$\mathcal{E}(\xi,t) = \frac{1}{2}\int_{\Omega^+}\Psi(x)^2\xi^2(t)\ dxdy + \frac{D}{2}\varepsilon^\alpha\int_0^t\int_{\Omega^+}\Psi(x)^2\Big\{\varepsilon^{-2}|\partial_y\xi|^2 +$$
$$|\partial_x\xi|^2\Big\}\ dxdyd\tau + \frac{K}{2}\int_0^{+\infty}\xi^2(t)|_{y=1}\Psi^2(x)\ dx \le -\int_0^t\int_{\Omega^+}\Psi(x)^2R^\varepsilon\xi\ dxdyd\tau -$$
$$\int_0^t\int_0^{+\infty} g^\varepsilon|_{y=1}\xi|_{y=1}\Psi^2(x)\ dxd\tau + 2D\varepsilon^\alpha\int_0^t\int_{\Omega^+}\Psi(x)^2\xi^2\ dxdyd\tau + \frac{1}{2}\int_{\Omega^+}\Psi(x)^2(\xi_0^\varepsilon)^2(t)\ dxdy. \tag{57}$$

Proof See [11] .

This simple proposition allows us to prove

Proposition 2 *In the setting of this Section we have*

$$\|\Psi(x)(c^\varepsilon - c_0^{eff})\|_{L^\infty(0,T;L^2((0,+\infty)\times(0,1))} \le \varepsilon^{1-\alpha/2}\frac{F^0}{\sqrt{D}} \tag{58}$$

$$\|\Psi(x)\partial_x(c^\varepsilon - c_0^{eff})\|_{L^2(0,T;L^2((0,+\infty)\times(0,1))} \le \varepsilon^{1-\alpha}\frac{F^0}{D} \tag{59}$$

$$\|\Psi(x)\partial_y(c^\varepsilon - c_0^{eff})\|_{L^2(0,T;L^2((0,+\infty)\times(0,1))} \le \varepsilon^{2-\alpha}\frac{F^0}{D}, \tag{60}$$

$$where \quad F^0 = C_1^F\|\partial_x c_0^{eff}\|_{L^2(O_T)} + C_2^F\|\partial_t c_0^{eff}\|_{L^2(O_T)} \le C_3^F\varepsilon^{-\alpha/4} \tag{61}$$

Proof It is exactly the same as the corresponding proof from [11], just F^0 is slightly different.

Corollary 2

$$\|c^\varepsilon - c_0^{eff}\|_{L^\infty(0,T;L^2_{loc}((0,+\infty)\times(0,1))} \le C\varepsilon^{1-3\alpha/4} \tag{62}$$

Remark 1 For $\alpha > 4/3$ the estimate (62) is of no interest.

4 The formal 2-scale expansion leading to Taylor's dispersion

The estimate obtained in the previous Section isn't satisfactory. At the other hand, it is known that the Taylor dispersion model gives a very good 1D approximation. With this motivation we briefly explain how to obtain formally the higher precision effective models and notably the variant of Taylor's dispersion formula, by the 2-scale asymptotic expansion.

We start with the problem (47)-(51) and search for c^ε in the form

$$c^\varepsilon = c^0(x,t;\varepsilon) + \varepsilon^{2-\alpha} c^1(x,y,t) + \varepsilon^{2(2-\alpha)} c^2(x,y,t) + \dots \tag{63}$$

After introducing (63) into the equation (47) we get

$$\varepsilon^0 \Big\{ \partial_t c^0 + Q(1-y^2)\partial_x c^0 - D\partial_{yy} c^1 \Big\} + \varepsilon^{2-\alpha} \Big\{ \partial_t c^1 + Q(1-y^2)\partial_x c^1 - D\varepsilon^{2(\alpha-1)} \partial_{xx} c^0 - D\varepsilon^\alpha \partial_{xx} c^1 - D\partial_{yy} c^2 \Big\} = O(\varepsilon^{2(2-\alpha)}) \tag{64}$$

In order to have (64) for every $\varepsilon \in (0, \varepsilon_0)$, all coefficients in front of the powers of ε should be zero.

The problem corresponding to the order ε^0 is

$$\begin{cases} -D\partial_{yy} c^1 = -Q(1/3 - y^2)\partial_x c^0 - \big(\partial_t c^0 + 2Q\partial_x c^0/3\big) \ \text{ on } \ (0,1), \\ \partial_y c^1 = 0 \text{ on } \ y = 0 \ \text{ and } \ -D\partial_y c^1 = K\partial_t c^0 \text{ on } \ y = 1 \end{cases} \tag{65}$$

for every $(x,t) \in (0,+\infty) \times (0,T)$. By Fredholm's alternative, the problem (65) has a solution if and only if

$$(1+K)\partial_t c^0 + 2Q\partial_x c^0/3 = 0 \quad \text{ in } \ (0,L) \times (0,T). \tag{66}$$

Unfortunately our initial and boundary data are incompatible and the hyperbolic equation (66) has a discontinuous solution. Since the asymptotic expansion for c^ε involves derivatives of c^0, the equation (66) doesn't suit our needs. In [2] the difficulty was overcome by supposing compatible initial and boundary data. We proceed by following an idea from [17] and suppose that

$$(1+K)\partial_t c^0 + 2Q\partial_x c^0/3 = O(\varepsilon^{2-\alpha}) \quad \text{ in } \ (0,+\infty) \times (0,T). \tag{67}$$

The hypothesis (67) will be justified *a posteriori*, after getting an equation for c^0.

