Noname manuscript No.
(will be inserted by the editor)

Convergence of iterative coupling for coupled flow and geomechanics

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Received: date / Accepted: date

Abstract In this paper we study solving iteratively the coupling of flow and mechanics. We demonstrate the stability and convergence of two widely used schemes: the undrained split method and the fixed stress split method. To our knowledge this is the first time that such results have been rigorously obtained and published in the scientific literature. In addition, we propose a new stress split method, with faster convergence rate than known schemes. These results are specially important today due to the interest in hydraulic fracturing ([1], [3], [4] and [5]), in oil and gas shale reservoirs.

Keywords Iterative splitting · Geomechanics · Biot system · Darcy flow equation

PACS PACS 02.30.Jr · PACS 02.30.Vv · PACS 43.20.Bi · PACS 43.20.Tb · PACS 47.56.+r · PACS 92.40.ke

Mathematics Subject Classification (2000) MSC 35Q35 · MSC 76M25 · MSC 76S05 · MSC 86Axx

1 Introduction

There are three approaches frequently employed in coupling flow and mechanics in porous media i.e. in the coupling of fluid flow and the mechanical response of the reservoir's solid structure. They are referred to as **fully implicit, loose or explicit coupling and iterative coupling**. The **fully implicit** involves solving all of the governing equations simultaneously and requires complex and expensive solvers. The **loosely or explicitly coupled** is less accurate and requires estimates of when to update the mechanical response. Iterative coupling is a sequential procedure where either the flow or the mechanics is solved first followed by solving the other problem using the latest solution information. At each time step the procedure is iterated until the solution converges within an acceptable tolerance. There are four well-known iterative coupling procedures, referred to as the **undrained split**, the **fixed stress split**, the **drained split** and the **fixed strain split** iterative methods. Kim et al have shown using a von Neumann stability analysis in [2] that the latter two methods exhibit stability problems, whereas the **undrained split** and the **fixed stress split** method are stable. Their results does not include convergence estimates nor rates of convergence.

In this paper we derive stability, convergence and the rate of convergence for the **undrained split** and the **fixed stress split**. More precisely we prove that the two methods define

This paper is dedicated to the **60th anniversary of C.J. van Duijn**, because of his impact of applying rigorous mathematics to real world problems.
The research of A.M. was partially supported by the GNR MOMAS (Modélisation Mathématique et Simulations numériques liées aux problèmes de gestion des déchets nucléaires) (PACEN/CNRS, ANDRA, BRGM, CEA, EDF, IRSN). He would like to thank *Institute for Computational Engineering and Science (ICES)*, UT Austin for hospitality in April 2009, 2010 and 2011. The research by M. F. Wheeler was partially supported by the U.S. Department of Energy, Office of Science, Office of Basic Energy Sciences through *DOE Energy Frontier Research Center: The Center for Frontiers of Subsurface Energy Security (CFSES)* under Contract No. DE-SC0001114.

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a contraction map with respect to correctly chosen metrics. The **undrained split** is shown to have the same contraction constant as the **fixed stress split**. In addition, we propose a new method, with even smaller contraction constant. Convergence of discrete schemes and computational results will appear in a forthcoming paper.

We study the simplest model of real applied importance: the quasi-static Biot system. The important parameters and unknowns are given in the Table 1

SYMBOL	*QUANTITY*	*UNITY*
$\mathbf{u}$	displacement	m
p	fluid pressure	Pa
σ^{por}	total poroelasticity tensor	Pa
$e(\mathbf{u}) = (\nabla \mathbf{u} + \nabla^\tau \mathbf{u})/2$	linearized strain tensor	dimensionless
ϕ	porosity	dimensionless
$\mathscr{K}$	permeability	Darcy
$\mathbf{v}^D$	Darcy's velocity	m/sec
α	Biot's coefficient	dimensionless
ρ_s	solid phase density	kg/m^3
ρ_f	fluid phase density	kg/m^3
$\rho_b = \phi\rho_f + (1-\phi)\rho_s$	bulk density	kg/m^3
η	fluid viscosity	$kg/$m sec
M	Biot's modulus	Pa
$\mathscr{G}$	Gassman rank-4 tensor	Pa
m	fluid mass per bulk volume	kg/m^3
$\rho_{f,0}$	reference state fluid density	kg/m^3
$B_f = \rho_{f,0}/\rho_f$	formation volume factor	dimensionless

