Theory of the dynamic Biot-Allard equations and their link to the quasi-static Biot system[a]

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We undertake establishing well-posedness of the dynamic Biot-Allard equations. It is obtained using the precise properties of the dynamic permeability matrix following the homogenization derivation of the model. By taking the singular limit of the contrast coefficient, the quasi-static Biot system can be obtained from the dynamic Biot equations. These results can be used to formulate an efficient computational algorithm for solving dynamic Biot-Allard equations for subsurface flows with the characteristic reservoir time scales larger than the intrinsical characteristic time. This result appears to be completely new in the literature on Biot's theory. We conclude by showing that in the case of periodic deformable porous media the dynamic permeability has the required properties.

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I. INTRODUCTION

Effective deformation and filtration in a deformable porous medium is described by the Biot-Allard equations

$$\rho \partial_{tt}\mathbf{u} - \text{ Div } \{A^H e(\mathbf{u}) - \alpha p\} + \partial_t \int_0^t \mathcal{A}(\frac{(t-\zeta)\eta}{\rho_f \ell^2})(\rho_f \mathbf{F}(x,\zeta) - \nabla p(x,\zeta) - \rho_f \partial_{\zeta\zeta}\mathbf{u}(x,\zeta)) \, d\zeta = \rho \mathbf{F}, \qquad (1)$$

$$\partial_t \Big(Mp + \text{ div } (\alpha \mathbf{u}) \Big) + \text{ div } \{ \int_0^t \mathcal{A}(\frac{(t-\zeta)\eta}{\rho_f \ell^2})(\mathbf{F}(x,\zeta) - \frac{1}{\rho_f}\nabla p(x,\zeta) - \partial_{\zeta\zeta}\mathbf{u}(x,\zeta)) \, d\zeta \} = 0 \qquad (2)$$

where $e(*)$ stands for the symmetrized gradient (strain tensor), $\mathbf{u}$ is the effective solid phase displacement, p is the effective pressure and the parameters are defined in Table I

SYMBOL	*QUANTITY*	*UNITY*
ρ_s	solid grain density	kg/m^3
ρ_f	pore fluid density	kg/m^3
φ	porosity	$0 < \varphi < 1$
$\rho = \rho_f \varphi + \rho_s(1-\varphi)$	effective mass density	kg/m^3
η	pore fluid viscosity	$kg/$m sec
ℓ	typical pore size	m
α	Biot's pressure-storage coupling tensor	dimensionless
M	combined porosity and compressibility of the fluid and solid	dimensionless
A^H	Gassman's fourth order effective elasticity tensor	Pa
$\rho_f \ell^2/\eta$	intrinsical characteristic time	sec
$\mathcal{A}$	dynamic permeability tensor	dimensionless
Λ	characteristic Gassman's coefficient	Pa
E_f	pore fluid bulk modulus	Pa

TABLE I. *Effective coefficients for the Biot-Allard equations*

The equations (1)-(2) represent the solid displacement - pressure real time formulation of the dynamic Biot's equation, usually written in the frequency formulation (see the collection of Biot's papers[1]). Note that the dynamical permeability corresponds to the inverse Laplace transform of the inverse of Biot's viscodynamic operator.

These equations are obtained from the linearized first principles fluid/structure pore level interaction, using homogenization approach. We suppose small deformations, small fluid compressibility and small Reynolds numbers, which allows us to make the following important simplifications:

1. dropping the inertial term in the Navier-Stokes equations,

2. supposing an incompressible or a slightly compressible pore fluid,

3. using a linear elastic model (Navier's equations) to describe the solid skeleton and

4. linearization of the fluid/solid interface coupling conditions.

Furthermore, the initial porous medium configuration is heterogeneous at the pore level but statistically homogeneous at macroscopic level. It is supposed that there are two connected phases, a solid and a fluid one. The solid phase is deformable.

A representative example of such geometry is the *periodic* porous medium with connected fluid and solid phases. It is obtained by a periodic arrangement of the pores and effective coefficients can be determined using periodic representative volume elements.

First we define the geometrical structure inside the unit cell $\mathcal{Y} = (0,1)^3$. Following Allaire[2] we make the following assumptions on the geometry:

A1: $\mathcal{Y}_s$ (the solid part) is a closed subset of $\bar{\mathcal{Y}}$ of strictly positive measure and $\mathcal{Y}_f = \mathcal{Y} \backslash \mathcal{Y}_s$ (the pore containing the fluid). Then $\mathcal{Y}_f$ is supposed to be an open connected periodic set of strictly positive measure, with a smooth boundary.

A2: We make periodic repetition of $\mathcal{Y}_s$ over $\mathbb{R}^n$ and, For sufficiently small $\ell > 0$, set $\mathcal{Y}^\ell_{S_i} = \ell(\mathcal{Y}_s + k)$, $k \in \mathbb{Z}^n$. Let $T_\ell = \{k \in \mathbb{Z}^3 | \mathcal{Y}^\ell_{S_k} \subseteq \overline{\Omega}_L\}$. The construction yields a closed solid skeleton $\Omega^\ell_s = \bigcup_{k \in T_\ell} \mathcal{Y}^\ell_{S_k}$, the fluid structure interface $\Gamma^\ell = \partial\Omega^\ell_s \setminus \partial\Omega_L$ and the pore volume filled with a fluid $\Omega^\ell_f = \Omega \setminus \Omega^\ell_s$. Both Ω^ℓ_s and Ω^ℓ_f are connected and Γ^ℓ is a smooth surface.

Figure 1 shows a typical pore satisfying (**A1**).

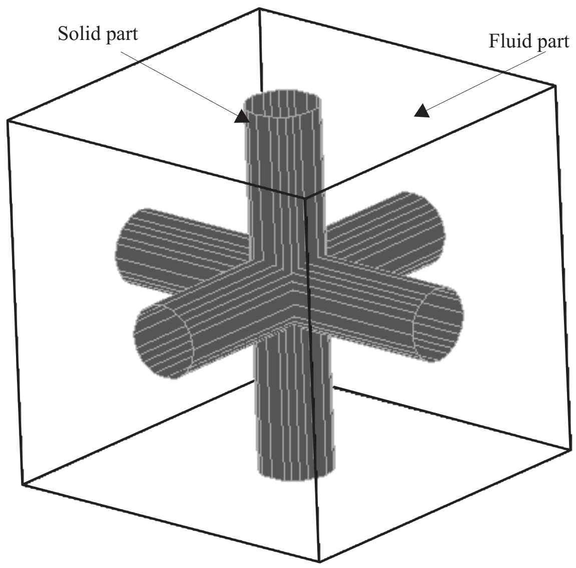

FIG. 1. The pore RVE

The linear fluid-structure equations in the reference configuration read

$$\rho_f \frac{\partial^2 \mathbf{u}_f^\ell}{\partial t^2} + \nabla p^\ell = \eta \Delta \frac{\partial \mathbf{u}_f^\ell}{\partial t} + \rho_f \mathbf{F} \ \text{ in } \quad \Omega_f^\ell \tag{3}$$

$$\frac{p^\ell}{\rho_f E_f} + \ \text{div}\ \mathbf{u}_f^\ell = 0 \ \text{ in } \quad \Omega_f^\ell \tag{4}$$

$$\rho_s \frac{\partial^2 \mathbf{u}_s^\ell}{\partial t^2} = \ \text{div}(Ae(\mathbf{u}_s^\ell)) + \rho_s \mathbf{F} \ \text{ in } \quad \Omega_s^\ell \tag{5}$$

$$\mathbf{u}_s^\ell = \mathbf{u}_f^\ell \ \text{ on } \quad \Gamma^\ell \quad \text{(displacement continuity at the interface)} \tag{6}$$

$$(-p^\ell I + 2\eta e(\frac{\partial \mathbf{u}_f^\ell}{\partial t}))\mathbf{n} = Ae(\mathbf{u}_s^\ell)\mathbf{n} \ \text{ on } \Gamma^\ell. \tag{7}$$

We note that the 4th order tensor A contains the elasticity coefficients of the solid skeleton and that we have supposed a linear coupling, evaluated in (6)-(7) at the non-deformed interface Γ^ℓ.