Hence (65) reduces to

$$\begin{cases} -D\partial_{yy} c^1 = -Q(1/3 - y^2)\partial_x c^0 + K\partial_t c^0 \text{ on } \ (0,1), \\ \partial_y c^1 = 0 \text{ on } \ y = 0 \ \text{ and } \ -D\partial_y c^1 = K\partial_t c^0 \text{ on } \ y = 1 \end{cases} \tag{68}$$

for every $(x,t) \in (0,+\infty) \times (0,T)$, and we have

$$c^1(x,y,t) = \frac{Q}{D}\Big(\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180}\Big)\partial_x c^0 + \frac{K}{D}\Big(\frac{1}{6} - \frac{y^2}{2}\Big)\partial_t c^0 + C_0(x,t), \tag{69}$$

where C_0 is an arbitrary function.

Let us go to the next order. Then we have

$$\begin{cases} -D\partial_{yy} c^2 = -Q(1-y^2)\partial_x c^1 + D\varepsilon^{2(\alpha-1)}\partial_{xx} c^0 - \partial_t c^1 \\ +D\varepsilon^\alpha \partial_{xx} c^1 - \varepsilon^{\alpha-2}\big((1+K)\partial_t c^0 + 2Q\partial_x c^0/3\big) \quad \text{ on } \ (0,1), \\ \partial_y c^2 = 0 \text{ on } \ y = 0 \ \text{ and } \ -D\partial_y c^2 = K\partial_t c^1 \text{ on } \ y = 1 \end{cases} \tag{70}$$

for every $(x,t) \in (0,+\infty) \times (0,T)$. The problem (70) has a solution if and only if

$$\partial_t c^0 + 2Q\partial_x c^0/3 + K(\partial_t c^0 + \varepsilon^{2-\alpha}\partial_t c^1|_{y=1}) + \varepsilon^{2-\alpha}\partial_t\Big(\int_0^1 c^1 \ dy\Big) - \varepsilon^\alpha D\partial_{xx} c^0 + Q\varepsilon^{2-\alpha}\partial_x\Big(\int_0^1 (1-y^2)c^1 \ dy\Big) - D\varepsilon^2 \partial_{xx}\Big(\int_0^1 c^1 dy\Big) = 0 \quad \text{ in } \ (0,+\infty) \times (0,T). \tag{71}$$

(71) is the equation for c^0 . For $\alpha > 0$ the last term at the left hand side is of smaller order, as we will see when estimating the error. Consequently, we drop it.

Next, in order to get the simplest possible equation for c^0 we choose C_0 such that $\partial_{tt}c^0$ and $\partial_{xt}c^0$ do not appear in the effective equation. After a short calculation we find that

$$C_0(x,t) = \frac{1}{3D}\frac{K^2}{1+K}\partial_t c^0 - \frac{2Q}{45D}\frac{K(7K+2)}{(1+K)^2}\partial_x c^0. \tag{72}$$

Now c^1 takes the form

$$c^1(x,y,t) = \frac{Q}{D}\Big(\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180} - \frac{2}{45}\frac{K(7K+2)}{(1+K)^2}\Big)\partial_x c^0 + \frac{K}{D}\Big(\frac{1}{6} + \frac{1}{3}\frac{K}{1+K} - \frac{y^2}{2}\Big)\partial_t c^0. \tag{73}$$

and the equation (71) becomes

$$(1+K)\partial_t c^0 + \frac{2Q}{3}\partial_x c^0 = \varepsilon^\alpha \tilde{D}\partial_{xx}c^0 \quad \text{in } (0,+\infty)\times(0,T). \tag{74}$$

with

$$\tilde{D} = D + \frac{8}{945}\frac{Q^2}{D}\varepsilon^{2(1-\alpha)} + \frac{4Q^2}{135D}\frac{K(7K+2)}{(1+K)^2}\varepsilon^{2(1-\alpha)} \tag{75}$$

Now the problem (70) becomes

$$\begin{cases} -D\partial_{yy}c^2 = -\frac{Q^2}{D}\partial_{xx}c^0\Big\{\frac{8}{945} + (1-y^2)\Big(\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180}\Big)\Big\} + \\ \partial_{xt}c^0\frac{QK}{D}\Big\{\frac{2}{45} - (1-y^2)\Big(\frac{1}{6} - \frac{y^2}{2}\Big)\Big\} + \Big(-\frac{2KQ}{45D}\frac{1+6K}{1+K} + \frac{2QK}{45D}\frac{K(7K+2)}{(1+K)^2}\Big)\partial_{xt}c^0 - \\ \Big(\frac{K^2}{3D} - \frac{K^3}{3D(1+K)}\Big)\partial_{tt}c^0 - \Big(\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180}\Big)\partial_{xt}c^0\frac{Q}{D} + \frac{Q}{D}\Big(\frac{1}{3} - y^2\Big)\Big(\frac{2Q}{45}\partial_{xx}c^0\frac{K(7K+2)}{(1+K)^2} - \\ \frac{K^2}{3(1+K)}\partial_{xt}c^0\Big) - \Big(\frac{1}{6} - \frac{y^2}{2}\Big)\partial_{tt}c^0\frac{K}{D} \quad \text{on } (0,1), \quad \partial_y c^2 = 0 \text{ on } y=0 \\ \text{and } \; -D\partial_y c^2 = \Big(\frac{2KQ}{45D} - \frac{2QK}{45D}\frac{K(7K+2)}{(1+K)^2}\Big)\partial_{xt}c^0 - \Big(\frac{K^2}{3D} - \frac{K^3}{3D(1+K)}\Big)\partial_{tt}c^0 \text{ on } y=1. \end{cases} \tag{76}$$