Table 1 *Unknowns and effective coefficients*

The quasi-static Biot equations ([6]) are an elliptic-parabolic system of PDEs, valid in the poroelastic cube $\Omega = (0,L)^3$, for every $t \in (0,T)$, which reads:

$$\sigma^{por} - \sigma_0 = \mathscr{G} e(\mathbf{u}) - \alpha(p-p_0)I; \tag{1}$$

$$-\text{ div } \{\sigma^{por}\} = \rho_b \mathbf{g}; \tag{2}$$

$$\mathbf{v}^D = \frac{\mathscr{K}}{B_f \eta}(\rho_f \mathbf{g} - \nabla p); \tag{3}$$

$$m = m_0 + \rho_{f,0}\alpha \text{ div } \mathbf{u} + \frac{\rho_{f,0}}{M}(p-p_0); \tag{4}$$

$$\partial_t\Big(\frac{1}{M}p + \text{ div } (\alpha \mathbf{u})\Big) + \text{ div } \{\mathbf{v}^D\} = f; \tag{5}$$

$$p|_{t=0} = p_0; m|_{t=0} = m_0; \mathbf{u}|_{t=0} = 0; \sigma^{por}|_{t=0} = \sigma_0; \tag{6}$$

$$\{\mathbf{u}, p\} \quad \text{is periodic in } \mathbf{x} \quad \text{with period } L. \tag{7}$$

Obviously, we can suppose without loss in generality that $p_0 = 0$.

For convenience we have assumed periodic boundary conditions. Boundary conditions for the general situation involving displacement and traction as well as for pressure and flux, prescribed on portions of the boundary , respectively , can be treated by the same analysis as presented here. We make the following hypothesis on the effective coefficients

(**H1**) B_f, η, M , $\rho_{f,0}$ and ρ_s are positive constants.

(**H2**) $\mathscr{K}$ is a symmetric uniformly positive definite matrix, with the smallest eigenvalue k and largest eigenvalue k^*. Furthermore, for any symmetric matrix B we have

$$\mathscr{G}B : B \geq a|B|^2 + K_{dr}(TrB)^2, \tag{8}$$

where K_{dr} is the drained bulk modulus.

(**H3**) m_0, p_0, f and σ_0 are smooth L-periodic function with respect to x.

Following [2], we further assume

(**H4**) ρ_b is independent of time and equal to

$$\rho_b \mathbf{g} = -\text{ div } \sigma_0. \tag{9}$$

We will prove the existence and uniqueness for the system (1)-(7) using the convenient iterative methods, used in practice.

2 Convergence of the iterative methods

2.1 "Undrained Split" iterative method

The *undrained split* iterative method consists in imposing constant fluid mass during the structure deformation. This means that we will calculate two pressures: $p^{n+1/2}$ at the half-time step and then p^{n+1}. We set

$$p^{n+1/2} = p^n - \alpha M \text{ div } (\mathbf{u}^{n+1/2} - \mathbf{u}^n). \tag{10}$$

Then, using the hypothesis (**H4**), our iterative process reads as follows

$$-\text{ div } \{\mathscr{G} e(\mathbf{u}^{n+1}) + M\alpha^2 \text{ div } \mathbf{u}^{n+1} I\} = -\nabla\{\alpha p^n + M\alpha^2 \text{ div } \mathbf{u}^n\}; \tag{11}$$

$$\frac{1}{M}\partial_t p^{n+1} + \text{ div } \{\frac{\mathscr{K}}{B_f\eta}(\rho_f \mathbf{g} - \nabla p^{n+1})\} = -\text{ div } (\alpha \partial_t \mathbf{u}^{n+1}) + f; \tag{12}$$

$$\{\mathbf{u}^{n+1}, p^{n+1}\}|_{t=0} = 0 \quad \text{on } \Omega; \tag{13}$$

$$\{\mathbf{u}^{n+1}, p^{n+1}\} \quad \text{is periodic in } \mathbf{x} \quad \text{with period } L. \tag{14}$$

We introduce the functional spaces

$$V_T = \{\mathbf{z} \in C([0,T]; H^1_{per}(\Omega)^3 \cap L^2_0(\Omega)^3) \mid \partial_t e(\mathbf{z}) \in L^2(\Omega)^9\} \tag{15}$$

$$W_T = \{r \in H^1(\Omega \times (0,T)) \mid r \in C([0,T]; H^1_{per}(\Omega))\}. \tag{16}$$

Theorem 1 *Let us suppose hypothesis (***H1***)-(***H4***) and let $\mathscr{S}$ be the operator mapping $\{\mathbf{u}^n, p^n\}$ to $\{\mathbf{u}^{n+1}, p^{n+1}\}$. Th en $\mathscr{S}$ admits a unique fixed point from $V_T \times W_T$ satisfying (1)-(7).*

Proof Let us introduce the following notation for the fluid mass per unit bulk volume:

$$m^n = m_0 + \rho_{f,0}\alpha \text{ div } \mathbf{u}^n + \frac{\rho_{f,0}}{M} p^n.$$