In the references[3],[4],[5],[9],[10] either Laplace's transform in time is applied to system (3)-(7) or the excitation by an external harmonic source with frequency ω is supposed. After identifying the characteristic pore size $\varepsilon = \ell/L$ (the ratio between two length scales) as the small parameter, the technique of homogenization[4] was applied. It allowed Burridge and Keller in[3] and Sanchez-Palencia et al in[4] to derive from (3)-(7) the homogenized systems in slow variable x and the fast variable $y = x/\varepsilon$ (see also references therein). Most of the published work was formal, but there are rigorous homogenization derivations in the fundamental book[4] and in the article[5].

The rigorous homogenization of (3)-(7) in space and in time variables is in[6] and[7]. The case of an inviscid fluid filling the pores is treated in[8]. Furthermore in these references the separation between slow and fast scales was undertaken, which allowed reducing the two-scale homogenized equations to the Biot equations from[1]. In[8] the scales separation yielded Biot's equation for a porous medium filled by an inviscid fluid. The viscous case is more complicated, and in[6] and[7], Biot's equations were justified but with a tensorial viscodynamic operator.

In the case of pores filled by a viscous fluid, the result depends on the *contrast of property number* $C = \eta T/\Lambda = \eta/(\Lambda\omega)$. If $C = O(\varepsilon^2)$ the homogenization approach gives *a diphasic macroscopic behavior of the fluid-solid mixture*, described by the *diphasic Biot system.* This case was given a particular attention in[3],[4],[5],[7],[9],[10]. This regime arises in most applications and we suppose in text which follows that $C = O(\varepsilon^2)$. The rigorous convergence result, for fixed frequency, and using the newly introduced two-scale convergence is in article[5] by Nguetseng. We explain the asymptotic analysis result following[7], were the system (6)-(7) was homogenized in space and in time variables and the form of the equations presented here was given.

The technique of homogenization involves the following steps:

1. We write the unknowns $\mathbf{u}_f^\ell, \mathbf{u}_s^\ell$ and p^ℓ as functions of ε , of the slow spatial scale x/L and of the fast spatial scale $y/L = x/\ell$. Then the unknowns are expanded in a power series in ε :

$$\begin{cases} \mathbf{u}_f^\ell = \mathbf{u}_f^0(x,y,t) + \varepsilon \mathbf{u}_f^1(x,y,t) + \dots, \\ \mathbf{u}_s^\ell = \mathbf{u}_s^0(x,y,t) + \varepsilon \mathbf{u}_s^1(x,y,t) + \dots, \\ p^\ell = p^0(x,y,t) + \varepsilon p^1(x,y,t) + \dots. \end{cases} \tag{8}$$

   Next, in the system (6)-(7) the differential operators are expressed in both slow and fast spatial variables and the perturbative series (8) are introduced into the rescaled equations. After equating terms with equal powers of ε the two-scale homogenized equations are obtained. They contain $\mathbf{u}_f^0, \mathbf{u}_s^0, \mathbf{u}_s^1, p^0$ and p^1 as unknown functions and number of spatial variables is doubled. The scale separation is needed.

2. We follow[7] and find out that

$$\mathbf{u}_s^0 = \mathbf{u}(x,t), \quad p^0(x,y,t) = \chi_{\mathcal{Y}_f}(y) p(x,t) \quad \text{and}$$
$$\mathbf{u}_s^1 = p(x,t)\mathbf{w}^0(y) + \sum_{i,j=1}^{3} \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\mathbf{w}^{ij}(y),$$

where $\mathbf{w}^{ij}$ correspond to the homogenization of the empty elastic deformable porous medium and depend only on the porous medium geometry. The periodic cell average of the corresponding stress $A(e_y(\mathbf{w}^{ij}) + \text{ sym } \mathbf{e}^i \otimes \mathbf{e}^j)$ gives the positive definite 4th order Gassman tensor A^H. Biot's tensor α is given by $\alpha_{ij} = \displaystyle\int_{\mathcal{Y}_f} \text{div}_y \mathbf{w}^{ij} \, dy$. $\mathbf{w}^0$ is displacement of the elastic solid structure of the unit pore to which the unit pressure is applied at the fluid/solid interface. It satisfies $\int_{\mathcal{Y}_f} \text{div}_y \mathbf{w}^0 \, dy < 0$.

3. The fluid may be regarded at the leading order as incompressible at y scale. $\{\partial_t \mathbf{u}_f^0, p^1\}$ satisfies the non-stationary Stokes variable in y variable in the pore. The forcing term is $\rho_f \mathbf{F} - \nabla_x p(x,t)$. The solid-fluid interface condition is $\partial_t \mathbf{u}_f^0 = \partial_t \mathbf{u}$. The separation of the fast and slow variables yields

$$\partial_t \mathbf{u}_f^0 = \partial_t \mathbf{u} + \frac{1}{\eta} \int_0^t Q(y, \frac{t-\zeta}{\rho_f \ell^2}\eta)(\rho_f \mathbf{F} - \nabla_x p(x,t) - \rho_f \partial_{\zeta\zeta} \mathbf{u}) \, d\zeta, \tag{9}$$

where the jth column $\mathbf{q}^j$ of the matrix Q satisfies the non-stationary incompressible pore Stokes system (50)-(52). The dynamic permeability $\mathcal{A}(t)$ is defined as $\mathcal{A}(t) = \displaystyle\int_{\mathcal{Y}_f} Q(y,t) \, dy$. Note that the complex vector valued Laplace transform of $\partial_t(\mathbf{u}_f^0 - \mathbf{u})$ satisfies Darcy's law, with $\hat{\mathcal{A}}(\omega)$ being the permeability.

4. The structure momentum equation (1) is obtained from the compatibility condition for the solvability of the unit cell solid part Navier's equations for $\mathbf{u}^2$. In addition to the classical homogenization of an elastic porous structure (see[4]), the pore fluid applies the contact interface force $(p^1 I - 2\eta e_y(\partial_t \mathbf{u}_f^0))\mathbf{n}$ on the solid structure. Averaging all terms linked to $\mathbf{u}_s^1$ in the compatibility condition and using (9), yields equation (1).

5. The effective pressure equation (2) is obtained by averaging the next order of expansion in the incompressibility condition. Average of $\text{div}_y \mathbf{u}_f^1$ over $\mathcal{Y}_f$ is equal to the average of -$\text{div}_y \mathbf{u}_s^1$ over $\mathcal{Y}_s$. Inserting the separation of scales formula for $\mathbf{u}_s^1$ yields equation (2). It is important to note that we have the same Biot coefficient matrix α in the term

div(αp) in (1) and in div($\alpha\partial_t\mathbf{u}$) in (2). Next, $M = -\frac{1}{\Lambda}\int_{\mathcal{Y}_f} \mathrm{div}_y \mathbf{w}^0 \, dy + \frac{\varphi}{\rho_f E_f}$ and even for an incompressible fluid, there would be a term -($\int_{\mathcal{Y}_f} \mathrm{div}_y \mathbf{w}^0 \, dy)\partial_t p$, corresponding to the effective compressibility of the solid-fluid mixture.

Therefore the effective behavior is described by the effective solid phase displacement $\mathbf{u}$ and the effective pressure p. they are defined at **every point** of Ω_L and we do not distinguish the solid and fluid phases any more. The filtration velocity is

$$\partial_t \mathbf{u} + \frac{1}{\eta}\int_0^t \mathcal{A}(\frac{t-\zeta}{\rho_f \ell^2}\eta)(\rho_f \mathbf{F} - \nabla_x p(x,t) - \rho_f \partial_{\zeta\zeta}\mathbf{u}) \, d\zeta.$$

Concerning the approximation, it is proved in[7] that

$$\sqrt{\rho_f}\chi_{\mathcal{Y}_f}(\frac{x}{\varepsilon})(\mathbf{u}_f^\ell - \mathbf{u}_f^0(x,\frac{x}{\varepsilon},t)) + \sqrt{\rho_s}(\mathbf{u}_s^\ell - \mathbf{u})\chi_{\mathcal{Y}_s}(\frac{x}{\varepsilon}) \to 0 \;\text{ in }\; C([0,T];L^2(\Omega)^3), \;\text{as}\; \varepsilon = \frac{\ell}{L} \to 0.$$