If we choose c^2 such that $\int_0^1 c^2\, dy = 0$, then

$$\begin{aligned} c^2(x,y,t) = \varepsilon^{2-2\alpha}\Big\{ -\frac{Q^2}{D^2}\partial_{xx}c^0\Big(\frac{281}{453600} + \frac{23}{1512}y^2 - \frac{37}{2160}y^4 + \frac{1}{120}y^6 - \frac{1}{672}y^8\Big) + \\ \frac{Q}{D^2}\partial_{xt}c^0\Big(\frac{31}{7560} - \frac{7}{360}y^2 + \frac{y^4}{72} - \frac{y^6}{360}\Big) - \frac{Q}{D^2}\Big(-\frac{y^4}{12} + \frac{y^2}{6} - \frac{7}{180}\Big)\Big(\frac{2Q}{45}\partial_{xx}c^0\frac{K(7K+2)}{(1+K)^2} - \\ \frac{K^2}{3(1+K)}\partial_{xt}c^0\Big) + \frac{QK}{D^2}\partial_{xt}c^0\Big(\frac{y^6}{60} - \frac{y^4}{18} + \frac{11y^2}{180} - \frac{11}{945}\Big) + \frac{K}{2D^2}\partial_{tt}c^0\Big(-\frac{y^4}{12} + \frac{y^2}{6} - \frac{7}{180}\Big) + \\ \Big(\Big(\frac{KQ}{45D^2} - \frac{QK}{45D^2}\frac{K(7K+2)}{(1+K)^2}\Big)\partial_{xt}c^0 - \Big(\frac{K^2}{6D^2} - \frac{K^3}{6D^2(1+K)}\Big)\partial_{tt}c^0\Big)\Big(\frac{1}{3} - y^2\Big)\Big\} \end{aligned} \tag{77}$$

5 First Correction

The estimate (62) isn't satisfactory. In order to get a better approximation we take the correction constructed using the formal 2-scale expansion in Section 4.

Let $0 \le \alpha < 2$. We start by the $\mathcal{O}(\varepsilon^2)$ approximation and consider the function

$$\begin{aligned} c_1^{eff}(x,y,t;\varepsilon) = c(x,t;\varepsilon) + \varepsilon^{2-\alpha}\zeta(t)\Big(\frac{Q}{D}\Big(\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180} - \frac{2}{45}\frac{K(7K+2)}{(1+K)^2}\Big) \\ \cdot\frac{\partial c}{\partial x}(x,t;\varepsilon) + \frac{K}{D}\Big(\frac{1}{6} + \frac{K}{3(1+K)} - \frac{y^2}{2}\Big)\partial_t c(x,t;\varepsilon)\Big) \end{aligned} \tag{78}$$

where c is the solution to the effective problem (74)-(75) with

$$c|_{x=0} = 0,\ c|_{t=0} = 1,\ \partial_x c \in L^2((0,+\infty)\times(0,T)), \tag{79}$$

The cut-off in time ζ is given by (29) and we use to eliminate the time-like boundary layer appearing at $t = 0$. These effects are not visible in the formal expansion.

Let $\mathcal{L}^\varepsilon$ be the differential operator given by (46). Following the formal expansion from Section 4, we know that $\mathcal{L}^\varepsilon$ applied to the correction without boundary layer functions and cut-offs would give $F_1^\varepsilon + F_2^\varepsilon + F_3^\varepsilon + F_4^\varepsilon + F_5^\varepsilon$, where

$$\begin{cases} F_1^\varepsilon = \partial_{xx}c\dfrac{Q^2}{D}\varepsilon^{2-\alpha}\Big\{\dfrac{8}{945} + (1-y^2)(\dfrac{y^2}{6} - \dfrac{y^4}{12} - \dfrac{7}{180})\Big\} \\ F_2^\varepsilon = \partial_{xt}c\dfrac{QK}{D}\varepsilon^{2-\alpha}\Big\{-\dfrac{2}{45} + (1-y^2)(\dfrac{1}{6} - \dfrac{y^2}{2})\Big\} \\ F_3^\varepsilon = \varepsilon^{2-\alpha}(\dfrac{y^2}{6} - \dfrac{y^4}{12} - \dfrac{7}{180})\partial_{xt}c\dfrac{Q}{D} \\ F_4^\varepsilon = \varepsilon^{2-\alpha}(\dfrac{1}{6} - \dfrac{y^2}{2})\partial_{tt}c\dfrac{K}{D} \\ F_5^\varepsilon = -\varepsilon^{2-\alpha}\{(\dfrac{2KQ}{45D} - \dfrac{2QK}{45D}\dfrac{K(7K+2)}{(1+K)^2})\partial_{xt}c^0 - (\dfrac{K^2}{3D} - \dfrac{K^3}{3D(1+K)})\partial_{tt}c^0\} \\ F_6^\varepsilon = -\varepsilon^{2-\alpha}\{\dfrac{Q}{D}(\dfrac{1}{3} - y^2)\big(\dfrac{2Q}{45}\partial_{xx}c^0\dfrac{K(7K+2)}{(1+K)^2} - \dfrac{K^2}{3(1+K)}\partial_{xt}c^0\big)\} \end{cases} \tag{80}$$

These functions aren't integrable up to $t = 0$ and we need a cut off ζ in order to deal with them.