Next, let the invariant distance d_{us} be given by

$$d_{us}^2\biggl((\mathbf{u},p),0\biggr) = \frac{k}{B_f\eta M} \max_{0\le t\le T} ||\nabla p(t)||^2_{L^2(\Omega)^3} +$$
$$\frac{2a\alpha^2}{K_{dr}+M\alpha^2}||e(\partial_t \mathbf{u})||^2_{L^2(\Omega\times(0,T))^9} + \alpha^2|| \text{ div } \partial_t \mathbf{u}||^2_{L^2(\Omega\times(0,T))}$$
$$+||\underbrace{\partial_t(\frac{1}{M}p+\alpha \text{ div } \mathbf{u})}_{\partial_t m/\rho_{f,0}}||^2_{L^2(\Omega\times(0,T))} \tag{17}$$

on the closed subspace

$$\mathscr{Q} = \{(\mathbf{A},B) \in V_T\times W_T \;|\mathbf{A}|_{t=0} = 0;\; B|_{t=0} = 0\;\} \tag{18}$$

of the functional space $V_T \times W_T$. We see that the operator $\mathscr{S}$, such that $\mathscr{S}(\mathbf{u}^n, p^n) = (\mathbf{u}^{n+1}, p^{n+1})$, maps $\mathscr{Q}$ into itself.

Step 1. We use the notation $\delta\mathbf{u}_t^n = \partial_t(\mathbf{u}^n - \mathbf{u}^{n-1})$, $\delta m_t^n = \partial_t(m^n - m^{n-1})$ and $\delta p_t^n = \partial_t(p^n - p^{n-1})$. Then (11) holds true for the differences $\delta\mathbf{u}^{n+1}, \delta\mathbf{u}^n, \delta p^n$. We take the time derivative of (11) and test the resulting equation by $\mathbf{z} = \delta\mathbf{u}_t^{n+1}$. Applying Green's formula we have

$$\int_\Omega (\mathscr{G}e(\delta\mathbf{u}_t^{n+1}) : e(\delta\mathbf{u}_t^{n+1}) + M\alpha^2 | \text{ div } \delta\mathbf{u}_t^{n+1}|^2)\, dx =$$
$$\frac{\alpha M}{\rho_{f,0}}\int_\Omega \delta m_t^n \text{ div } \delta\mathbf{u}_t^{n+1}\, dx \le \frac{\alpha^2 M\varepsilon}{2}\int_\Omega | \text{ div } \delta\mathbf{u}_t^{n+1}|^2\, dx +$$
$$\frac{M}{2\varepsilon\rho_{f,0}^2}\int_\Omega |\delta m_t^n|^2\, dx, \quad \forall\varepsilon > 0.$$

Using the hypothesis (**H2**) we obtain

$$a\int_\Omega |e(\delta\mathbf{u}_t^{n+1})|^2\, dx + (K_{dr} + M\alpha^2(1-\frac{\varepsilon}{2}))\int_\Omega | \text{ div } \delta\mathbf{u}_t^{n+1}|^2\, dx$$
$$\le \frac{M}{2\varepsilon\rho_{f,0}^2}\int_\Omega |\delta m_t^n|^2\, dx.$$

Consequently, we conclude that the following estimate

$$\frac{a\alpha^2}{K_{dr}+M\alpha^2(1-\frac{\varepsilon}{2})}\int_\Omega |e(\delta\mathbf{u}_t^{n+1})|^2\, dx + \alpha^2\int_\Omega | \text{ div } \delta\mathbf{u}_t^{n+1}|^2\, dx$$
$$\le \frac{M\alpha^2}{2\varepsilon(K_{dr}+M\alpha^2(1-\frac{\varepsilon}{2}))}\int_\Omega |\frac{\delta m_t^n}{\rho_{f,0}}|^2\, dx.$$

The coefficient in front of $||\delta m_t^n/\rho_{f,0}||_{L^2(\Omega)}$ is smallest for $\varepsilon = K_{dr}/(M\alpha^2)+1$ and the above estimate becomes

$$\frac{2a\alpha^2}{K_{dr}+M\alpha^2}\int_\Omega |e(\delta\mathbf{u}_t^{n+1})|^2\, dx + \alpha^2\int_\Omega | \text{ div } \delta\mathbf{u}_t^{n+1}|^2\, dx$$
$$\le \left(\frac{M\alpha^2}{K_{dr}+M\alpha^2}\right)^2\int_\Omega |\frac{\delta m_t^n}{\rho_{f,0}}|^2\, dx. \tag{19}$$

Step 2.

Testing (12) with δp_t^{n+1} and applying (**H2**) we get

$$\int_0^t\int_\Omega \frac{\delta m_\tau^{n+1}}{\rho_{f,0}}\delta p_\tau^{n+1}\, dxd\tau + \frac{1}{2}\int_\Omega \frac{k}{B_f\eta}|\nabla\delta p^{n+1}(t)|^2\, dx \le 0,$$