For mathematical considerations we need the dimensionless form of the system (1)-(2). If the characteristic size of the Young modulus in A^H is Λ and the time scale T_c, then the observation length L of Terzaghi (see[3]) for the problem is $L = \ell\sqrt{\Lambda T_c/\eta}$ and the characteristic pressure $P = \sqrt{\Lambda\eta/T_c}$. The dimensionless form of (1)-(2) is given by

$$\kappa\partial_{tt}\mathbf{u}^0 - \text{ Div }\{A^0 D(\mathbf{u}^0) - \alpha p^0\} + \partial_t \int_0^t \mathcal{A}(\frac{t-\zeta}{\kappa_f})(\psi_f \mathbf{F}(x,\zeta)$$
$$-\nabla p^0(x,\zeta) - \kappa_f \partial_{\zeta\zeta}\mathbf{u}^0(x,\zeta)) \, d\zeta = \psi \mathbf{F}(x,t), \tag{10}$$

$$M^0\partial_t p^0 + \text{ div }\{\int_0^t \mathcal{A}(\frac{t-\zeta}{\kappa_f})(\frac{\psi_f}{\kappa_f}\mathbf{F} - \frac{1}{\kappa_f}\nabla p^0 - \partial_{\zeta\zeta}\mathbf{u}^0) \, d\zeta\} + \text{ div }\{\alpha\partial_t \mathbf{u}^0\} = 0 \tag{11}$$

where $M^0 = M\Lambda$, $A^0 = A^H/\Lambda$, $\psi_f = T_c F_0\sqrt{\rho_f \ell \, \eta}$, $\psi_s = T_c F_0 \sqrt{\rho_s \ell \, \eta}$ and $\psi = \psi_f\varphi + \psi_s(1 - \varphi)$. Contrast coefficients are $\kappa_f = \frac{\rho_f \ell^2}{\eta T_c}$ and $\kappa_s = \frac{\rho_s \ell^2}{\eta T_c}$, respectively. $\kappa = \kappa_f\varphi + \kappa_s(1-\varphi)$.

II. A PRIORI ESTIMATES FOR THE DYNAMIC BIOT EQUATIONS

We study the system (10), (11) in the cube $\Omega = (0,1)^3$, for $t \in (0,T)$. For simplicity we assume homogeneous initial conditions and periodic boundary conditions.

For a Hilbert space X, let $C_0^\infty(\mathbb{R}_+;X)$ denotes infinitely differentiable functions defined on $\mathbb{R}$ with values in X and with compact support on $\mathbb{R}_+ = (0,+\infty)$.

With notation $\Re\tau$ for the real part of $\tau \in \mathbb{C}$, we introduce the complex Laplace transform in time of a vector valued function f and denote it $\hat{f}$. It is defined for $\tau \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} : \text{Re } \lambda > \lambda_0 > 0\}$. We apply it to the system (10), (11) to have

Theorem 1. *Let us suppose* $\mathbf{F} \in C_0^\infty(\mathbb{R}_+; L^2(\Omega)^3)$ *and*

: *(H1)* A^0 *is a symmetric positive definite 4th order tensor.*

: *(H2)* M^0 *is a positive constant.*

: *(H3)* α *is a symmetric matrix.*

: *(H4) The Laplace transform of* $\hat{\mathcal{A}}$ *of the dynamic permeability matrix* $\mathcal{A}$ *is a complex symmetric (but not Hermitian) matrix and satisfies*

$$\Re\{\hat{\mathcal{A}}(\tau)\xi\overline{\xi}\} \geq C\frac{\lambda_1 + \Re\tau}{|\lambda_1+\tau|^2}|\xi|^2, \ \forall \xi \in \mathbb{C}^3, \tag{12}$$

$$\Re\{(\tau + c_1)\hat{\mathcal{A}}(\tau)\xi\overline{\xi}\} \geq C|\xi|^2, \ \ \textit{for some} \ \ c_1 \geq 0 \ \ \textit{and} \ \ \forall \xi \in \mathbb{C}^3, \tag{13}$$

$$\Re\{\tau(\kappa I - \tau\kappa_f^2\hat{\mathcal{A}}(\tau\kappa_f))\xi\overline{\xi} + \hat{\mathcal{A}}(\kappa_f\tau)\beta\overline{\beta} + (\alpha - \tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))\beta\overline{\xi} - \overline{(\alpha - \tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))\overline{\xi}\beta}\}$$
$$\geq \kappa_s(1-\varphi)\Re\tau|\xi|^2 + C\frac{\kappa_f\Re\tau}{|\lambda_1 + \kappa_f\tau|^2}|\beta|^2, \ \forall \xi, \beta \in \mathbb{C}^3, \tag{14}$$

$$||\hat{\mathcal{A}}(\tau)||_\infty \leq C\frac{1}{|\lambda_1+\tau|}, \tag{15}$$

for some $\lambda_1 > 0$.

Then we have the following a priori estimate:

$$\Re\tau\int_\Omega\{\kappa_s|\tau\hat{\mathbf{u}}^0|^2 + |e(\hat{\mathbf{u}}^0)|^2 + |\hat{p}^0|^2\}\ dx + \frac{\Re\tau\kappa_f}{|\lambda_1+\tau\kappa_f|^2}\int_\Omega|\nabla\hat{p}^0|^2\ dx \leq C||(1+|\tau|)\hat{\mathbf{F}}||^2_{L^2(\Omega)}. \tag{16}$$

Remark 1. *As discussed in the Appendix, the hypothesis (H1)-(H4) are natural. In particular, the matrix* $\hat{\mathcal{A}}$ *is not uniformly positive definite with respect to* τ *and (14)-(15) give its precise behavior. It is known from the literature, that in the case* $\Re\tau = 0$*, the impedance* $\hat{\mathcal{A}}(i\,\mathrm{Im}\,\tau)$ *goes to zero for large values of* $\mathrm{Im}\,\tau$*. For details we refer to*[9] *and*[10] .

Proof. Application of the Laplace transform to (10) and (11) with $\tau \in \mathbb{C}_+$ yields

$$-\ \mathrm{Div}\ \{A^0e(\hat{\mathbf{u}}^0)\} +\ \mathrm{Div}\ \{(\alpha - \tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))\hat{p}^0\} +$$
$$(\kappa I -\ \tau\kappa_f^2\hat{\mathcal{A}}(\tau\kappa_f))\tau^2\hat{\mathbf{u}}^0 = (\psi I - \tau\kappa_f\psi_f\hat{\mathcal{A}}(\tau\kappa_f))\hat{\mathbf{F}}(x,\tau), \tag{17}$$

$$M^0\tau\hat{p}^0 +\ \mathrm{div}\ \{(\alpha - \tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))\tau\hat{\mathbf{u}}^0\} -\ \mathrm{div}\ \{\hat{\mathcal{A}}(\tau\kappa_f)\nabla\hat{p}^0\} = -\ \mathrm{div}\ \{\hat{\mathcal{A}}(\tau\kappa_f)\psi_f\hat{\mathbf{F}}(x,\tau)\}. \tag{18}$$

Now we test equation (17) with $\overline{\tau\hat{\mathbf{u}}^0}$, take the complex conjugate of (18), test it with $\hat{p}^0$ and sum up the obtained variational equalities, to get

$$\tau\int_\Omega ((\kappa I-\tau\kappa_f^2\hat{\mathcal{A}}(\tau\kappa_f))\tau\hat{\mathbf{u}}^0\overline{\tau\hat{\mathbf{u}}^0}\,dx+\overline{\tau}\int_\Omega A^0 e(\hat{\mathbf{u}}^0):e(\overline{\hat{\mathbf{u}}^0})\,dx+\overline{\tau}M^0\int_\Omega|\hat{p}^0|^2\,dx+$$
$$\int_\Omega\overline{\hat{\mathcal{A}}(\tau\kappa_f)}\nabla\overline{\hat{p}}^0\nabla\hat{p}^0\,dx+\int_\Omega\Big((\alpha-\tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))\nabla\hat{p}^0\overline{\tau\hat{\mathbf{u}}^0}-\overline{(\alpha-\tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))}\overline{\tau\hat{\mathbf{u}}^0}\nabla\hat{p}^0\Big)\,dx$$
$$=\int_\Omega\psi_f\overline{\hat{\mathcal{A}}(\tau\kappa_f)\hat{\mathbf{F}}(x,\tau)}\nabla\hat{p}^0\,dx+\int_\Omega(\psi I-\tau\kappa_f\psi_f\hat{\mathcal{A}}(\tau\kappa_f))\hat{\mathbf{F}}(x,\tau)\overline{\tau\hat{\mathbf{u}}^0}\,dx. \qquad (19)$$