After applying $\mathcal{L}^\varepsilon$ to c_1^{eff}, we find out that

$$\mathcal{L}^\varepsilon(c_1^{eff}) = \zeta(t)\sum_{j=1}^{6}F_j^\varepsilon + \Big(\varepsilon^\alpha\partial_{xx}c(\tilde{D} - D) + Q(1/3 - y^2)\partial_x c - K\partial_t c\Big)(1-\zeta(t))$$
$$+\zeta'(t)\varepsilon^{2-\alpha}\Big(\partial_x c\frac{Q}{D}\{\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180} - \frac{2}{45}\frac{K(7K+2)}{(1+K)^2}\} +$$
$$\frac{K}{D}(\frac{1}{6} + \frac{K}{3(1+K)} - \frac{y^2}{2})\partial_t c\Big) \equiv \Phi_1^\varepsilon \ \text{ and } \ -\mathcal{L}^\varepsilon(c_1^{eff}) = \mathcal{L}^\varepsilon(c^\varepsilon - c_1^{eff}) = -\Phi_1^\varepsilon \tag{81}$$

At the lateral boundary $y = 1$ we have

$$-D\varepsilon^{\alpha-2}\partial_y c_1^{eff}|_{y=1} = \zeta(t)K\partial_t c \tag{82}$$

$$K\partial_t c_1^{eff}|_{y=1} = K\partial_t c + K\varepsilon^{2-\alpha}\zeta(t)\big(\frac{Q}{D}\frac{2}{45}(1 - \frac{K(7K+2)}{(1+K)^2})\partial_{xt}c - \frac{K}{3D}\partial_{tt}c\frac{1}{1+K}\big) -$$
$$K\zeta'(t)\varepsilon^{2-\alpha}\big(\partial_x c\frac{Q}{D}\frac{2}{45}(1 - \frac{K(7K+2)}{(1+K)^2}) - \frac{K}{3D}\partial_t c\frac{1}{1+K}\big) \tag{83}$$

Now $c^\varepsilon - c_1^{eff}$ satisfies the system

$$\mathcal{L}^\varepsilon(c^\varepsilon - c_1^{eff}) = -\Phi_1^\varepsilon \ \text{ in } \ \Omega^+ \times (0,T) \tag{84}$$

$$-D\varepsilon^{\alpha-2}\partial_y(c^\varepsilon - c_1^{eff})|_{y=1} = K\partial_t(c^\varepsilon - c_1^{eff})|_{y=1} + g^\varepsilon|_{y=1} \ \text{ on } \ (0,+\infty)\times(0,T) \tag{85}$$

$$\partial_y(c^\varepsilon - c_1^{eff})|_{y=0} = 0 \ \text{ on } \ (0,+\infty)\times(0,T) \tag{86}$$

$$(c^\varepsilon - c_1^{eff})|_{t=0} = 0 \ \text{ on } \ \Omega^+ \text{ and } (c^\varepsilon - c_1^{eff})|_{x=0} = \eta_0^\varepsilon \text{ on } \ (0,1)\times(0,T). \tag{87}$$

with

$$g^\varepsilon = K\varepsilon^{2-\alpha}\zeta(t)\big(\frac{Q}{D}\frac{2}{45}(1-\frac{K(7K+2)}{(1+K)^2})\partial_{xt}c - \frac{K}{3D}\partial_{tt}c\frac{1}{1+K}\big) -$$
$$K\zeta'(t)\varepsilon^{2-\alpha}\big(\partial_x c\frac{Q}{D}\frac{2}{45}(1-\frac{K(7K+2)}{(1+K)^2}) - \frac{K}{3D}\partial_t c\frac{1}{1+K}\big) + (1-\zeta)K\partial_t c \tag{88}$$
$$\text{and}\quad \eta_0^\varepsilon = -\varepsilon^{2-\alpha}\zeta(t)\partial_x c|_{x=0}\big(\frac{y^2}{6}-\frac{y^4}{12}-\frac{7}{180}-\frac{2}{45}\frac{K(7K+2)}{(1+K)^2}\big)\frac{Q}{D}. \tag{89}$$

Now we should estimate Φ_1^ε to see if the right hand side is smaller than in Section 3. We have

Proposition 3 *Let $O_T = \Omega^+ \times (0,T)$. Let $\varphi \in H^1(O_T)$, $\varphi = 0$ at $x = 0$. Then we have*

$$|\int_0^t\int_{\Omega^+}\zeta F_1^\varepsilon\varphi\; dxdyd\tau| \le C\varepsilon^{3(2-\alpha)/2}\|\zeta(\tau)\partial_{xx}c\|_{L^2(O_t))}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}$$
$$\le C\big(\varepsilon^{3-5\alpha/2}H(1-\alpha)+\varepsilon^{1-\alpha/2}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{90}$$
$$|\int_0^t\int_{\Omega^+}\zeta(\tau)F_3^\varepsilon\varphi\; dxdyd\tau| \le C\varepsilon^{3(2-\alpha)/2}\|\zeta(\tau)\partial_{xt}c\|_{L^2(O_t)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \le$$
$$C\big(\varepsilon^{3-5\alpha/2}H(1-\alpha)+\varepsilon^{1-\alpha/2}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{91}$$
$$|\int_0^t\int_{\Omega^+}(1-\zeta)\partial_{xx}c\varepsilon^{\alpha}(\tilde D - D)\varphi\; dxdyd\tau| \le C\varepsilon^{2-3\alpha/2}\|\varepsilon^{\alpha/2}\partial_x\varphi\|_{L^2(O_t)}\cdot$$
$$\|(1-\zeta)\partial_x c\|_{L^2(O_t)} \le C\varepsilon^{2-3\alpha/2}\|\varepsilon^{\alpha/2}\partial_x\varphi\|_{L^2(O_t)} \tag{92}$$
$$|\int_0^t\int_{\Omega^+}(1-\zeta)Q(1/3-y^2)\partial_x c\varphi\; dxdyd\tau| \le C\varepsilon^{1-\alpha/2}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\cdot$$
$$\|(1-\zeta)\partial_x c\|_{L^2(O_t)} \le C\varepsilon^{1-\alpha/2}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{93}$$