implying

$$\int_0^t\int_\Omega |\frac{\delta m_\tau^{n+1}}{\rho_{f,0}}|^2\, dxd\tau = \int_0^t\int_\Omega \frac{\delta m_\tau^{n+1}}{\rho_{f,0}}\alpha \text{ div } \delta\mathbf{u}_\tau^{n+1}\, dxd\tau +$$
$$\int_0^t\int_\Omega \frac{\delta m_\tau^{n+1}}{\rho_{f,0}}\frac{\delta p_\tau^{n+1}}{M}\, dxd\tau \le \int_0^t\int_\Omega \frac{\delta m_\tau^{n+1}}{\rho_{f,0}}\alpha \text{ div } \delta\mathbf{u}_\tau^{n+1}\, dxd\tau$$
$$-\frac{1}{2}\int_\Omega \frac{k}{B_f\eta M}|\nabla\delta p^{n+1}(t)|^2\, dx$$

and

$$\int_0^t\int_\Omega |\frac{\delta m_\tau^{n+1}}{\rho_{f,0}}|^2\, dxd\tau \le \int_0^t\int_\Omega |\alpha \text{ div } \delta\mathbf{u}_\tau^{n+1}|^2\, dxd\tau -$$
$$\int_\Omega \frac{k}{B_f\eta M}|\nabla\delta p^{n+1}(t)|^2\, dx. \tag{20}$$

Integrating from 0 to t inequality (19) and combining it with (20) gives

$$\int_0^t\int_\Omega |\frac{\partial_t m^{n+1}}{\rho_{f,0}}|^2\, dxd\tau + \int_\Omega \frac{k}{B_f\eta M}|\nabla p^{n+1}(t)|^2\, dx +$$
$$\frac{2a\alpha^2}{K_{dr}+M\alpha^2}\int_0^t\int_\Omega |e(\partial_t\mathbf{u}^{n+1})|^2\, dxd\tau$$
$$\le \left(\frac{M\alpha^2}{K_{dr}+M\alpha^2}\right)^2\int_0^t\int_\Omega |\frac{\partial_t m^n}{\rho_{f,0}}|^2\, dxd\tau. \tag{21}$$

Another direct consequence of (19)-(20) is the following estimate

$$\int_0^t\int_\Omega |\alpha \text{ div } \delta\mathbf{u}_\tau^{n+1}|^2\, dxd\tau \le$$
$$\left(\frac{M\alpha^2}{K_{dr}+M\alpha^2}\right)^2\int_0^t\int_\Omega |\alpha \text{ div } \delta\mathbf{u}_\tau^n|^2\, dxd\tau \tag{22}$$

We note that (21)-(22) implies

$$d_{us}\biggl((\mathbf{u}^{n+1},p^{n+1}) - (\mathbf{u}^n,p^n)\biggr) \le$$
$$\gamma d_{us}\biggl((\mathbf{u}^n,p^n) - (\mathbf{u}^{n-1},p^{n-1})\biggr) \tag{23}$$

with $\gamma = \dfrac{M\alpha^2}{K_{dr}+M\alpha^2} < 1$. Hence $\mathscr{S}$ is a contraction mapping on $\mathscr{Q}$ and by the **contraction mapping principle**, it has a unique fixed point in $\mathscr{Q}$. The theorem is proved.□

Remark 1 One can pose the question if the natural energy norm defines a contraction.

Again we write equations (11) and (12) for differences $\delta \mathbf{u}^{n+1}, \delta \mathbf{u}^n, \delta p^n$ and δp^{n+1}. We test (11) by $\delta \mathbf{u}_t^{n+1}$, (12) by δp^{n+1} and sum up the variational formulations. It yields

$$\int_\Omega \left(\mathscr{G} e(\delta \mathbf{u}^{n+1}(t)) : e(\delta \mathbf{u}^{n+1}(t)) + \frac{M}{4} (\frac{\delta m^{n+1}(t)}{\rho_{f,0}})^2 \right) dx$$
$$+ \int_0^t \int_\Omega \frac{k}{B_f \eta} |\nabla \delta p^{n+1}|^2 \, dx d\tau \le 2M\alpha (|| \frac{\delta m^n}{\rho_{f,0}} ||_{L^2(\Omega \times (0,t))}$$
$$+ || \frac{\delta m^{n+1}}{\rho_{f,0}} ||_{L^2(\Omega \times (0,t))}) || \text{ div } \delta \mathbf{u}_\tau^{n+1} ||_{L^2(\Omega \times (0,t))}. \tag{24}$$

Therefore, we see that the standard energy estimate is not self-contained and requires an estimate for $\partial_t \mathbf{u}^{n+1}$. It was established in Theorem 1 using the higher order derivative estimates. It enables us to establish the fast convergence even without having the contraction property. In the estimates which follow we develop the appropriate estimate.