Using (14)-(15) yields

$$\Re\tau\int_\Omega\{\kappa_s|\tau\hat{\mathbf{u}}^0|^2+|e(\hat{\mathbf{u}}^0)|^2+|\hat{p}^0|^2\}\,dx+\frac{\Re\tau\kappa_f}{|\lambda_1+\tau\kappa_f|^2}\int_\Omega|\nabla\hat{p}^0|^2\,dx\le$$
$$C||\hat{\mathbf{F}}||_{L^2(\Omega)}\Big(||\frac{\sqrt{\Re\tau\kappa_f}\nabla\hat{p}^0}{|\lambda_1+\tau\kappa_f|}||_{L^2(\Omega)}+||\tau\hat{\mathbf{u}}^0||_{L^2(\Omega)}\Big). \qquad (20)$$

(20) implies estimate (16). □

In order to return to the original time variable we need to invert the Laplace transform. In the vector valued setting we use the following result:

Let X be a Hilbert space, and let $H^2(\mathbb{C}_+,X)$ be the subset of the space of holomorphic functions defined by

$$H^2(\mathbb{C}_+,X)=\{\,h:\mathbb{C}_+\to X\text{ such that}$$
$$||h||^2_{H^2(\mathbb{C}_+,X)}=\sup_{x>0}\int_{\mathbb{R}}||h(x+is)||^2_X\,ds<+\infty\}.$$

Then we have

Theorem 2. *(vector valued Paley-Wiener theorem from[11], page 48) Let X be a Hilbert space. Then the map $f\to\hat{f}|_{\mathbb{C}}$ is an isometric isomorphism of $L^2(\mathbb{R}_+,X)$ onto $H^2(\mathbb{C}_+,X)$.*

In our situation $X=L^2(\Omega)$ and by Theorem 2 estimate (16) yields

Corollary 1. *We have the following a priori estimate*

$$\int_0^T\int_\Omega\{\kappa_s|\partial_t\mathbf{u}^0|^2+|e(\mathbf{u}^0)|^2+|p^0|^2\}\,dxdt+$$
$$\int_0^T\int_\Omega|\nabla\frac{1}{\sqrt{\kappa_f}}\int_0^t e^{-\lambda_1(t-\zeta)/\kappa_f}p^0(x,\zeta)\,d\zeta|^2\,dxdt\le C\int_0^{+\infty}\int_\Omega(|\mathbf{F}|^2+|\partial_t\mathbf{F}|^2)\,dxdt. \qquad (21)$$

In order to avoid negative Sobolev norm in time for ∇p^0 in (21) we do further calculations. Namely we multiply the unknowns $\{\hat{\mathbf{u}}^0,\hat{p}^0\}$ in (17) and (18) by $(\lambda_1+\kappa_f\tau)$, respectively, and apply the same argument as in Theorem 1 to them. Using estimate (16) we obtain

Proposition 1. *Under the assumptions of Theorem 1, we have*

$$\int_\Omega \{\kappa_s|\tau^2\hat{\mathbf{u}}^0|^2+|e(\tau\hat{\mathbf{u}}^0)|^2+|\tau\hat{p}^0|^2\}\ dx+\int_\Omega|\nabla\hat{p}^0|^2\ dx\le\frac{C(\kappa_f,\kappa_s)}{\Re\tau}||(1+|\tau|^2)\hat{\mathbf{F}}||^2_{L^2(\Omega)}. \quad (22)$$

and

$$\int_0^T\int_\Omega\left(\kappa_s|\partial_{tt}\mathbf{u}^0|^2+|e(\partial_t\mathbf{u}^0)|^2+|\partial_t p^0|^2\right)\ dxdt+\int_0^T\int_\Omega|\nabla p^0|^2\ dxdt\le$$
$$C(\kappa_f,\kappa_s)\int_0^{+\infty}\sum_{m=0}^2\int_\Omega|\partial_t^m\mathbf{F}|^2\ dxdt. \quad (23)$$

It is important to control derivatives independently of κ_f. It is achieved by proving the estimate (16) for $\{\tau\hat{\mathbf{u}}^0,\tau\hat{p}^0\}$.

Proposition 2. *Let* $\int_\Omega\mathbf{F}\ dx=0$*, for every* $t>0$*. Under the assumptions of Theorem 1, we have*

$$\int_\Omega\{\kappa_s|\tau^2\hat{\mathbf{u}}^0|^2+|e(\tau\hat{\mathbf{u}}^0)|^2+|e(\hat{\mathbf{u}}^0)|^2+|\hat{\mathbf{u}}^0|^2+$$
$$\tau\hat{p}^0|^2+|\hat{p}^0|^2+|\nabla\hat{p}^0|^2\}\ dx\le C||(1+|\tau|^2)\hat{\mathbf{F}}||^2_{L^2(\Omega)}. \quad (24)$$

and

$$\int_0^T\int_\Omega\{\kappa_s|\partial_{tt}\mathbf{u}^0|^2+|e(\partial_t\mathbf{u}^0)|^2+|e(\mathbf{u}^0)|^2+|\mathbf{u}^0|^2+|\partial_t p^0|^2+|\nabla p^0|^2\}\ dxdt$$
$$\le C\int_0^{+\infty}\sum_{m=0}^2\int_\Omega|\partial_t^m\mathbf{F}|^2\ dxdt. \quad (25)$$

Proof. Arguing as in the proof of Theorem 1, we obtain

$$\Re\tau\int_\Omega\{\kappa_s|\tau^2\hat{\mathbf{u}}^0|^2+|e(\tau\hat{\mathbf{u}}^0)|^2+|e(\hat{\mathbf{u}}^0)|^2+|\tau\hat{p}^0|^2+|\hat{p}^0|^2\}\ dx\le C||(1+|\tau|^2)\hat{\mathbf{F}}||^2_{L^2(\Omega)}. \quad (26)$$

Since $\int_\Omega\hat{\mathbf{F}}\ dx=0$, after integrating equation (17) over Ω and using (77), we obtain $\int_\Omega\hat{\mathbf{u}}^0\ dx=0$. Korn's inequality implies

$$\int_\Omega|\hat{\mathbf{u}}^0|^2\ dx\le C\int_\Omega|e(\hat{\mathbf{u}}^0)|^2\ dx.$$

Next, we test equation (18) with $\overline{\hat{p}}^0$ and use (26) and (12) to conclude that

$$\frac{\lambda_1+\Re\tau\kappa_f}{|\lambda_1+\tau\kappa_f|^2}\int_\Omega|\nabla\hat{p}^0|^2\ dx\le C||(1+|\tau|^2)\hat{\mathbf{F}}||^2_{L^2(\Omega)}. \quad (27)$$

Finally, testing equation (18) with $\tau \overline{\hat{p}}^0$ and using (26) and (13) yields

$$\int_\Omega |\nabla \hat{p}^0|^2 \, dx \leq C ||(1 + |\tau|^2) \hat{\mathbf{F}}||^2_{L^2(\Omega)}. \tag{28}$$

□

With the estimates (21), (23) and (25) proving existence and uniqueness for the dynamic Biot system (10), (11) with homogeneous initial conditions and periodic boundary conditions, is straightforward. We conclude the Section by stating our main result

Theorem 3. *Under the assumptions of Theorem 1, the problem (10), (11), supplemented by homogeneous initial conditions and periodic boundary conditions, has a unique variational solution* $\{\mathbf{u}^0, p^0\} \in H^1((0,T) \times \Omega)^3 \cap H^2(0,T; L^2(\Omega))^3 \times H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$.

III. FROM THE DYNAMIC BIOT SYSTEM TO THE QUASI-STATIC BIOT SYSTEM

In this Section we show that the quasi-static Biot equations can be used iteratively to solve the dynamic Biot equations. We study the behavior of the system (10), (11) in the limit $\kappa_f, \kappa_s \to 0$. Note that κ_f (and consequently also κ_s) stands for the ratio between the intrinsical characteristic time and the characteristic reservoir time scale. In reservoir engineering they are small. An exemple is given at Table II. For the data from Table II

SYMBOL	*QUANTITY*	*CHARACTERISTIC VALUE*
Λ	Young's modulus	7e9 Pa
ρ_f	fluid density	1e3 kg/m^3
ρ_s	solid grain density	2.65e3 kg/m^3
η	fluid viscosity	1e−3 kg/m sec
ℓ	typical pore size	1e−5 m
L	observation length	5000 m
ε	small parameter	$\varepsilon = \ell/L = 0.2\text{e}{-8}$
E_f	pore fluid bulk modulus	1e6 Pa

TABLE II. *Typical reservoir data description* one finds

$$T_c = 0.41 \text{ days}\ ,\ \kappa_f = 0.28e-8,\ \kappa_s = 0.742e-8, \quad \kappa_{co} = O(1) = \psi_f = O(1) = \psi_s.$$

For more details we refer to[12]. We note that in simulation of the noise protection tiny poroelastic layers $\kappa_f = O(1) = \kappa_s$ and one has to consider the full hyperbolic-parabolic Biot-Allard system with memory.