$$|\int_0^t\int_{\Omega^+}\zeta'(\frac{t}{D\varepsilon})\varepsilon^{2-\alpha}\big(\partial_x c\frac{Q}{D}\frac{2}{45}(1-\frac{K(7K+2)}{(1+K)^2}) - \frac{K}{3D}\partial_t c\frac{1}{1+K}\big)\cdot$$
$$\varphi\; dxdyd\tau| \le C\varepsilon^{3-3\alpha/2}\|\zeta'\partial_x c\|_{L^2(O_t)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \le$$
$$C\big(\varepsilon^{3-5\alpha/2}H(1-\alpha)+\varepsilon^{1-\alpha/2}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{94}$$

Proof Let us note that in (90)-(91) and (93)-(94) the averages of the polynomials in y are zero. We write them in the form $P(y) = \partial_y P_1(y)$, where P_1 has zero traces at $y = 0, 1$, and after partial integration and applying the results from Section 2, giving us the precise regularity, obtain the estimates. Since $(1-\zeta)\partial_{xx}c$ isn't square integrable, we use the x-derivative in order to obtain (92). □

Proposition 4 *Let $O_T = \Omega^+ \times (0,T)$. Let $\varphi \in H^1(O_T)$, $\varphi = 0$ at $x = 0$. Then we have*

$$
\begin{aligned}
&|\int_0^t \int_{\Omega^+} \zeta F_2^\varepsilon \varphi \; dxdyd\tau| \leq C\varepsilon^{3(1-\alpha/2)} \|\zeta \partial_{xt} c\|_{L^2(O_t)} \|\varepsilon^{\alpha/2-1}\partial_y \varphi\|_{L^2(O_t)} \\
&\quad \leq C\big(\varepsilon^{3-5\alpha/2} H(1-\alpha) + \varepsilon^{1-\alpha/2} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1}\partial_y \varphi\|_{L^2(O_t)} \qquad (95)
\end{aligned}
$$

$$
\begin{aligned}
&|\int_0^t \int_{\Omega^+} \zeta F_4^\varepsilon \varphi \; dxdyd\tau| \leq C\varepsilon^{3-3\alpha/2} \|\zeta \partial_{tt} c\|_{L^2(O_t)} \|\varepsilon^{\alpha/2-1}\partial_y \varphi\|_{L^2(O_t)} \leq \\
&\quad C\big(\varepsilon^{3-5\alpha/2)} H(1-\alpha) + \varepsilon^{1-\alpha/2} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1}\partial_y \varphi\|_{L^2(O_t)} \qquad (96)
\end{aligned}
$$

$$
\begin{aligned}
&|\int_0^t \int_{\Omega^+} \zeta F_6^\varepsilon \varphi \; dxdyd\tau| \leq C\varepsilon^{3-3\alpha/2} (\|\zeta \partial_{tt} c\|_{L^2(O_t)} + \|\zeta \partial_{xt} c\|_{L^2(O_t)}) \|\varepsilon^{\alpha/2-1}\partial_y \varphi\|_{L^2(O_t)} \leq \\
&\quad C\big(\varepsilon^{3-5\alpha/2)} H(1-\alpha) + \varepsilon^{1-\alpha/2} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1}\partial_y \varphi\|_{L^2(O_t)} \qquad (97)
\end{aligned}
$$

$$
\begin{aligned}
&|\int_0^t \int_0^{+\infty} \zeta \partial_{xt} c \varepsilon^{2-\alpha} (\int_0^1 \varphi \; dy - \varphi|_{y=1}) \; dxd\tau| \leq C\varepsilon^{2-\alpha} \|\partial_{xt} c\|_{L^2(0,t;L^2((0,+\infty)))} \| \int_0^1 \varphi \; dy - \varphi|_{y=1} \|_{L^2(O_t)} \\
&\quad \leq C\big(\varepsilon^{3-5\alpha/2} H(1-\alpha) + \varepsilon^{1-\alpha/2} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1}\partial_y \varphi\|_{L^2(O_t)} \qquad (98)
\end{aligned}
$$

$$
\begin{aligned}
&|\int_0^t \int_0^{+\infty} \zeta(t) \partial_{tt} c \varepsilon^{2-\alpha} (\int_0^1 \varphi \; dy - \varphi|_{y=1}) \; dxd\tau| \leq C\big(\varepsilon^{3-5\alpha/2} H(1-\alpha) + \\
&\quad \varepsilon^{1-\alpha/2} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1}\partial_y \varphi\|_{L^2(O_t)} \qquad (99)
\end{aligned}
$$

$$
\begin{aligned}
&|\int_0^t \int_0^{+\infty} (1-\zeta(t)) \partial_t c (\int_0^1 \varphi \; dy - \varphi|_{y=1}) \; dxd\tau| \leq \\
&\quad C\big(\varepsilon^{3-5\alpha/2} H(1-\alpha) + \varepsilon^{1-\alpha/2} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1}\partial_y \varphi\|_{L^2(O_t)} \qquad (100)
\end{aligned}
$$