We use the obvious inequality

$$\int_0^t \int_\Omega |\delta m^n|^2 \, dx d\tau \le \frac{t^2}{2} \int_0^t \int_\Omega |\partial_\tau m^n|^2 \, dx d\tau$$

and (17) to see that (24) implies

$$\max_{0 \le t \le T} \int_\Omega \Big(\mathscr{G} e(\delta \mathbf{u}^{n+1}(t)) : e(\delta \mathbf{u}^{n+1}(t)) +$$
$$\frac{M}{4} (\frac{\delta m^{n+1}(t)}{\rho_{f,0}})^2 \Big) \, dx + \int_0^T \int_\Omega \frac{k}{B_f \eta} |\nabla \delta p^{n+1}(\tau)|^2 \, dx d\tau$$
$$\le MT^2 (d_{us}^2((\mathbf{u}^{n+1}, p^{n+1}) - (\mathbf{u}^n, p^n)) +$$
$$d_{us}^2((\mathbf{u}^{n-1}, p^{n-1}) - (\mathbf{u}^n, p^n))). \tag{25}$$

(25), together with Theorem 1, yields fast convergence in the natural energy norm.

2.2 "Fixed Stress Split" iterative method

The *fixed stress split* iterative method consists in imposing constant volumetric mean total stress. This means that the $\sigma_v = \sigma_{v,0} + K_{dr} \text{ div } \mathbf{u} - \alpha(p - p_0)$ is kept constant at the half-time step. Our iterative process reads as follows

$$\left(\frac{1}{M} + \frac{\alpha^2}{K_{dr}} \right) \partial_t p^{n+1} + \text{ div } \{ \frac{\mathscr{K}}{B_f \eta} (\rho_f \mathbf{g} - \nabla p^{n+1}) \} =$$
$$- \frac{\alpha}{K_{dr}} \partial_t \sigma_v^n + f = f - \alpha \text{ div } \partial_t \mathbf{u}^n + \frac{\alpha^2}{K_{dr}} \partial_t p^n; \tag{26}$$
$$- \text{ div } \{ \mathscr{G} e(\mathbf{u}^{n+1}) \} + \alpha \nabla p^{n+1} = 0; \tag{27}$$
$$\{ \mathbf{u}^{n+1}, p^{n+1} \}|_{t=0} = 0 \quad \text{on } \Omega; \tag{28}$$
$$\{ \mathbf{u}^{n+1}, p^{n+1} \} \quad \text{is periodic in } \mathbf{x} \quad \text{with period } L. \tag{29}$$

Remark 2 We remark that the fixed stress approach is useful in employing reservoir simulators in that (26) can be extended to treat the mass balance equations arising in black oil or compositional flows.

Theorem 2 *Let us suppose hypothesis (**H1**)-(**H4**) and let $\mathscr{S}$ be the operator mapping $\{\mathbf{u}^n, p^n\}$ to $\{\mathbf{u}^{n+1}, p^{n+1}\}$. Then $\mathscr{S}$ admits a unique fixed point from $V_T \times W_T$ satisfying (1)-(7).*

Proof Let us introduce the following notation for volumetric mean total stress

$$\sigma_v = \sigma_{v,0} + K_{dr} \text{ div } \mathbf{u} - \alpha(p - p_0).$$

Then (26) and (27) hold true for the differences $\delta \mathbf{u}^{n+1}, \delta \mathbf{u}^n, \delta p^n$ and δp^{n+1}, with $f = 0$ and $g = 0$.

Step 1. We multiply the variant of (26), valid for δp^{n+1} and $\delta \partial_t \sigma_v^n$, by $\partial_t \delta p^{n+1}$ and get

$$\left(\frac{1}{M\alpha^2} + \frac{1}{K_{dr}} \right) \int_0^t \int_\Omega |\alpha \partial_\tau \delta p^{n+1}|^2 \, dx d\tau +$$
$$\int_\Omega \frac{k}{2B_f \eta} |\nabla \delta p^{n+1}(t)|^2 \, dx \le - \frac{\alpha}{K_{dr}} \int_0^t \int_\Omega \partial_\tau \delta p^{n+1} \delta \partial_\tau \sigma_v^n \, dx d\tau$$
$$\le \frac{\varepsilon}{2} \int_0^t \int_\Omega |\alpha \delta \partial_\tau p^{n+1}|^2 \, dx d\tau +$$
$$\frac{1}{2\varepsilon K_{dr}} \int_0^t \int_\Omega (\partial_\tau \delta \sigma_v^n)^2 \, dx d\tau, \ \forall \varepsilon > 0.$$

Again, the coefficient in front of $|| \delta \partial_t \sigma_v^n ||_{L^2(\Omega \times (0,t))}$ is smallest for $\varepsilon = 1/K_{dr} + 1/(M\alpha^2)$ and the above estimate becomes

$$\left(\frac{1}{M\alpha^2} + \frac{1}{K_{dr}} \right) \int_0^t \int_\Omega |\alpha \delta \partial_\tau p^{n+1}|^2 \, dx d\tau + \bar{k} \int_\Omega |\nabla \delta p^{n+1}(t)|^2 \, dx$$
$$\le \frac{M\alpha^2}{K_{dr}(M\alpha^2 + K_{dr})} \int_0^t \int_\Omega (\partial_\tau \delta \sigma_v^n)^2 \, dx d\tau, \tag{30}$$

where $\bar{k} = \dfrac{k}{B_f \eta}$.