Formally in the singular limit $\kappa_f, \kappa_s \to 0$ one obtains the quasi-static Biot system:

$$-\text{ Div } \{A^0 e(\mathbf{u}^{QS})\} + \text{ Div } \{\alpha p^{QS}\} = \psi \mathbf{F}(x,t), \tag{29}$$

$$M^0 \partial_t p^{QS} + \text{ div } \{K(\psi_f \mathbf{F} - \nabla p^{QS}) + \alpha \partial_t \mathbf{u}^{QS}\} = 0 \tag{30}$$

with periodic boundary conditions and the homogeneous initial condition for the pressure $p^{QS}|_{t=0} = 0$. Appearance of the Darcy permeability tensor $K = \int_0^{+\infty} \mathcal{A}(z)\ dz$ is linked to the fact that for every t positive and a bounded continuous function $\mathbf{g}$ defined for $t \geq 0$, and with values in $\mathbb{R}^3$ one has

$$\lim_{\kappa_f \to 0} \int_0^t \mathcal{A}(\frac{t-z}{\kappa_f}) \frac{\mathbf{g}(z)}{\kappa_f}\ dz = (\int_0^{+\infty} \mathcal{A}(z)\ dz) \mathbf{g}(t) = K\mathbf{g}(t). \tag{31}$$

Proving that the quasi-static Biot system has a unique solution with high regularity in time is straightforward. We have

Lemma 1. *Let $\mathbf{F} \in C_0^\infty(\mathbb{R}_+; L_0^2(\Omega)^3)$, A^0 be a symmetric positive definite $4th$ order tensor and K be a positive definite symmetric matrix. Assume M^0 is a positive constant and α be a symmetric matrix. Then the problem (29)-(30) has a unique solution $\{\mathbf{u}^{QS}, p^{QS}\} \in H^k(0,T; H^1_{per}(\Omega))^4$, $\int_\Omega \mathbf{u}^{QS}\ dx = 0$, for all $k \in \mathbb{N}$.*

In order to obtain the iterative procedure we introduce the unknowns

$$\mathbf{u}^c = \frac{\mathbf{u}^0 - \mathbf{u}^{QS}}{\kappa_f} \quad \text{ and } \quad p^c = \frac{p^0 - p^{QS}}{\kappa_f}.$$

Using (10), (11) and (29), (30) we see that $\{\mathbf{u}^c, p^c\}$ satisfy the system

$$\begin{gathered} \kappa \partial_{tt} \mathbf{u}^c - \text{ Div } \{A^0 D(\mathbf{u}^c) - \alpha p^c\} - \partial_t \int_0^t \mathcal{A}(\frac{t-z}{\kappa_f})(\nabla p^c(x,z) + \kappa_f \partial_{zz} \mathbf{u}^c(x,z))\ dz = \\ -\frac{\kappa}{\kappa_f} \partial_{tt} \mathbf{u}^{QS} + \frac{1}{\kappa_f} \partial_t \int_0^t \mathcal{A}(\frac{t-z}{\kappa_f}) \Big(\nabla p^{QS} + \kappa_f \partial_{zz} \mathbf{u}^{QS} - \psi_f \mathbf{F}(x,z) \Big)\ dz, \end{gathered} \tag{32}$$

$$\begin{gathered} M^0 \partial_t p^c - \text{ div } \{\int_0^t \mathcal{A}(\frac{t-z}{\kappa_f})(\frac{1}{\kappa_f} \nabla p^c + \partial_{zz} \mathbf{u}^c)\ dz\} + \text{ div } \{\alpha \partial_t \mathbf{u}^c\} = \\ \frac{1}{\kappa_f} \text{ div } \{K(\psi_f \mathbf{F} - \nabla p^{QS})\} + \frac{1}{\kappa_f} \text{ div } \{\int_0^t \mathcal{A}(\frac{t-z}{\kappa_f})(\frac{1}{\kappa_f} \nabla p^{QS} + \partial_{zz} \mathbf{u}^{QS} - \frac{\psi_f}{\kappa_f} \mathbf{F})\ dz\}. \end{gathered} \tag{33}$$

with homogeneous initial conditions and periodic boundary conditions.

Theorem 4. *Let us suppose that the assumptions of Theorem 1 and Lemma 1 are satisfied and in addition assume $\hat{\mathcal{A}}$ is an analytic function such that*

$$\frac{d}{d\tau}\hat{\mathcal{A}}(0) = -\int_0^{+\infty} t\mathcal{A}(t)\ dt. \tag{34}$$

Then in the limit $\kappa_f, \kappa_s \to 0$ we have

$$\mathbf{u}^c = \frac{\mathbf{u}^0 - \mathbf{u}^{QS}}{\kappa_f} \to \mathbf{u}^{cor} \quad in \quad H^1((0,T)\times\Omega)^3 \tag{35}$$

$$and \qquad p^c = \frac{p^0 - p^{QS}}{\kappa_f} \to p^{cor} \quad in \quad H^1((0,T)\times\Omega), \tag{36}$$

where $\{\mathbf{u}^{cor}, p^{cor}\}$ is the solution for the problem

$$- \ Div\ \{A^0 e(\mathbf{u}^{cor})\} + \ Div\ \{\alpha p^{cor}\} = -(\varphi + (1-\varphi)\frac{\rho_s}{\rho_f})\partial_{tt}\mathbf{u}^{QS} -$$

$$\partial_t K(\psi_f \mathbf{F}(x,t) - \nabla p^{QS}(x,t)), \tag{37}$$

$$M^0\partial_t p^{cor} + \ div\ \{-K\nabla p^{cor} + \alpha\partial_t \mathbf{u}^{cor}\} = \ div\ \{K\partial_{tt}\mathbf{u}^{QS}\}$$

$$+ \ div\ \{(\int_0^{+\infty} z\mathcal{A}(z)\ dz)\partial_t(\psi_f\mathbf{F}(x,t) - \nabla p^{QS}(x,t))\}. \tag{38}$$

Remark 2. *Using $K = \hat{\mathcal{A}}(0)$, we see that (34) implies*

$$\lim_{\kappa_f \to 0}\frac{1}{\kappa_f}\{\int_0^t \mathcal{A}(\frac{t-z}{\kappa_f})\frac{\mathbf{g}(z)}{\kappa_f}\ dz - K\mathbf{g}(t)\} = -(\int_0^{+\infty}\tau\mathcal{A}(z)\ dz)\partial_t\mathbf{g},$$

for every $g \in H^1_0(\mathbb{R}_+)$.

Corollary 2. $\{\mathbf{u}^c, p^c\} = \{\mathbf{u}^{QS}, p^{QS}\} + \kappa_f\{\mathbf{u}^{cor}, p^{cor}\} + o(\kappa_f)$ *in the topology of* $H^1((0,T)\times\Omega)^4$.