Corollary 3 *Let $\varphi \in H^1(O_T)$, $\varphi = 0$ at $x = 0$. Let Φ_1^ε be given by (81) and g^ε by (88). Then we have*

$$
\begin{aligned}
&|\int_0^t \int_{\Omega^+} \Phi_1^\varepsilon \varphi \; dxdyd\tau + \int_0^t \int_0^{+\infty} g^\varepsilon|_{y=1} \varphi|_{y=1} \; dxd\tau| \leq C\big(\varepsilon^{1-\alpha/2} H(1-\alpha) \\
&\quad + \varepsilon^{2-3\alpha/2} H(\alpha-1)\big) \Big\{ \|\varepsilon^{\alpha/2-1}\partial_y \varphi\|_{L^2(O_t)} + \|\varepsilon^{\alpha/2}\partial_x \varphi\|_{L^2(O_t)} \Big\} \qquad (101)
\end{aligned}
$$

Next we should correct the values at $x = 0$ and apply Proposition 1. Due to the presence of the term containing the first order derivative in x, the boundary layer corresponding to our problem doesn't enter into the theory from [12] and one should generalize it. The generalization in the case of the periodic boundary conditions at the lateral boundary is in the paper [15]. In our knowledge, the generalization to the case of Neumann's boundary conditions at the lateral boundary, was never published. It seems that the results from [15] apply also to this case ([16]). In order to avoid developing the new theory for the boundary layer, we simply use the boundary layer for the Neumann problem for Laplace operator:

$$
\begin{cases}
-\Delta_{y,z}\beta = 0 \quad \text{for} \;\; (z,y) \in \Omega^+. \\
-\partial_y \beta = 0 \;\; \text{for} \;\; y = 1, \quad \text{and for} \;\; y = 0, \\
\beta = \dfrac{y^2}{6} - \dfrac{y^4}{12} - \dfrac{7}{180} \;\; \text{for} \;\; z = 0.
\end{cases} \qquad (102)
$$

It is known (see e.g. [12]) that there exists a constant $\gamma_0 > 0$ such that the solution β for (102) satisfies the estimates

$$
\int_z^{+\infty} \int_0^1 |\nabla_{y,z}\beta|^2 \; dydz \leq c_0 e^{-\gamma_0 z}, \quad z > 0 \qquad (103)
$$

$$
|\beta(y,z)| \leq c_0 e^{-\gamma_0 z}, \quad \forall (y,z) \in \Omega^+ \qquad (104)
$$

Then the transport term is ignored and a large error in the forcing term is created. The error is concentrated at small times and by eliminating them we would obtain a good estimate.

In order to use this particular point, we prove the following proposition :

Proposition 5 *Let $\Psi(x)=1/(1+x)$. Let g^ε and Φ^ε be bounded functions such that $\Psi g^\varepsilon \in H^1(\Omega^+\times(0,T))$ and $\Psi\Phi^\varepsilon \in L^2(\Omega^+\times(0,T))$. Let ξ, $\Psi\xi\in C^{0,\alpha_0}([0,T];L^2(\Omega^+))$, $\Psi\nabla_{x,y}\xi\in L^2(\Omega^+\times(0,T))$, be a bounded function which satisfies the system*

$$\mathcal{L}^\varepsilon(\xi)=-\Phi^\varepsilon \quad in \quad \Omega^+\times(0,T) \tag{105}$$

$$-D\varepsilon^{\alpha-2}\partial_y\xi|_{y=1}=K\partial_t\xi|_{y=1}+g^\varepsilon|_{y=1} \quad and \quad \partial_y\xi|_{y=0}=0 \quad on \quad (0,+\infty)\times(0,T) \tag{106}$$

$$\xi|_{t=0}=0 \qquad on \quad \Omega^+ \quad and\, \xi|_{x=0}=0 \; on \; (0,1)\times(0,T). \tag{107}$$

Then we have the following energy estimate

$$\begin{aligned}\mathcal{E}(t^m\xi,t)=t^{2m}\int_{\Omega^+}\Psi(x)^2\xi^2(t)\;dxdy+D\varepsilon^\alpha\int_0^t\int_{\Omega^+}\Psi(x)^2\tau^{2m}\Big\{\varepsilon^{-2}|\partial_y\xi|^2+\\ |\partial_x\xi|^2\Big\}\;dxdyd\tau+Kt^{2m}\int_0^{+\infty}\xi^2(t)|_{y=1}\Psi^2(x)\;dx\le\\ C_1|\int_0^t\int_{\Omega^+}\tau^{2m}\Psi(x)^2\Phi^\varepsilon\xi\;dxdyd\tau+\int_0^t\int_0^{+\infty}\tau^{2m}g^\varepsilon|_{y=1}\xi|_{y=1}\Psi^2(x)\;dxd\tau|+\\ C_2D\varepsilon^\alpha\int_0^t\int_{\Omega^+}\tau^{2m}\Psi(x)^2\xi^2\;dxdyd\tau,\;\forall m\ge1.\end{aligned} \tag{108}$$

Proof It is along the same lines as the corresponding proof from [11] . □

Next, in order to use this estimate we should refine the estimates from Propositions 3 and 4 . First we note that the estimate (34) changes to

$$\begin{aligned}\|t^m\partial_{tt}c\|_{L^q((0,T)\times(0,+\infty))}+\|t^m\partial_{tx}c\|_{L^q((0,T)\times(0,+\infty))}+\|t^m\partial_{xx}c\|_{L^q((0,T)\times(0,+\infty))}\\ \le C_q(m)(\gamma\bar{D})^{1/(2q)-1}.\end{aligned} \tag{109}$$