Step 2.

Next we take the time derivative of (27), valid for $\{\delta \mathbf{u}^{n+1}, \delta p^{n+1}\}$, and test the resulting equation by $\partial_t \delta \mathbf{u}^{n+1}$. It yields

$$\int_\Omega \mathscr{G} e(\partial_t \delta \mathbf{u}^{n+1}) : e(\partial_t \delta \mathbf{u}^{n+1}) = \alpha \int_\Omega \partial_t \delta p^{n+1} \text{ div } \partial_t \mathbf{u}^{n+1} \, dx,$$

which implies

$$2a(\frac{K_{dr}}{M\alpha^2} + 1) \int_\Omega |e(\partial_t \delta \mathbf{u}^{n+1})|^2 \, dx +$$
$$2(\frac{1}{M\alpha^2} + \frac{1}{K_{dr}}) \int_\Omega |K_{dr} \text{ div } \partial_t \delta \mathbf{u}^{n+1}|^2 \, dx \le$$
$$2(\frac{1}{M\alpha^2} + \frac{1}{K_{dr}}) \int_\Omega \alpha \partial_t \delta p^{n+1} K_{dr} \text{ div } \delta \partial_t \mathbf{u}^{n+1} \, dx. \tag{31}$$

After summing up (30) and (31), one has

$$\int_0^t \int_\Omega (\partial_\tau \delta \sigma_v^{n+1})^2 \, dx d\tau + K_{dr}^2 \int_0^t \int_\Omega | \text{ div } \partial_t \delta \mathbf{u}^{n+1}|^2 \, dx d\tau$$
$$+ 2aK_{dr} \int_0^t \int_\Omega |e(\partial_t \delta \mathbf{u}^{n+1})|^2 \, dx d\tau +$$
$$\frac{\bar{k} M\alpha^2 K_{dr}}{M\alpha^2 + K_{dr}} \int_\Omega |\nabla \delta p^{n+1}(t)|^2 \, dx \le$$
$$\left(\frac{M\alpha^2}{M\alpha^2 + K_{dr}} \right)^2 \int_0^t \int_\Omega (\partial_\tau \delta \sigma_v^n)^2 \, dx d\tau. \tag{32}$$

Now we proceed as in Subsection 2.1 and obtain the result: The expression on the left hand side defines the invariant distance d_{fs} by

$$d^2_{fs}\Big((\mathbf{u},p),0\Big)=\frac{\bar{k}M\alpha^2K_{dr}}{M\alpha^2+K_{dr}}\max_{0\le t\le T}||\nabla p(t)||^2_{L^2(\Omega)}+$$
$$2aK_{dr}||e(\partial_t\mathbf{u})||^2_{L^2(\Omega\times(0,T))^3}+K^2_{dr}||\text{div }\partial_t\mathbf{u}||^2_{L^2(\Omega\times(0,T))}$$
$$+||\underbrace{\partial_t(-\alpha p+K_{dr}\text{ div }\mathbf{u})}_{\partial_t\sigma_v}||^2_{L^2(\Omega\times(0,T))}, \tag{33}$$

on the closed subspace

$$\mathscr{Q}=\{(\mathbf{A},B)\in V_T\times W_T\ |\mathbf{A}|_{t=0}=0\ \text{ and }\ B|_{t=0}=0\ \}. \tag{34}$$

We see that that the operator $\mathscr{S}$, such that $\mathscr{S}(\mathbf{u}^n,p^n)=(\mathbf{u}^{n+1},p^{n+1})$, maps $\mathscr{Q}$ into itself.

We find out that

$$d_{fs}\Big((\mathbf{u}^{n+1},p^{n+1})-(\mathbf{u}^n,p^n)\Big)\le$$
$$\gamma_{FS}d_{fs}\Big((\mathbf{u}^n,p^n)-(\mathbf{u}^{n-1},p^{n-1})\Big) \tag{35}$$

with $\gamma_{FS}=\dfrac{M\alpha^2}{K_{dr}+M\alpha^2}<1$. Hence $\mathscr{S}$ is a contraction mapping on $\mathscr{Q}$ and by the **contraction mapping principle**, it has a unique fixed point in $\mathscr{Q}$.□