Proof: First we remark that integrating the equation (17) over Ω gives $\int_\Omega \mathbf{u}^0\ dx = 0$. Application of the Laplace transform to (32) and (33) with $\tau \in \mathbb{C}_+$ yields

$$- \text{ Div } \{A^0 e(\hat{\mathbf{u}}^c)\} + \text{ Div } \{(\alpha - \tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))\hat{p}^c\} + (\kappa I - \ \tau\kappa_f^2\hat{\mathcal{A}}(\tau\kappa_f))\tau^2\hat{\mathbf{u}}^c =$$

$$-(\frac{\kappa}{\kappa_f}I - \tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))\tau^2\hat{\mathbf{u}}^{QS} - \tau\psi_f\hat{\mathcal{A}}(\tau\kappa_f)\hat{\mathbf{F}}(x,\tau) + \tau \text{ div } \{\hat{\mathcal{A}}(\tau\kappa_f)\hat{p}^{QS}\}, \tag{39}$$

$$M^0\tau\hat{p}^c + \text{ div } \{(\alpha - \tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))\tau\hat{\mathbf{u}}^c\} - \text{ div } \{\hat{\mathcal{A}}(\tau\kappa_f)\nabla\hat{p}^c\} =$$

$$-\frac{1}{\kappa_f}\text{ div } \{(\hat{\mathcal{A}}(\tau\kappa_f) - K)(\psi_f\hat{\mathbf{F}}(x,\tau) - \nabla\hat{p}^{QS})\} + \text{ div } \{\hat{\mathcal{A}}(\tau\kappa_f)\tau^2\hat{\mathbf{u}}^{QS}\}. \tag{40}$$

Using the arguments of Proposition 2 we conclude that

$$\int_\Omega\{\kappa_s|\tau^2\hat{\mathbf{u}}^c|^2 + |e(\tau\hat{\mathbf{u}}^c)|^2 + |e(\hat{\mathbf{u}}^c)|^2 + |\hat{\mathbf{u}}^c|^2 + |\tau\hat{p}^c|^2 + |\hat{p}^c|^2 +$$

$$|\nabla\hat{p}^c|^2\}\ dx \le C(1+|\tau|^2)^2\Big(||\hat{\mathbf{F}}||^2_{L^2(\Omega)} + ||\tau\nabla\hat{p}^{QS}||^2_{L^2(\Omega)} + ||\tau^2\hat{\mathbf{u}}^{QS}||^2_{L^2(\Omega)}\Big), \tag{41}$$

where C is independent of κ_f. Therefore the sequence $\{\hat{\mathbf{u}}^c, \hat{p}^c\}$ contains a $H^2(\mathbb{C}_+, L^2(\Omega))^3$-weakly convergent subsequence which converges to the quadruple $\{\hat{\mathbf{u}}^{c,0}, \hat{p}^{c,0}\}$ in the following sense

$$\begin{cases} \tau\hat{\mathbf{u}}^c \rightharpoonup \tau\hat{\mathbf{u}}^{c,0}, \ \hat{\mathbf{u}}^c \rightharpoonup \hat{\mathbf{u}}^{c,0} \ \text{ and } \ e(\tau\hat{\mathbf{u}}^c) \rightharpoonup e(\tau\hat{\mathbf{u}}^{c,0}), \\ \nabla\hat{p}^c \rightharpoonup \nabla\hat{p}^{c,0}, \ \hat{p}^c \rightharpoonup \hat{p}^{c,0} \ \text{ and } \ \tau\hat{p}^c \rightharpoonup \tau\hat{p}^{c,0}, \end{cases} \tag{42}$$

as $\kappa_f \to 0$.

In addition we have the following convergences in $H^2(\mathbb{C}_+, L^2(\Omega))^3$ as $\kappa_f \to 0$:

$$\frac{\kappa}{\kappa_f} I - \tau\kappa_f \hat{\mathcal{A}}(\tau\kappa_f))\tau^2\hat{\mathbf{u}}^{QS} \to (\varphi + (1-\varphi)\frac{\rho_s}{\rho_f})\tau^2\mathbf{u}^{QS}, \tag{43}$$

$$\tau\psi_f \hat{\mathcal{A}}(\tau\kappa_f)\hat{\mathbf{F}}(x,\tau) \to \psi_f \hat{\mathcal{A}}(0)\tau\hat{\mathbf{F}}(x,\tau), \tag{44}$$

$$\tau \text{ div } \{\hat{\mathcal{A}}(\tau\kappa_f)\hat{p}^{QS}\} \to \text{ div } \{\hat{\mathcal{A}}(0)\tau\hat{p}^{QS}\}, \tag{45}$$

$$\frac{1}{\kappa_f} \text{ div } \{(\hat{\mathcal{A}}(\tau\kappa_f) - K)(\psi_f\hat{\mathbf{F}}(x,\tau) - \nabla\hat{p}^{QS})\} \to \text{ div } \{\frac{d}{d\tau}\hat{\mathcal{A}}(0)\tau(\psi_f\hat{\mathbf{F}}(x,\tau) - \nabla\hat{p}^{QS})\}, \tag{46}$$

$$\text{div } \{\hat{\mathcal{A}}(\tau\kappa_f)\tau^2\hat{\mathbf{u}}^{QS}\} \to \text{ div } \{\hat{\mathcal{A}}(0)\tau^2\hat{\mathbf{u}}^{QS}\}. \tag{47}$$

Therefore $\{\hat{\mathbf{u}}^{c,0}, \hat{p}^{c,0}\}$ satisfies the system (37)-(38), with homogeneous initial conditions and periodic boundary conditions. Because of uniqueness for the quasi-static Biot system, we conclude that $\{\hat{\mathbf{u}}^{c,0}, \hat{p}^{c,0}\} = \{\hat{\mathbf{u}}^{cor}, \hat{p}^{cor}\}$ and the convergence of the whole sequence.

Strong convergences follows as usual, by using the solutions as test functions. □

Remark 3. *We see that the quasi-static Biot equations solver could be used to solve the dynamic Biot equations. The quasi-static Biot equations are solved efficiently using iterative coupling procedures, where either the flow or the mechanics is solved first followed by solving the other problem using the latest solution information. We refer to[13] for presentation of the four widely used methods and their von Neumann stability analysis. The convergence and convergence rates for two widely used schemes, the undrained split method and the fixed stress split method was recently proved in[15].*

Remark 4. *It is easy to see that* $\{\mathbf{u}^W, p^W\} = \{\mathbf{u}^{QS} + \kappa_f\mathbf{u}^{cor}, p^{QS} + \kappa_f p^{cor}\}$ *satisfies at order* $O(\kappa_f)$ *the system*

$$\kappa\partial_{tt}\mathbf{u}^W - \ Div\ \{A^0 e(\mathbf{u}^W)\} + \ Div\ \{\alpha p^W - \kappa_f K\partial_t p^W\} = \psi\mathbf{F} - \kappa_f\psi_f K\partial_t\mathbf{F}, \tag{48}$$

$$M^0\partial_t p^W + \partial_t \ div\ \{\alpha\mathbf{u}^W - \kappa_f K\partial_t\mathbf{u}^W\} - \ div\ \{K\nabla p^W - \kappa_f(\int_0^{+\infty} z\mathcal{A}(z)\ dz)\partial_t\nabla p^W\} =$$
$$\psi_f \ div\ \{\kappa_f(\int_0^{+\infty} z\mathcal{A}(z)\ dz)\partial_t\mathbf{F} - K\mathbf{F}\}. \tag{49}$$

IV. APPENDIX: ELLIPTICITY ESTIMATES FOR THE DYNAMIC PERMEABILITY

As stated above, the Biot-Allard equations can be obtained using homogenization if one supposes the statistical homogeneity of the pore structure. In particular we suppose a periodic porous medium defined by (**A1**)-(**A2**).

Proposition 3. *(see[7]) Let us suppose (**A1**)-(**A2**). Then the hypotheses (**H1**)-(**H3**) from Theorem 1 are valid.*

We now demonstrate the properties of the dynamic permeability assumed as the hypothesis (**H4**) in Theorem 1 are valid.

In order to calculate the dynamic permeability, we consider the permeability auxiliary problem:

$$\partial_t \mathbf{q}^i - \Delta \mathbf{q}^i + \nabla \pi_q^i = 0 \ \text{ in } \ \mathcal{Y}_f \times (0,T) \tag{50}$$

$$\text{div } \mathbf{q}^i = 0 \ \text{ in } \ \mathcal{Y}_f \times (0,T), \quad \mathbf{q}^i = 0 \ \text{ on } \ (\partial\mathcal{Y}_f \setminus \partial\mathcal{Y}) \times (0,T) \tag{51}$$

$$\{\mathbf{q}^i, \pi_q^i\} \quad \text{are 1-periodic in } y, \quad \mathbf{q}^i|_{\{t=0\}} = \mathbf{e}^i \text{ on } \mathcal{Y}_f. \tag{52}$$

We note that $\mathbf{q}^i$ depends only on the geometry. We calculate $\mathbf{q}^i$ using the corresponding spectral problem:

$$-\triangle \mathbf{w} + \nabla \rho = \lambda \mathbf{w} \quad \text{in} \quad \mathcal{Y}_f \tag{53}$$

$$\text{div } \ \mathbf{w} = 0 \quad \text{in} \quad \mathcal{Y}_f, \quad \mathbf{w} = 0 \ \text{ on } \ \partial\mathcal{Y}_f \backslash \partial\mathcal{Y} \tag{54}$$

$$\mathbf{w} \ \text{ is } \ H^1(\mathcal{Y}) - \text{periodic and } \ \rho \ \text{ is } \ L^2(\mathcal{Y}) - \ \text{periodic.} \tag{55}$$

By the elementary spectral theory the eigenfunctions for (53)-(55) form an orthonormal basis $\{\mathbf{f}^k\}_{k\geq 1}$ for $L^2(\mathcal{Y}_f)^3$ and an orthogonal basis for $V_{per} = \{\mathbf{z} \in H^1_{per} : \mathbf{z} = 0 \text{ on } \partial\mathcal{Y}_f \backslash \partial\mathcal{Y}$ and $\text{div } \mathbf{z} = 0 \text{ in } \mathcal{Y}_f\}$. Note that $H^1_{per} = \{\mathbf{z} \in H^1(\mathcal{Y}_f)^3 : \mathbf{z} \text{ is } H^1(\mathcal{Y})\text{-periodic } \}$.