Hence one gains $\varepsilon^{\alpha/4}$ (respectively $\varepsilon^{1/2-\alpha/4}$) for the L^2-norm. In analogy with Propositions 3 and 4 we have

Proposition 6 *Let $O_T=\Omega^+\times(0,T)$. Let $\varphi\in H^1(O_T)$, $\varphi=0$ at $x=0$ and $m>1$. Then we have*

$$\begin{aligned}|\int_0^t\int_0^\infty\int_0^1\tau^m\zeta F_1^\varepsilon\varphi\;dxdyd\tau|\le C\varepsilon^{3(2-\alpha)/2}\|\tau^m\partial_{xx}c\|_{L^2(O_t)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\\ \le C\big(\varepsilon^{3-9\alpha/4}H(1-\alpha)+\varepsilon^{3/2-3\alpha/4}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\end{aligned} \tag{110}$$

$$\begin{aligned}|\int_0^t\int_0^\infty\int_0^1\tau^m\zeta F_3^\varepsilon\varphi\;dxdyd\tau|\le C\varepsilon^{3(2-\alpha)/2}\|\tau^m\partial_{xt}c\|_{L^2(O_t)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\le\\ C\big(\varepsilon^{3-9\alpha/4}H(1-\alpha)+\varepsilon^{3/2-3\alpha/4}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\end{aligned} \tag{111}$$

$$\begin{aligned}|\int_0^t\int_{\Omega^+}\zeta\tau^m F_2^\varepsilon\varphi\;dxdyd\tau|\le C\varepsilon^{3(1-\alpha/2)}\|\tau^m\zeta\partial_{xt}c\|_{L^2(O_t)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\\ \le C\big(\varepsilon^{3-9\alpha/4}H(1-\alpha)+\varepsilon^{3/2-3\alpha/4}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\end{aligned} \tag{112}$$

$$|\int_0^t \int_{\Omega^+} \zeta\tau^m F_6^\varepsilon \varphi \ dxdyd\tau| \leq C\varepsilon^{3-3\alpha/2}\Big(\|\zeta\tau^m \partial_{tt}c\|_{L^2(O_t)} + \|\zeta\tau^m \partial_{xx}c\|_{L^2(O_t)}\Big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \leq$$
$$C\big(\varepsilon^{3-9\alpha/4)}H(1-\alpha) + \varepsilon^{3/2-3\alpha/4}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{113}$$

$$|\int_0^t \int_{\Omega^+} \zeta\tau^m F_4^\varepsilon \varphi \ dxdyd\tau| \leq C\varepsilon^{3-3\alpha/2}\|\zeta\tau^m \partial_{tt}c\|_{L^2(O_t)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \leq$$
$$C\big(\varepsilon^{3-9\alpha/4)}H(1-\alpha) + \varepsilon^{3/2-3\alpha/4}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{114}$$

$$|\int_0^t \int_0^{+\infty} \zeta\tau^m \partial_{xt}c\varepsilon^{2-\alpha}(\int_0^1 \varphi \ dy - \varphi|_{y=1}) \ dxd\tau| \leq C\varepsilon^{2-\alpha}\|\tau^m\partial_x c\|_{L^2(O_t)}\|\int_0^1 \varphi \ dy - \varphi|_{y=1}\|_{L^2(O_t)}$$
$$\leq C\big(\varepsilon^{3-9\alpha/4}H(1-\alpha) + \varepsilon^{3/2-3\alpha/4}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{115}$$

$$|\int_0^t \int_0^{+\infty} \zeta(t)\tau^m \partial_{tt}c\varepsilon^{2-\alpha}(\int_0^1 \varphi \ dy - \varphi|_{y=1}) \ dxd\tau| \leq$$
$$C\varepsilon^{3(1-\alpha/2)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{116}$$

Proof These estimates are straightforward consequences of Propositions 3 and 4 . □

We gain more with other terms:

Proposition 7 *Let $\varphi \in H^1(O_T)$, $\varphi = 0$ at $x = 0$. Then we have*

$$|\int_0^t \int_0^\infty \int_0^1 (1-\zeta)\tau^m \partial_{xx}c\varepsilon^{2-\alpha}\varphi \ dxdyd\tau| \leq C\varepsilon^{2-3\alpha/2}\|(1-\zeta)\tau^m\partial_x c\|_{L^2(O_t)}\|\varepsilon^{\alpha/2}\partial_x\varphi\|_{L^2(O_t)}$$
$$\leq C\big(\varepsilon^{m\alpha+2-3\alpha/2}H(1-\alpha) + \varepsilon^{m(2-\alpha)+2-3\alpha/2}H(\alpha-1)\big)\|\varepsilon^{\alpha/2}\partial_x\varphi\|_{L^2(O_t)} \tag{117}$$

$$|\int_0^t \int_0^\infty \int_0^1 (1-\zeta)\tau^m Q(1/3-y^2)\partial_x c\varphi \ dxdyd\tau| \leq C\varepsilon^{1-\alpha/2}\|(1-\zeta)\tau^m\partial_x c\|_{L^2(O_t)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \leq$$
$$C\big(\varepsilon^{m\alpha+1-\alpha/2}H(1-\alpha) + \varepsilon^{m(2-\alpha)+1-\alpha/2}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{118}$$

$$|\int_0^t \int_{\Omega^+} \zeta'(\frac{t}{D\varepsilon})\tau^m\varepsilon^{2-\alpha}\big\{\partial_x c\frac{Q}{D}\big\{\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180}\big\} - \frac{K}{2D}(\frac{1}{3} - y^2)\partial_t c\big\}\cdot$$
$$\varphi \ dxdyd\tau| \leq C\varepsilon^{3-3\alpha/2}(\|\zeta'\tau^m\partial_x c\|_{L^2(O_t)} + \|\zeta'\tau^m\partial_t c\|_{L^2(O_t)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \leq$$
$$C\big(\varepsilon^{3-3\alpha/2+\alpha(m-1)}H(1-\alpha) + \varepsilon^{3-3\alpha/2+(2-\alpha)(m-1)}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{119}$$