2.3 Optimized "Fixed Stress Split" iterative method

The *fixed stress split* iterative method consisted in imposing constant volumetric mean total stress. In this section we work with the quantity $\sigma_\beta=\sigma_0+K_{dr}\text{ div }\mathbf{u}-\dfrac{\beta K_{dr}}{\alpha}(p-p_0)$. Our iterative process reads as follows

$$\Big(\frac{1}{M}+\beta\Big)\partial_t p^{n+1}+\text{ div }\{\frac{\mathscr{K}}{B_f\eta}(\rho_f\mathbf{g}-\nabla p^{n+1})\}=$$
$$-\frac{\alpha}{K_{dr}}\partial_t\sigma^n_\beta+f=f-\alpha\text{ div }\partial_t\mathbf{u}^n+\beta\partial_t p^n; \tag{36}$$
$$-\text{ div }\{\mathscr{G}e(\mathbf{u}^{n+1})\}+\alpha\nabla p^{n+1}=0; \tag{37}$$
$$\{\mathbf{u}^{n+1},\,p^{n+1}\}|_{t=0}=0\quad\text{on }\Omega; \tag{38}$$
$$\{\mathbf{u}^{n+1},\,p^{n+1}\}\quad\text{is periodic in }\ \mathbf{x}\quad\text{with period }L. \tag{39}$$

Remark 3 We remark that this method is new. It interesting because it gives the fastest convergence.

Theorem 3 *Let us suppose hypothesis (***H1***)-(***H4***) and $\beta\ge\alpha^2/(2K_{dr})$. Let $\mathscr{S}$ be the operator mapping $\{\mathbf{u}^n,p^n\}$ to $\{\mathbf{u}^{n+1},p^{n+1}\}$. Then $\mathscr{S}$ is a contraction and admits a unique fixed point from $V_T\times W_T$ satisfying (1)-(7). The contraction constant is smallest for $\beta=\alpha^2/(2K_{dr})$ and takes value $\gamma_W=\dfrac{M\alpha^2}{M\alpha^2+2K_{dr}}$.*

Proof Let us introduce the following notation for "artificial" volumetric mean total stress

$$\sigma_\beta=\sigma_0+K_{dr}\text{ div }\mathbf{u}-\frac{\beta K_{dr}}{\alpha}(p-p_0).$$

Then (36) and (37) hold true for the differences $\delta\mathbf{u}^{n+1},\delta\mathbf{u}^n,\delta p^n$ and δp^{n+1}, with $f=0$ and $g=0$.

Step 1. We multiply the variant of (36), valid for δp^{n+1} and $\delta\partial_t\sigma^n_\beta$, by $\partial_t\delta p^{n+1}$ and get

$$\Big(\frac{1}{M}+\beta\Big)\frac{\alpha^2}{\beta^2K^2_{dr}}\int_0^t\int_\Omega|\frac{\beta K_{dr}}{\alpha}\partial_\tau\delta p^{n+1}|^2\,dxd\tau+$$
$$\int_\Omega\frac{k}{2B_f\eta}|\nabla\delta p^{n+1}(t)|^2\,dx\le-\frac{\alpha}{K_{dr}}\int_0^t\int_\Omega\partial_\tau\delta p^{n+1}\delta\partial_\tau\sigma^n_\beta\,dxd\tau$$
$$\le\frac{\varepsilon}{2}\frac{\alpha^2}{\beta^2K^2_{dr}}\int_0^t\int_\Omega|\frac{\beta K_{dr}}{\alpha}\delta\partial_\tau p^{n+1}|^2\,dxd\tau+$$
$$\frac{\alpha^2}{2\varepsilon K^2_{dr}}\int_0^t\int_\Omega(\partial_\tau\delta\sigma^n_\beta)^2\,dxd\tau,\ \forall\varepsilon>0.$$

Again, the coefficient in front of $||\delta\partial_t\sigma^n_\beta||_{L^2(\Omega\times(0,t))}$ is smallest for $\varepsilon=\beta+1/M$ and the above estimate becomes

$$\int_0^t\int_\Omega|\frac{\beta K_{dr}}{\alpha}\delta\partial_\tau p^{n+1}|^2\,dxd\tau+\bar{k}_1\int_\Omega|\nabla\delta p^{n+1}(t)|^2\,dx$$
$$\le\Big(\frac{\beta}{\beta+1/M}\Big)^2\int_0^t\int_\Omega(\partial_\tau\delta\sigma^n_\beta)^2\,dxd\tau, \tag{40}$$

where $\bar{k}_1=\dfrac{k}{B_f\eta}(\dfrac{\beta K_{dr}}{\alpha})^2\dfrac{1}{\beta+1/M}$.

Step 2.

Next we take the time derivative of (37), valid for $\{\delta\mathbf{u}^{n+1},\delta p^{n+1}\}$, and test the resulting equation by $\partial_t\delta\mathbf{u}^{n+1}$. It yields

$$\int_\Omega\mathscr{G}e(\partial_t\delta\mathbf{u}^{n+1}):e(\partial_t\delta\mathbf{u}^{n+1})=\alpha\int_\Omega\partial_t\delta p^{n+1}\text{ div }\partial_t\delta\mathbf{u}^{n+1}\,dx,$$

which implies

$$2a\frac{\beta K^2_{dr}}{\alpha^2}\int_\Omega|e(\partial_t\delta\mathbf{u}^{n+1})|^2\,dx+$$
$$2\frac{\beta K_{dr}}{\alpha^2}\int_\Omega|K_{dr}\text{ div }\partial_t\delta\mathbf{u}^{n+1}|^2\,dx\le$$
$$2\int_\Omega\frac{\beta K_{dr}}{\alpha}\partial_t\delta p^{n+1}K_{dr}\text{ div }\delta\partial_t\mathbf{u}^{n+1}\,dx. \tag{41}$$