Furthermore λ_1, the minimum eigenvalue of the Stokes operator (53), is positive and $\lambda_k \to +\infty$, as $k \to +\infty$. By the elementary variational parabolic theory we have the following separation of variables expansion for $\mathbf{q}^j$:

$$\mathbf{q}^j(y,t) = \sum_{k=1}^{+\infty} e^{-\lambda_k t} \mathbf{f}^k(y) \int_{\mathcal{Y}_f} f_j^k(z)\, dz. \tag{56}$$

The series (56) converges in $C([0,T]; L^2(\mathcal{Y}_f))$ and in $L^2(0,T; V_{per})$.

Consequently we have

$$||\mathbf{q}^i(t)||_{L^2(\mathcal{Y}_f)} \leq \varphi^{1/2} e^{-\lambda_1 t}, \quad 1 \leq i \leq 3, \quad \varphi = |\mathcal{Y}_f|. \tag{57}$$

The dynamic permeability matrix is defined by

$$\mathcal{A}_{ij}(t) = \int_{\mathcal{Y}_f} q^i_j(y,t)\ dy \tag{58}$$

$$\text{and} \quad K_{ij} = \hat{\mathcal{A}}_{ij}(0) = \int_0^{+\infty} \mathcal{A}_{ij}(t)\ dt \text{ is Darcy's permeability.} \tag{59}$$

The Laplace transform of the matrix $\mathcal{A}$, given by (58) is

$$\hat{\mathcal{A}}_{ij}(\tau) = \int_{\mathcal{Y}_f} \hat{q}^i_j(y,\tau)\ dy, \quad \text{with } K_{ij} = \hat{\mathcal{A}}_{ij}(0) \tag{60}$$

$$\text{and} \int_0^{+\infty} t\mathcal{A}_{ij}(t)\ dt = \int_0^{+\infty} \int_{\mathcal{Y}_f} q^i_j(y,t) w^k_i(y)\ dydt. \tag{61}$$

We have

$$\hat{\mathcal{A}}_{ij}(\tau) = \int_{\mathcal{Y}_f} \hat{q}^i_j(y,\tau)\ dy = \overline{\tau} \int_{\mathcal{Y}_f} \hat{\mathbf{q}}^i \overline{\hat{\mathbf{q}}^j}\ dy + \int_{\mathcal{Y}_f} \nabla_y \hat{\mathbf{q}}^i \nabla_y \overline{\hat{\mathbf{q}}^j}\ dy =$$

$$\tau \int_{\mathcal{Y}_f} \hat{\mathbf{q}}^i \hat{\mathbf{q}}^j\ dy + \int_{\mathcal{Y}_f} \nabla_y \hat{\mathbf{q}}^i \nabla_y \hat{\mathbf{q}}^j\ dy, \quad 1 \leq i,j \leq 3, \tag{62}$$

which implies that $\hat{\mathcal{A}}$ is a complex symmetric matrix (but not a Hermitian matrix).

Furthermore, we have

$$\tau \hat{\mathcal{A}}_{ij}(\tau) = |\tau|^2 \int_{\mathcal{Y}_f} \hat{\mathbf{q}}^i \overline{\hat{\mathbf{q}}^j}\ dy + \tau \int_{\mathcal{Y}_f} \nabla_y \hat{\mathbf{q}}^i \nabla_y \overline{\hat{\mathbf{q}}^j}\ dy. \tag{63}$$

Let $\xi \in \mathbb{C}^3$, let $\mathbf{e}^\xi = \sum_{j=1}^3 \xi_j \mathbf{e}^j$ and let $\hat{\mathbf{q}}^\xi(\tau) = \sum_{j=1}^3 \xi_j \hat{\mathbf{q}}^j(\tau)$. Then $\hat{\mathbf{q}}^\xi$ is the solution to the problem

$$\tau \hat{\mathbf{q}}^\xi - \Delta \hat{\mathbf{q}}^\xi + \nabla \hat{\pi}^\xi_q = \mathbf{e}^\xi \ \text{ in } \ \mathcal{Y}_f, \tag{64}$$

$$\text{div } \hat{\mathbf{q}}^\xi = 0 \ \text{ in } \ \mathcal{Y}_f, \quad \hat{\mathbf{q}}^\xi = 0 \ \text{ on } \ (\partial\mathcal{Y}_f \setminus \partial\mathcal{Y}), \tag{65}$$

$$\{\hat{\mathbf{q}}^\xi, \hat{\pi}^\xi_q\} \quad \text{are 1-periodic in } y. \tag{66}$$

We use the spectral basis $\{\mathbf{f}^k\}_{k\geq 1}$ to write expansions for $\hat{\mathbf{q}}^\xi$, $\mathbf{w}^j = \hat{\mathbf{q}}^j(0)$ and for the matrices

$\hat{\mathcal{A}}$ and K:

$$\hat{\mathbf{q}}^{\xi}(y,\tau)=\sum_{k=1}^{+\infty}\frac{\mathbf{f}^k(y)}{\lambda_k+\tau}\int_{\mathcal{Y}_f}\mathbf{e}^{\xi}\cdot\mathbf{f}^k(z)\,dz, \tag{67}$$

$$\mathbf{w}^j(y)=\sum_{k=1}^{+\infty}\frac{\mathbf{f}^k(y)}{\lambda_k}\int_{\mathcal{Y}_f}f_j^k(z)\,dz, \tag{68}$$

$$\hat{\mathcal{A}}(\tau)\xi\overline{\xi}=\sum_{k=1}^{+\infty}\frac{1}{\lambda_k+\tau}\Big|\int_{\mathcal{Y}_f}\mathbf{e}^{\xi}\cdot\mathbf{f}^k(z)\,dz\Big|^2, \tag{69}$$

$$\text{and}\quad K\xi\overline{\xi}=\sum_{k=1}^{+\infty}\frac{1}{\lambda_k}\Big|\int_{\mathcal{Y}_f}\mathbf{e}^{\xi}\cdot\mathbf{f}^k(z)\,dz\Big|^2. \tag{70}$$

Proposition 4. *Let $\mathcal{A}$ be given by (58) and its Laplace transform $\hat{\mathcal{A}}$ by (60). Then the estimate (15) from Theorem 1 holds true.*

Furthermore, the matrix $\hat{\mathcal{A}}$ is positive definite for every $\tau\in\mathbb{C}_+=\{\tau\in\mathbb{C}:\ \Re\tau>0\}$ in the sense that the following estimates hold:

$$\Re\{\hat{\mathcal{A}}(\tau)\xi\overline{\xi}\}\geq C\frac{\Re\tau}{|\lambda_1+\tau|^2}|\xi|^2+\sum_{k=1}^{+\infty}\frac{\lambda_k}{|\lambda_k+\tau|^2}\Big|\int_{\mathcal{Y}_f}\mathbf{e}^{\xi}\cdot\mathbf{f}^k(z)\,dz\Big|^2,\ \forall\xi\in\mathbb{C}^3, \tag{71}$$

$$\Re\{\hat{\mathcal{A}}(\tau)\xi\overline{\xi}\}\geq C\frac{\lambda_1+\Re\tau}{|\lambda_1+\tau|^2}|\xi|^2,\ \forall\xi\in\mathbb{C}^3, \tag{72}$$

$$\Re\{\tau\hat{\mathcal{A}}(\tau)\xi\overline{\xi}\}+\Re\{\hat{\mathcal{A}}(\tau)\xi\overline{\xi}\}\geq K\xi\overline{\xi},\ \forall\xi\in\mathbb{C}^3. \tag{73}$$

Proof: Estimate (15) is obvious. Next we prove (71):

$$\Re\{\hat{\mathcal{A}}(\tau)\xi\overline{\xi}\}=\sum_{k=1}^{+\infty}\frac{\lambda_k+\Re\tau}{|\lambda_k+\tau|^2}\Big|\int_{\mathcal{Y}_f}\mathbf{e}^{\xi}\cdot\mathbf{f}^k(z)\,dz\Big|^2\geq$$

$$\frac{\lambda_1^2\Re\tau}{|\lambda_1+\tau|^2}\sum_{k=1}^{+\infty}\frac{1}{\lambda_k^2}\Big|\int_{\mathcal{Y}_f}\mathbf{e}^{\xi}\cdot\mathbf{f}^k(z)\,dz\Big|^2+\sum_{k=1}^{+\infty}\frac{\lambda_k}{|\lambda_k+\tau|^2}\Big|\int_{\mathcal{Y}_f}\mathbf{e}^{\xi}\cdot\mathbf{f}^k(z)\,dz\Big|^2,\quad\forall\xi\in\mathbb{C}^3. \tag{74}$$

Estimate (74) implies estimate (71).