Before applying Proposition 5 and getting the final estimate, we should correct the trace at $x = 0$. It is done by adding

$$\bar{c}_1^{eff} = -\varepsilon^{2-\alpha}\zeta(t)(\beta^\varepsilon + \mathcal{B}e^{-x/\varepsilon})\partial_x c\frac{Q}{D}, \tag{120}$$

where $\beta^\varepsilon(x,y) = \beta(x/\varepsilon, y)$ is the boundary layer function given by (102) and $\mathcal{B} = -\dfrac{2}{45}\dfrac{K(7K+2)}{(1+K)^2}$. Then for $\xi^\varepsilon = c^\varepsilon - c_1^{eff} - \bar{c}_1^{eff}$ we have

$$\mathcal{L}^\varepsilon(\xi) = -\Phi^\varepsilon = -\Phi_1^\varepsilon + \partial_t\zeta\varepsilon^{2-\alpha}\partial_x c\frac{Q}{D}\beta^\varepsilon + \varepsilon^{2-\alpha}\beta^\varepsilon\zeta(t)\big\{\partial_{xt}c\frac{Q}{D} -$$
$$\varepsilon^\alpha\partial_{xxx}cQ\big\} + \partial_x\beta^\varepsilon\frac{Q^2}{D}(1-y^2)\zeta\varepsilon^{2-\alpha}\partial_x c - \varepsilon^{2-\alpha}Q\partial_{xx}c\zeta(t)\big(2\varepsilon^\alpha\partial_x\beta^\varepsilon -$$
$$\beta^\varepsilon(1-y^2)\frac{Q}{D}\big) + \varepsilon^{2-\alpha}\frac{Q}{D}\mathcal{L}^\varepsilon(\mathcal{B}e^{-x/\varepsilon}\partial_x c) \ \text{ in } \ \Omega^+ \times (0,T) \tag{121}$$

$$-D\varepsilon^{\alpha-2}\partial_y\xi^\varepsilon|_{y=1} = K\partial_t\xi|_{y=1} + g^\varepsilon|_{y=1} - K\varepsilon^{2-\alpha}\zeta\frac{Q}{D}\partial_{xt}c(\beta^\varepsilon + \mathcal{B}e^{-x/\varepsilon})|_{y=1} \tag{122}$$

$$\text{and } \ \partial_y\xi^\varepsilon|_{y=0} = 0 \ \text{ on } \ (0,+\infty)\times(0,T) \tag{123}$$

$$\xi^\varepsilon|_{t=0} = 0 \quad \text{ on } \ \Omega^+ \quad \text{and } \xi^\varepsilon|_{x=0} = 0 \text{ on } \ (0,1)\times(0,T). \tag{124}$$

We need an estimate for new terms. The estimates are analogous to those in [11] and we just remark that all new terms are of order $\mathcal{O}(\varepsilon^{m-2})$ and, consequently, we can simply ignore them. In order to explain why they are of lower order we estimate a typical term:

$$\int_0^T \int_0^{+\infty} |\tau^m \partial_x c\beta^\varepsilon|^2 \, dx d\tau \leq C \int_0^T \int_0^{+\infty} \tau^{2m} \exp\{-\frac{2\gamma_0 x}{\varepsilon}\} \exp\{-\frac{(x-\tau\bar{Q})^2}{2\gamma \bar{D}\tau}\} \, \frac{dx d\tau}{\gamma\tau\bar{D}}$$
$$\leq C \int_0^T \tau^{2m} (\varepsilon D\tau)^{-1/2} \exp\{-C_0\tau/\varepsilon\} \, dx d\tau \leq C\varepsilon^{2m-4}. \qquad (125)$$

Now the application of Proposition 5 is straightforward and after considering various powers we get

Theorem 3 *Let c be given by (79), let c_1^{eff} be given by (78) and $\bar{c}_1^{eff}$ by (120). Then we have*

$$\|t^3(c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff})\|_{L^\infty(0,T;L^2_{loc}(\Omega^+))} \leq C\big(\varepsilon^{3-9\alpha/4}H(1-\alpha)+$$
$$\varepsilon^{3(1-\alpha/2)/2}H(\alpha-1)\big) \qquad (126)$$

$$\|t^3\partial_y\big(c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff}\big)\|_{L^2(0,T;L^2_{loc}(\Omega^+))} \leq$$
$$C\varepsilon^{1-\alpha/2}\big(\varepsilon^{3-9\alpha/4}H(1-\alpha) + \varepsilon^{3(1-\alpha/2)/2}H(\alpha-1)\big) \qquad (127)$$

$$\|t^2\partial_x\big(c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff}\big)\|_{L^2(0,T;L^2_{loc}(\Omega^+))} \leq$$
$$C\varepsilon^{-\alpha/2}\big(\varepsilon^{3-9\alpha/4}H(1-\alpha) + \varepsilon^{3(1-\alpha/2)/2}H(\alpha-1)\big) \qquad (128)$$

Proving Corollary 1 follows the lines of the analogous construction form the article [11] and it is a direct consequence of Theorem 3.

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