We note that the right hand side of (41) represents the integral of the product of the terms from the definition of $\partial_t\sigma_\beta$. After summing up (40) and (41) and using the definition of σ_β, one has

$$\int_0^t\int_\Omega(\partial_\tau\delta\sigma^{n+1}_\beta)^2\,dxd\tau+(2\frac{\beta K_{dr}}{\alpha^2}$$
$$-1)\int_0^t\int_\Omega|K_{dr}\text{ div }\partial_t\delta\mathbf{u}^{n+1}|^2\,dxd\tau+$$
$$\frac{2a\beta K^2_{dr}}{\alpha^2}\int_0^t\int_\Omega|e(\partial_t\delta\mathbf{u}^{n+1})|^2\,dxd\tau+\bar{k}_1\int_\Omega|\nabla\delta p^{n+1}(t)|^2\,dx$$
$$\le\Big(\frac{\beta}{\beta+1/M}\Big)^2\int_0^t\int_\Omega(\partial_\tau\delta\sigma^n_\beta)^2\,dxd\tau. \tag{42}$$

The estimate (42) yields a contraction map only if $\beta \geq \alpha^2/(2K_{dr})$. The contraction constant is smallest for $\beta = \alpha^2/(2K_{dr})$. Now we proceed as in Subsection 2.1 and obtain the result: The expression on the left hand side defines the invariant distance d_W by

$$d_W^2\Big((\mathbf{u},p),0\Big) = \frac{2a\beta K_{dr}^2}{\alpha^2}||e(\partial_t \mathbf{u})||^2_{L^2(\Omega\times(0,T))^3} +$$

$$\bar{k}_1 \max_{0\leq t\leq T}||\nabla p(t)||^2_{L^2(\Omega)} + (2\frac{\beta K_{dr}}{\alpha^2}-1)||K_{dr}\text{div } \partial_t \mathbf{u}||^2_{L^2(\Omega\times(0,T))}$$

$$+||\underbrace{\partial_t(-\frac{\beta K_{dr}}{\alpha}p + K_{dr}\text{ div }\mathbf{u})}_{\partial_t \sigma_\beta}||^2_{L^2(\Omega\times(0,T))}, \qquad (43)$$

on the closed subspace

$$\mathscr{Q} = \{(\mathbf{A},B)\in V_T\times W_T \,|\mathbf{A}|_{t=0}=0 \text{ and } B|_{t=0}=0\,\}. \qquad (44)$$

We see that that the operator $\mathscr{S}$, such that $\mathscr{S}(\mathbf{u}^n,p^n) = (\mathbf{u}^{n+1},p^{n+1})$, maps $\mathscr{Q}$ into itself.

We find out that

$$d_w\Big((\mathbf{u}^{n+1},p^{n+1})-(\mathbf{u}^n,p^n)\Big) \leq$$

$$\gamma_{FSW}\, d_{fs}\Big((\mathbf{u}^n,p^n)-(\mathbf{u}^{n-1},p^{n-1})\Big) \qquad (45)$$

with $\gamma_{FSW} = \dfrac{\beta}{\beta+1/M} < 1$. Hence $\mathscr{S}$ is a contraction mapping on $\mathscr{Q}$ and by the **contraction mapping principle**, it has a unique fixed point in $\mathscr{Q}$.□

Acknowledgements The authors would like to thank the (anonymous) referee for careful reading of the paper and for the hint about getting the optimal contraction constant in Subsection 2.1.

References

1. R.H. Dean, J.H. Schmidt, Hydraulic-Fracture Predictions with a Fully Coupled Geomechanical Reservoir Simulator, SPE Journal, Vol. 14 (2009), p. 707-714.
2. J. Kim, H.A. Tchelepi, R. Juanes, Stability, Accuracy and Efficiency of Sequential Methods for Coupled Flow and Geomechanics, SPE International, SPE 119084, p. 1-19.
3. L. Ji, A. Settari, R.B. Sullivan, A Novel Hydraulic Fracturing Model Fully Coupled With Geomechanics and Reservoir Simulation, SPE Journal, Vol. 14 (2009), 423–430.
4. P. Samier, S. De Gennaro, Iterative Coupling of Geomechanics With Reservoir Simulation, SPE Reservoir Simulation Symposium, 26-28 February 2007, Houston, Texas, U.S.A.
5. A. Settari, F. Maurits, Coupled reservoir and geomechanical simulation system, SPE J, Vol. 3 (1998), 219-226.
6. I. Tolstoy, ed. Acoustics, elasticity, and thermodynamics of porous media, Twenty-one papers by M. A. Biot, Acoustical Society of America, New York (1992).