Next we have

$$\Re\{\hat{\mathcal{A}}(\tau)\xi\overline{\xi}\}=\sum_{k=1}^{+\infty}\frac{\lambda_k+\Re\tau}{|\lambda_k+\tau|^2}\Big|\int_{\mathcal{Y}_f}\mathbf{e}^{\xi}\cdot\mathbf{f}^k(z)\,dz\Big|^2\geq C\frac{\lambda_1+\Re\tau}{|\lambda_1+\tau|^2}K\xi\overline{\xi}. \tag{75}$$

Since the permeability tensor K is positive definite, (75) implies (72).

Finally, the remaining lower bound is

$$\Re\{(1+\tau)\hat{\mathcal{A}}(\tau)\xi\overline{\xi}\}=\sum_{k=1}^{+\infty}\Big(\frac{|\tau|^2+\lambda_k\Re\tau}{|\lambda_k+\tau|^2}+\frac{\lambda_k^2+\lambda_k\Re\tau}{|\lambda_k+\tau|^2}\frac{1}{\lambda_k}\Big)\Big|\int_{\mathcal{Y}_f}\mathbf{e}^{\xi}\cdot\mathbf{f}^k(z)\,dz\Big|^2\geq$$

$$\sum_{k=1}^{+\infty}\frac{1}{\lambda_k}\Big|\int_{\mathcal{Y}_f}\mathbf{e}^{\xi}\cdot\mathbf{f}^k(z)\,dz\Big|^2\geq K\xi\overline{\xi}. \tag{76}$$

□

Lemma 2. *Let the matrix $\mathcal{A}$ be given by (58) and let its Laplace's transform $\hat{\mathcal{A}}$ be given by (60). Then the matrix $\tau\kappa I - \tau\kappa_f^2\hat{\mathcal{A}}(\kappa_f\tau))$ is positive definite for every $\tau \in \mathbb{C}_+$ and we have*

$$\Re\{\tau(\kappa I - \tau\kappa_f^2\hat{\mathcal{A}}(\tau\kappa_f))\xi\overline{\xi}\} \geq \kappa_s(1-\varphi)\Re\tau|\xi|^2 + \kappa_f^2\sum_{k=1}^{+\infty}\frac{\lambda_k(\operatorname{Im}\tau)^2}{|\lambda_k+\kappa_f\tau|^2}|\int_{\mathcal{Y}_f}\mathbf{e}^\xi\cdot\mathbf{f}^k(z)\,dz|^2,\ \forall\xi\in\mathbb{C}^3. \tag{77}$$

Proof: We estimate the sesquilinear form $\tau^2\kappa_f^2\hat{\mathcal{A}}(\kappa_f\tau)\xi\overline{\xi}$, with $\tau\in\mathbb{C}_+$:

$$\Re\{\tau^2\kappa_f^2\hat{\mathcal{A}}(\kappa_f\tau)\xi\overline{\xi}\} = \kappa_f^2\sum_{k=1}^{+\infty}\frac{\lambda_k\Re\tau^2+\kappa_f\Re\tau|\tau|^2}{|\lambda_k+\kappa_f\tau|^2}|\int_{\mathcal{Y}_f}\mathbf{e}^\xi\cdot\mathbf{f}^k(z)\,dz|^2 =$$
$$(\Re\tau\kappa_f)\sum_{k=1}^{+\infty}\frac{\lambda_k\kappa_f\Re\tau+\kappa_f^2|\tau|^2}{|\lambda_k+\kappa_f\tau|^2}|\int_{\mathcal{Y}_f}\mathbf{e}^\xi\cdot\mathbf{f}^k(z)\,dz|^2 - \kappa_f^2\sum_{k=1}^{+\infty}\frac{\lambda_k(\operatorname{Im}\tau)^2}{|\lambda_k+\kappa_f\tau|^2}|\int_{\mathcal{Y}_f}\mathbf{e}^\xi\cdot\mathbf{f}^k(z)\,dz|^2 \leq$$
$$\Re\tau\kappa_f\sum_{k=1}^{+\infty}|\int_{\mathcal{Y}_f}\mathbf{e}^\xi\cdot\mathbf{f}^k(z)\,dz|^2 - \kappa_f^2\sum_{k=1}^{+\infty}\frac{\lambda_k(\operatorname{Im}\tau)^2}{|\lambda_k+\kappa_f\tau|^2}|\int_{\mathcal{Y}_f}\mathbf{e}^\xi\cdot\mathbf{f}^k(z)\,dz|^2$$
$$= \Re\tau\varphi\kappa_f|\xi|^2 - \kappa_f^2\sum_{k=1}^{+\infty}\frac{\lambda_k(\operatorname{Im}\tau)^2}{|\lambda_k+\kappa_f\tau|^2}|\int_{\mathcal{Y}_f}\mathbf{e}^\xi\cdot\mathbf{f}^k(z)\,dz|^2, \quad \forall\xi\in\mathbb{C}^3, \tag{78}$$

which implies (77). □

Corollary 3. *Estimates (12)-(15) from the hypothesis (H4) of Theorem 1 hold true.*

Proof: It remains only to prove (14). Let $\xi,\beta\in\mathbb{C}^3$. Then we have

$$\tau\hat{\mathcal{A}}(\tau)\beta\overline{\xi} - \overline{\tau\hat{\mathcal{A}}(\tau)\overline{\xi}}\beta = 2i\operatorname{Im}\tau\int_{\mathcal{Y}_f}\nabla_y\hat{\mathbf{q}}^\beta\nabla_y\overline{\hat{\mathbf{q}}^\xi}\,dy \quad \text{and}$$

$$\Re\{\tau(\kappa I - \tau\kappa_f^2\hat{\mathcal{A}}(\tau\kappa_f))\xi\overline{\xi} + \hat{\mathcal{A}}(\kappa_f\tau)\beta\overline{\beta} + (\alpha-\tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))\beta\overline{\xi} - \overline{(\alpha-\tau\kappa_f\hat{\mathcal{A}}(\tau\kappa_f))\overline{\xi}}\beta\} \geq$$
$$\kappa_s(1-\varphi)\Re\tau|\xi|^2 + C\kappa_f\frac{\Re\tau}{|\lambda_1+\kappa_f\tau|^2}|\beta|^2 + \sum_{k=1}^{+\infty}\frac{\lambda_k}{|\lambda_k+\kappa_f\tau|^2}\big((\kappa_f\operatorname{Im}\tau)^2|\int_{\mathcal{Y}_f}\mathbf{e}^\xi\cdot\mathbf{f}^k(z)\,dz|^2 +$$
$$|\int_{\mathcal{Y}_f}\mathbf{e}^\beta\cdot\mathbf{f}^k(z)\,dz|^2 - 2\operatorname{Im}\tau\kappa_f|\int_{\mathcal{Y}_f}\mathbf{e}^\xi\cdot\mathbf{f}^k(z)\,dz||\int_{\mathcal{Y}_f}\mathbf{e}^\beta\cdot\mathbf{f}^k(z)\,dz|\big),$$

which yields (14).□

Remark 5. *For the definition of the dynamic permeability in a general porous medium we refer to[16]. Heuristic estimates for a random porous medium are in[17] and[18]. For computation of the dynamic permeability for periodic, random and fractal porous media we refer to[9] and[10].*

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